SECTION P.6

Complex Numbers

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A3.

In Exercises PS1 to PS5, simplify the expression.

PS1. $(2 - 3x)(4 - 5x)$ [P.3]

PS2. $(2 - 5x)^2$ [P.3]

PS3. $\sqrt{96}$ [P.2]

PS4. $(2 + 3\sqrt{5})(3 - 4\sqrt{5})$ [P.2]

PS5. $\dfrac{5 + \sqrt{2}}{3 - \sqrt{2}}$ [P.2]

PS6. Which of the following polynomials, if any, does not factor over the integers? [P.4]

i. $81 - x^2$ **ii.** $9 + z^2$

Math Matters

It may seem strange to just invent new numbers, but that is how mathematics evolves. For instance, negative numbers were not an accepted part of mathematics until well into the thirteenth century. In fact, these numbers often were referred to as "fictitious numbers."

In the seventeenth century, René Descartes called square roots of negative numbers "imaginary numbers," an unfortunate choice of words, and started using the letter *i* to denote these numbers. These numbers were subjected to the same skepticism as negative numbers.

It is important to understand that these numbers are not *imaginary* in the dictionary sense of the word. This misleading word is similar to the situation of negative numbers being called *fictitious*.

If you think of a number line, then the numbers to the right of zero are positive numbers and the numbers to the left of zero are negative numbers. One way to think of an imaginary number is to visualize it as *up* or *down* from zero.

Introduction to Complex Numbers

Recall that $\sqrt{9} = 3$ because $3^2 = 9$. Now consider the expression $\sqrt{-9}$. To find $\sqrt{-9}$, we need to find a number c such that $c^2 = -9$. However, the square of any real number c (except zero) is a *positive* number. Consequently, we must expand our concept of number to include numbers whose squares are negative numbers.

Around the seventeenth century, a new number, called an *imaginary number*, was defined so that a negative number would have a square root. The letter i was chosen to represent the number whose square is -1.

Definition of *i*

The **imaginary unit,** designated by the letter i, is the number such that $i^2 = -1$.

The principal square root of a negative number is defined in terms of i.

Math Matters

The imaginary unit *i* is important in the field of electrical engineering. However, because the letter *i* is used by engineers as the symbol for electric current, these engineers use *j* for the complex unit.

Definition of an Imaginary Number

If a is a positive real number, then $\sqrt{-a} = i\sqrt{a}$. The number $i\sqrt{a}$ is called an **imaginary number.**

EXAMPLE

$\sqrt{-36} = i\sqrt{36} = 6i$ $\sqrt{-18} = i\sqrt{18} = 3i\sqrt{2}$

$\sqrt{-23} = i\sqrt{23}$ $\sqrt{-1} = i\sqrt{1} = i$

It is customary to write i in front of a radical sign, as we did for $i\sqrt{23}$, to avoid confusing $\sqrt{a}i$ with $\sqrt{ai}$.

Definition of a Complex Number

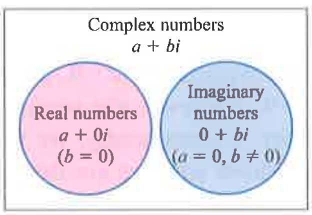
A **complex number** is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. The number a is the **real part** of $a + bi$, and b is the **imaginary part.**

EXAMPLE

$-3 + 5i$	• Real part: -3; imaginary part: 5
$2 - 6i$	• Real part: 2; imaginary part: -6
5	• Real part: 5; imaginary part: 0
$7i$	• Real part: 0; imaginary part: 7

Note from these examples that a real number is a complex number whose imaginary part is 0, and an imaginary number is a complex number whose real part is 0 and whose imaginary part is not 0.

Question • What are the real part and imaginary part of $3 - 5i$?



Note from the diagram at the left that the set of real numbers is a subset of the complex numbers and the set of imaginary numbers is a separate subset of the complex numbers. The set of real numbers and the set of imaginary numbers are disjoint sets.

Example 1 illustrates how to write a complex number in the **standard form** $a + bi$.

EXAMPLE 1 Write a Complex Number in Standard Form

Write $7 + \sqrt{-45}$ in the form $a + bi$.

Solution

$$\begin{aligned} 7 + \sqrt{-45} &= 7 + i\sqrt{45} \\ &= 7 + i\sqrt{9} \cdot \sqrt{5} \\ &= 7 + 3i\sqrt{5} \end{aligned}$$

▶ **Try Exercise 16, page 64**

▶ Addition and Subtraction of Complex Numbers

All the standard arithmetic operations that are applied to real numbers can be applied to complex numbers.

Answer • Real part: 3; imaginary part: -5.

Definition of Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are complex numbers, then

Addition $\quad (a + bi) + (c + di) = (a + c) + (b + d)i$

Subtraction $\quad (a + bi) - (c + di) = (a - c) + (b - d)i$

Basically, these rules say that to add two complex numbers, add the real parts and add the imaginary parts. To subtract two complex numbers, subtract the real parts and subtract the imaginary parts.

EXAMPLE 2 Add or Subtract Complex Numbers

Simplify. **a.** $(7 - 2i) + (-2 + 4i)$ **b.** $(-9 + 4i) - (2 - 6i)$

Solution

a. $(7 - 2i) + (-2 + 4i) = (7 + (-2)) + (-2 + 4)i = 5 + 2i$

b. $(-9 + 4i) - (2 - 6i) = (-9 - 2) + (4 - (-6))i = -11 + 10i$

Try Exercise 26, page 64

Multiplication of Complex Numbers

When complex numbers are multiplied, the term i^2 is frequently a part of the product. Recall that $i^2 = -1$. Therefore,

$$3i(5i) = 15i^2 = 15(-1) = -15$$
$$-2i(6i) = -12i^2 = -12(-1) = 12$$
$$4i(3 - 2i) = 12i - 8i^2 = 12i - 8(-1) = 8 + 12i$$

When multiplying square roots of negative numbers, first rewrite the radical expressions using i. For instance,

$$\sqrt{-6} \cdot \sqrt{-24} = i\sqrt{6} \cdot i\sqrt{24} \qquad \bullet\ \sqrt{-6} = i\sqrt{6},\ \sqrt{-24} = i\sqrt{24}$$
$$= i^2\sqrt{144} = -1 \cdot 12$$
$$= -12$$

Note from this example that it would have been incorrect to multiply the radicands of the two radical expressions. To illustrate:

$$\sqrt{-6} \cdot \sqrt{-24} \neq \sqrt{(-6)(-24)}$$

Caution

Recall that the definition of the product of radical expressions requires that the radicand be a positive number. Therefore, when multiplying expressions containing negative radicands, we first must rewrite the expression using *i* and a positive radicand.

Question • What is the product of $\sqrt{-2}$ and $\sqrt{-8}$?

Answer • $\sqrt{-2} \cdot \sqrt{-8} = i\sqrt{2} \cdot i\sqrt{8} = i^2\sqrt{16} = -1 \cdot 4 = -4$.

To multiply two complex numbers, we use the following definition.

Definition of Multiplication of Complex Numbers

If $a + bi$ and $c + di$ are complex numbers, then

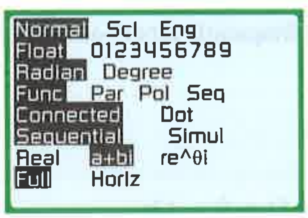
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Because every complex number can be written as a sum of two terms, it is natural to perform multiplication on complex numbers in a manner consistent with the operation defined on binomials and the definition $i^2 = -1$. By using this analogy, you can multiply complex numbers without memorizing the definition.

Integrating Technology

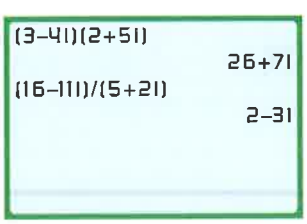
Some graphing calculators can be used to perform operations on complex numbers. Here are some typical screens for a TI-83/TI-83 Plus/TI-84 Plus graphing calculator.

Press MODE. Use the down arrow key to highlight $a + bi$.



Press ENTER 2nd [QUIT].

The following screen shows two examples of computations on complex numbers. To enter an i, use 2nd [i], which is located above the decimal point key.



EXAMPLE 3 Multiply Complex Numbers

Multiply.

a. $3i(2 - 5i)$ **b.** $(3 - 4i)(2 + 5i)$

Solution

a. $3i(2 - 5i) = 6i - 15i^2$

$= 6i - 15(-1)$ • Replace i^2 with -1.

$= 15 + 6i$ • Write in standard form.

b. $(3 - 4i)(2 + 5i) = 6 + 15i - 8i - 20i^2$

$= 6 + 15i - 8i - 20(-1)$ • Replace i^2 with -1.

$= 6 + 15i - 8i + 20$ • Simplify.

$= 26 + 7i$ • Write in standard form.

▸ Try Exercise 38, page 65

Division of Complex Numbers

Recall that the number $\frac{3}{\sqrt{2}}$ is not in simplest form because there is a radical expression in the denominator. Similarly, $\frac{3}{i}$ is not in simplest form because $i = \sqrt{-1}$. To write this expression in simplest form, multiply the numerator and denominator by i.

$$\frac{3}{i} \cdot \frac{i}{i} = \frac{3i}{i^2} = \frac{3i}{-1} = -3i$$

Here is another example.

$$\frac{3 - 6i}{2i} = \frac{3 - 6i}{2i} \cdot \frac{i}{i} = \frac{3i - 6i^2}{2i^2} = \frac{3i - 6(-1)}{2(-1)} = \frac{3i + 6}{-2} = -3 - \frac{3}{2}i$$

Recall that to simplify the quotient $\frac{2 + \sqrt{3}}{5 + 2\sqrt{3}}$, we multiply the numerator and denominator by the conjugate of $5 + 2\sqrt{3}$, which is $5 - 2\sqrt{3}$. In a similar manner, to find the quotient of two complex numbers, we multiply the numerator and denominator by the conjugate of the denominator.

The complex numbers $a + bi$ and $a - bi$ are called **complex conjugates** or **conjugates** of each other. The conjugate of the complex number z is denoted by $\overline{z}$. For instance,

$$\overline{2 + 5i} = 2 - 5i \quad \text{and} \quad \overline{3 - 4i} = 3 + 4i$$

Consider the product of a complex number and its conjugate. For instance,

$$\begin{aligned}(2 + 5i)(2 - 5i) &= 4 - 10i + 10i - 25i^2 \\ &= 4 - 25(-1) = 4 + 25 \\ &= 29\end{aligned}$$

Note that the product is a *real* number. This is always true.

Product of Complex Conjugates

The product of a complex number and its conjugate is a real number. That is, $(a + bi)(a - bi) = a^2 + b^2$.

Example

$(5 + 3i)(5 - 3i) = 5^2 + 3^2 = 25 + 9 = 34$

The next example shows how the quotient of two complex numbers is determined by using conjugates.

EXAMPLE 4 Divide Complex Numbers

Simplify: $\dfrac{16 - 11i}{5 + 2i}$

Solution

$$\begin{aligned}\frac{16 - 11i}{5 + 2i} &= \frac{16 - 11i}{5 + 2i} \cdot \frac{5 - 2i}{5 - 2i} \\ &= \frac{80 - 32i - 55i + 22i^2}{5^2 + 2^2} \\ &= \frac{80 - 32i - 55i + 22(-1)}{25 + 4} \\ &= \frac{80 - 87i - 22}{29} \\ &= \frac{58 - 87i}{29} \\ &= \frac{\cancel{29}(2 - 3i)}{\cancel{29}} = 2 - 3i\end{aligned}$$

• Multiply the numerator and denominator by the conjugate of the denominator.

▶ Try Exercise 52, page 65

Note

For real numbers, we can check a quotient using the fact that if $\frac{a}{b} = c$, then $bc = a$. For instance, $\frac{12}{4} = 3$ and $4 \cdot 3 = 12$. For complex number division, the same applies. Note in Example 4, $\frac{16 - 11i}{5 + 2i} = 2 - 3i$ and

$$(5+2i)(2-3i) = 10+4i-15i-6i^2 = 10-11i+6 = 16-11i$$

Powers of *i*

The following powers of i illustrate a pattern:

$$i^1 = i \qquad i^5 = i^4 \cdot i = 1 \cdot i = i$$
$$i^2 = -1 \qquad i^6 = i^4 \cdot i^2 = 1(-1) = -1$$
$$i^3 = i^2 \cdot i = (-1)i = -i \qquad i^7 = i^4 \cdot i^3 = 1(-i) = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 \qquad i^8 = (i^4)^2 = 1^2 = 1$$

Because $i^4 = 1$, $(i^4)^n = 1^n = 1$ for any integer n. Thus it is possible to evaluate powers of i by factoring out powers of i^4, as shown in the following.

$$i^{27} = (i^4)^6 \cdot i^3 = 1^6 \cdot i^3 = 1 \cdot (-i) = -i$$

The following theorem can also be used to evaluate powers of i.

Powers of *i*

If n is a positive integer, then $i^n = i^r$, where r is the remainder of the division of n by 4.

EXAMPLE 5 Evaluate a Power of *i*

Evaluate: i^{153}

Solution

Use the powers of i theorem.

$$i^{153} = i^1 = i \qquad \text{• Remainder of } 153 \div 4 \text{ is } 1.$$

▶ Try Exercise 64, page 65

EXERCISE SET P.6

Concept Check

In Exercises 1 to 4, determine the real part and the imaginary part of the complex number.

1. $12 - 17i$
2. $-37i$
3. $\sqrt{7}$
4. 0

In Exercises 5 to 8, write the conjugate of the complex number.

5. $4 - 9i$
6. $3 + i\sqrt{7}$
7. $-11i$
8. i

In Exercises 9 to 18, write the complex number in standard form.

9. $\sqrt{-81}$
10. $\sqrt{-64}$
11. $\sqrt{-98}$
12. $\sqrt{-27}$
13. $\sqrt{16} + \sqrt{-81}$
14. $\sqrt{25} + \sqrt{-9}$
15. $5 + \sqrt{-49}$
16. $6 - \sqrt{-1}$
17. $8 - \sqrt{-18}$
18. $11 + \sqrt{-48}$

In Exercises 19 to 58, simplify and write the complex number in standard form.

19. $(5 + 2i) + (6 - 7i)$
20. $(4 - 8i) + (5 + 3i)$
21. $(-2 - 4i) - (5 - 8i)$
22. $(3 - 5i) - (8 - 2i)$
23. $(1 - 3i) + (7 - 2i)$
24. $(2 - 6i) + (4 - 7i)$
25. $(-3 - 5i) - (7 - 5i)$
26. $(5 - 3i) - (2 + 9i)$

Indicates Try It Exercises

27. $8i - (2 - 8i)$

28. $3 - (4 - 5i)$

29. $5i \cdot 8i$

30. $(-3i)(2i)$

31. $\sqrt{-50} \cdot \sqrt{-2}$

32. $\sqrt{-12} \cdot \sqrt{-27}$

33. $3(2 + 5i) - 2(3 - 2i)$

34. $3i(2 + 5i) + 2i(3 - 4i)$

35. $(4 + 2i)(3 - 4i)$

36. $(6 + 5i)(2 - 5i)$

37. $(-3 - 4i)(2 + 7i)$

38. $(-5 - i)(2 + 3i)$

39. $(4 - 5i)(4 + 5i)$

40. $(3 + 7i)(3 - 7i)$

41. $\dfrac{6}{i}$

42. $\dfrac{-8}{2i}$

43. $\dfrac{6 + 3i}{i}$

44. $\dfrac{4 - 8i}{4i}$

45. $\dfrac{1}{7 + 2i}$

46. $\dfrac{5}{3 + 4i}$

47. $\dfrac{2i}{1 + i}$

48. $\dfrac{5i}{2 - 3i}$

49. $\dfrac{5 - i}{4 + 5i}$

50. $\dfrac{4 + i}{3 + 5i}$

51. $\dfrac{3 + 2i}{3 - 2i}$

52. $\dfrac{8 - i}{2 + 3i}$

53. $\dfrac{-7 + 26i}{4 + 3i}$

54. $\dfrac{-4 - 39i}{5 - 2i}$

55. $(3 - 5i)^2$

56. $(2 + 4i)^2$

57. $(1 + 2i)^2$

58. $(2 - i)^2$

In Exercises 59 to 66, evaluate.

59. i^{15}

60. i^{66}

61. $-i^{40}$

62. $-i^{51}$

63. $\dfrac{1}{i^{25}}$

64. $\dfrac{1}{i^{83}}$

65. i^{-34}

66. i^{-52}

In Exercises 67 to 70, evaluate $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ for the given values of a, b, and c. Write your answer as a complex number in standard form.

67. $a = 2, b = 4, c = 4$

68. $a = 4, b = -4, c = 2$

69. $a = 3, b = -3, c = 3$

70. $a = 2, b = 6, c = 6$

Enrichment Exercises

In Exercises 71 and 72, expand the power of the complex number.

71. $(1 - 3i)^3$

72. $(2 + i)^4$

73. Show that $\sqrt{i} = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$.

74. Show that $\sqrt{-i} = \dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$.

Exploring Concepts with Technology

Can You Trust Your Calculator?

You may think that your calculator always produces correct results in a *predictable* manner. However, the following experiment may change your opinion.

First note that the algebraic expression

$$p + 3p(1 - p)$$

is equal to the expression

$$4p - 3p^2$$

Use a graphing calculator to evaluate both expressions with $p = 0.05$. You should find that both expressions equal 0.1925. So far, we do not observe any unexpected results. Now replace p in each expression with the current value of that expression (0.1925 in this case). This is called *feedback* because we are feeding our output back into each expression as input. Each new evaluation is referred to as an *iteration*. This time, each expression takes on the value

(continued)

0.65883125. Still no surprises. Continue the feedback process. That is, replace p in each expression *with the current value* of that expression. Now each expression takes on the value 1.33314915207, as shown in the following table. The iterations were performed on a TI-85 calculator.

Iteration	$p + 3p(1 - p)$	$4p - 3p^2$
1	0.1925	0.1925
2	0.65883125	0.65883125
3	1.33314915207	1.33314915207

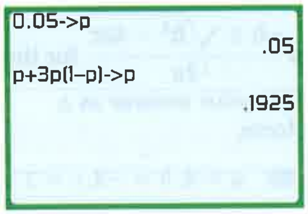
The following table shows that if we continue this feedback process on a calculator, the expressions $p + 3p(1 - p)$ and $4p - 3p^2$ will start to take on different values beginning with the fourth iteration. By the 37th iteration, the values do not even agree to two decimal places.

Iteration	$p + 3p(1 - p)$	$4p - 3p^2$
4	7.366232839E-4	7.366232838E-4
5	0.002944865294	0.002944865294
6	0.011753444481	0.0117534448
7	0.046599347553	0.046599347547
20	1.12135618652	1.12135608405
30	0.947163304835	0.947033128433
37	0.285727963839	0.300943417861

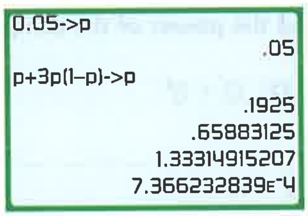
1. Use a calculator to find the first 20 iterations of $p + 3p(1 - p)$ and $4p - 3p^2$, with the initial value of $p = 0.5$.
2. Write a report on chaos and fractals. Include information on the "butterfly effect." An excellent source is *Chaos and Fractals, New Frontiers of Science* by Heinz-Otto Peitgen, Hartmut Jurgens, and Dietmar Saupe (New York: Springer-Verlag, 1992).
3. Equations of the form $p_{n+1} = p_n + rp_n(1 - p_n)$ are called *Verhulst population models.* Write a report on Verhulst population models.

Integrating Technology

To perform the iterations at the right with a TI graphing calculator, first store 0.05 in p and then store $p + 3p(1 - p)$ in p, as shown below.



Each time you press ENTER, the expression $p + 3p(1 - p)$ will be evaluated with p equal to the previous result.



CHAPTER P TEST PREP

The following test prep table summarizes essential concepts in this chapter. The references given in the right-hand column list Examples and Exercises that can be used to test your understanding of a concept.

P.1 The Real Number System

• The following sets of numbers are used extensively in algebra: Natural numbers $\{1, 2, 3, 4, \ldots\}$ Whole numbers $\{0, 1, 2, 3, 4, \ldots\}$ Integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ Rational numbers $\left\{\frac{p}{q}, \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\right\}$ {all terminating and repeating decimals} Irrational numbers {all nonterminating, nonrepeating decimals} Real numbers {all rational or irrational numbers}	See Example 1, page 3, and then try Exercises 1 and 2, page 70.