SECTION P.4

Greatest Common Factor
Factoring Trinomials
Special Factoring
Factor by Grouping
General Factoring

Factoring

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A2.

PS1. Simplify: $\frac{6x^3}{2x}$ [P.2]

PS2. Simplify: $(-12x^4)3x^2$ [P.2]

PS3. Express x^6 as a power of **a.** x^2 and **b.** x^3. [P.2]

In Exercises PS4 to PS6, replace the question mark to make a true statement.

PS4. $6a^3b^4 \cdot ? = 18a^3b^7$ [P.2]

PS5. $-3(5a - ?) = -15a + 21$ [P.1]

PS6. $2x(3x - ?) = 6x^2 - 2x$ [P.1]

Writing a polynomial as a product of polynomials is called **factoring.** Factoring is an important procedure that is often used to simplify fractional expressions and to solve equations.

In this section, we consider only the factorization of polynomials that have integer coefficients. Also, we are concerned only with **factoring over the integers.** That is, we search only for polynomial factors that have integer coefficients.

Greatest Common Factor

The first step in the factorization of any polynomial is to use the distributive property to factor out the **greatest common factor (GCF)** of the terms of the polynomial. Given two or more exponential expressions with the same prime number base or the same variable base, the GCF is the exponential expression with the smallest exponent. For example,

$$2^3 \text{ is the GCF of } 2^3, 2^5, \text{ and } 2^8 \quad \text{and} \quad a \text{ is the GCF of } a^4 \text{ and } a$$

The GCF of two or more monomials is the product of the GCFs of all the *common* bases. For example, to find the GCF of $27a^3b^4$ and $18b^3c$, factor the coefficients into prime factors and then write each common base with its smallest exponent.

$$27a^3b^4 = 3^3 \cdot a^3 \cdot b^4 \qquad 18b^3c = 2 \cdot 3^2 \cdot b^3 \cdot c$$

The only common bases are 3 and b. The product of these common bases with their smallest exponents is 3^2b^3. The GCF of $27a^3b^4$ and $18b^3c$ is $9b^3$.

The expressions $3x(2x + 5)$ and $4(2x + 5)$ have a common *binomial* factor, which is $2x + 5$. Thus the GCF of $3x(2x + 5)$ and $4(2x + 5)$ is $2x + 5$.

EXAMPLE 1 Factor Out the Greatest Common Factor

Factor out the GCF.

a. $12x^3y^4 - 24x^2y^5 + 18xy^6$ **b.** $(6x - 5)(4x + 3) - (4x + 3)(3x - 7)$

Solution

a. $12x^3y^4 - 24x^2y^5 + 18xy^6$

$= (6xy^4)2x^2 - (6xy^4)4xy + (6xy^4)3y^2$ • The GCF is $6xy^4$.

$= 6xy^4(2x^2 - 4xy + 3y^2)$ • Factor out the GCF.

(continued)

b. $(6x - 5)(4x + 3) - (4x + 3)(3x - 7)$

$= (4x + 3)[(6x - 5) - (3x - 7)]$ • The common binomial factor is $4x + 3$.

$= (4x + 3)(3x + 2)$

▸ **Try Exercise 16, page 48**

Factoring Trinomials

TO REVIEW

The FOIL method
See page 33.

Some trinomials of the form $x^2 + bx + c$ can be factored by a trial procedure. This method makes use of the FOIL method in reverse. For example, consider the following products.

$$(x + 3)(x + 5) = x^2 + 5x + 3x + (3)(5) = x^2 + 8x + 15$$

$$(x - 2)(x - 7) = x^2 - 7x - 2x + (-2)(-7) = x^2 - 9x + 14$$

$$(x + 4)(x - 9) = x^2 - 9x + 4x + (4)(-9) = x^2 - 5x - 36$$

The coefficient of x is the sum of the constant terms of the binomials.

The constant term of the trinomial is the product of the constant terms of the binomials.

Question • Is $(x - 2)(x + 7)$ the correct factorization of $x^2 - 5x - 14$?

Points to Remember to Factor $x^2 + bx + c$

1. The constant term c of the trinomial is the product of the constant terms of the binomials.
2. The coefficient b in the trinomial is the sum of the constant terms of the binomials.
3. If the constant term c of the trinomial is positive, the constant terms of the binomials have the same sign as the coefficient b in the trinomial.
4. If the constant term c of the trinomial is negative, the constant terms of the binomials have opposite signs.

EXAMPLE 2 Factor a Trinomial

Factor.

a. $x^2 + 7x - 18$ **b.** $x^2 + 7xy + 10y^2$

Solution

a. Find two integers whose product is -18 and whose sum is 7. The integers are -2 and 9: $-2(9) = -18$, $-2 + 9 = 7$.

$$x^2 + 7x - 18 = (x - 2)(x + 9)$$

b. Find two integers whose product is 10 and whose sum is 7. The integers are 2 and 5: $2(5) = 10$, $2 + 5 = 7$.

$$x^2 + 7xy + 10y^2 = (x + 2y)(x + 5y)$$

▸ **Try Exercise 22, page 48**

Note

In **b.**, the last term of the trinomial contains y^2, so the last term of each binomial factor has a y.

Answer • No. $(x - 2)(x + 7) = x^2 + 5x - 14$.

Sometimes it is impossible to factor a polynomial into the product of two polynomials having integer coefficients. Such polynomials are said to be **nonfactorable over the integers.** For example, $x^2 + 3x + 7$ is nonfactorable over the integers because there are no integers whose product is 7 and whose sum or difference is 3.

The trial method sometimes can be used to factor trinomials of the form $ax^2 + bx + c$, which do not have a leading coefficient of 1. We use the factors of a and c to form trial binomial factors. Factoring trinomials of this type may require testing many factors. To reduce the number of trial factors, make use of the following points.

Points to Remember to Factor $ax^2 + bx + c$, $a > 0$

1. If the constant term of the trinomial is positive, the constant terms of the binomials have the same sign as the coefficient b in the trinomial.
2. If the constant term of the trinomial is negative, the constant terms of the binomials have opposite signs.
3. If the terms of the trinomial do not have a common factor, then neither binomial will have a common factor.

EXAMPLE 3 Factor a Trinomial of the Form $ax^2 + bx + c$

Factor: $6x^2 - 11x + 4$

Solution

Because the constant term of the trinomial is positive and the coefficient of the x term is negative, the constant terms of the binomials will both be negative. We start by finding factors of the first term and factors of the constant term.

Factors of $6x^2$	Factors of 4 (both negative)
$x, 6x$	$-1, -4$
$2x, 3x$	$-2, -2$

Use these factors to write trial factors. Use the FOIL method to see whether any of the trial factors produce the correct middle term. If the terms of a trinomial do not have a common factor, then a binomial factor cannot have a common factor (point 3). Such trial factors need not be checked.

Trial Factors	Middle Term	
$(x - 1)(6x - 4)$	Common factor	• $6x$ and 4 have a common factor.
$(x - 4)(6x - 1)$	$-1x - 24x = -25x$	• This is not the correct middle term.
$(x - 2)(6x - 2)$	Common factor	• $6x$ and 2 have a common factor.
$(2x - 1)(3x - 4)$	$-8x - 3x = -11x$	• This is the correct middle term.

Thus $6x^2 - 11x + 4 = (2x - 1)(3x - 4)$.

▶ Try Exercise 28, page 48

If you have difficulty factoring a trinomial, you may wish to use the following theorem. It will indicate whether the trinomial is factorable over the integers.

Factorization Theorem

The trinomial $ax^2 + bx + c$, with integer coefficients a, b, and c, can be factored as the product of two binomials with integer coefficients if and only if $b^2 - 4ac$ is a perfect square.

EXAMPLE 4 Apply the Factorization Theorem

Determine whether each trinomial is factorable over the integers.

a. $4x^2 + 8x - 7$

b. $6x^2 - 5x - 4$

Solution

a. The coefficients of $4x^2 + 8x - 7$ are $a = 4$, $b = 8$, and $c = -7$. Applying the factorization theorem yields

$$b^2 - 4ac = 8^2 - 4(4)(-7) = 176$$

Because 176 is not a perfect square, the trinomial is nonfactorable over the integers.

b. The coefficients of $6x^2 - 5x - 4$ are $a = 6$, $b = -5$, and $c = -4$. Thus

$$b^2 - 4ac = (-5)^2 - 4(6)(-4) = 121$$

Because 121 is a perfect square, the trinomial is factorable over the integers. Using the methods we have developed, we find

$$6x^2 - 5x - 4 = (3x - 4)(2x + 1)$$

Try Exercise 34, page 48

Special Factoring

The product of a term and itself is called a **perfect square.** The exponents on variables of perfect squares are always even numbers. The **square root of a perfect square** is one of the two equal factors of the perfect square. To find the square root of a perfect square variable term, divide the exponent by 2. For the examples in Table P.3, assume that the variables represent positive numbers.

Table P.3 Perfect Squares and Square Roots

Term		Perfect Square	Square Root
7	$7 \cdot 7 =$	49	$\sqrt{49} = 7$
y	$y \cdot y =$	y^2	$\sqrt{y^2} = y$
$2x^3$	$2x^3 \cdot 2x^3 =$	$4x^6$	$\sqrt{4x^6} = 2x^3$
x^n	$x^n \cdot x^n =$	x^{2n}	$\sqrt{x^{2n}} = x^n$

The difference of two squares is a binomial expression of the form $a^2 - b^2$. For instance, the following are the differences of squares.

$x^2 - 9 = x^2 - 3^2$ • The square of x minus the square of 3

$y^2 - 64 = y^2 - 8^2$ • The square of y minus the square of 8

$25z^2 - 49 = (5z)^2 - 7^2$ • The square of $5z$ minus the square of 7

$x^6 - 81 = (x^3)^2 - 9^2$ • The square of x^3 minus the square of 9

The factors of the difference of two perfect squares are the sum and difference of the square roots of the perfect squares.

Factors of the Difference of Two Perfect Squares

$$a^2 - b^2 = (a + b)(a - b)$$

The difference of two perfect squares always factors over the integers. However, the sum of squares does not factor over the integers. For instance, $a^2 + b^2$ does not factor over the integers. As another example, $x^2 + 4$ is the sum of squares and does not factor over the integers. There are no integers whose product is 4 and whose sum is 0.

EXAMPLE 5 Factor the Difference of Squares

Factor.

a. $49x^2 - 144$ **b.** $a^4 - 81$

Solution

a. $49x^2 - 144 = (7x)^2 - 12^2$ • Write as the difference of squares.

$= (7x + 12)(7x - 12)$ • The binomial factors are the sum and the difference of the square roots of the squares.

b. $a^4 - 81 = (a^2)^2 - (9)^2$ • Write as the difference of squares.

$= (a^2 + 9)(a^2 - 9)$ • The binomial factors are the sum and the difference of the square roots of the squares.

$= (a + 3)(a - 3)(a^2 + 9)$ • $a^2 - 9$ is the difference of squares. Factor as $(a + 3)(a - 3)$. The sum of squares, $a^2 + 9$, does not factor over the integers.

▸ Try Exercise 50, page 48

The square of a binomial is a **perfect-square trinomial.** Here are some examples of perfect-square trinomials.

Square of a binomial		Perfect-square trinomial
$(x + 3)^2$	=	$x^2 + 6x + 9$
$(t - 4)^2$	=	$t^2 - 8t + 16$
$(3z + 5)^2$	=	$9z^2 + 30z + 25$
$(2x - 3y)^2$	=	$4x^2 - 12xy + 9y^2$

Every perfect-square trinomial can be factored by the trial method, but it generally is faster to factor perfect-square trinomials by using the following factoring formulas.

Factors of a Perfect-Square Trinomial

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

EXAMPLE 6 Factor a Perfect-Square Trinomial

Factor: $16m^2 - 40mn + 25n^2$

Solution

Because $16m^2 = (4m)^2$ and $25n^2 = (5n)^2$, try factoring $16m^2 - 40mn + 25n^2$ as the square of a binomial.

$$16m^2 - 40mn + 25n^2 \stackrel{?}{=} (4m - 5n)^2$$

Check:

$$\begin{aligned}(4m - 5n)^2 &= (4m - 5n)(4m - 5n)\\ &= 16m^2 - 20mn - 20mn + 25n^2\\ &= 16m^2 - 40mn + 25n^2\end{aligned}$$

The factorization checks. Therefore, $16m^2 - 40mn + 25n^2 = (4m - 5n)^2$.

▸ **Try Exercise 56, page 48**

Caution

It is important to check the proposed factorization. For instance, consider $x^2 + 13x + 36$. Because x^2 is the square of x and 36 is the square of 6, it is tempting to factor, using the perfect-square trinomial formulas, as $x^2 + 13x + 36 \stackrel{?}{=} (x + 6)^2$. Note that $(x + 6)^2 = x^2 + 12x + 36$, which is not the original trinomial. The correct factorization is $x^2 + 13x + 36 = (x + 4)(x + 9)$.

The product of the same three terms is called a **perfect cube.** The exponents on variables of perfect cubes are always divisible by 3. The **cube root of a perfect cube** is one of the three equal factors of the perfect cube. To find the cube root of a perfect cube variable term, divide the exponent by 3. See Table P.4.

Table P.4 Perfect Cubes and Cube Roots

Term		Perfect Cube	Cube Root
5	$5 \cdot 5 \cdot 5 =$	125	$\sqrt[3]{125} = 5$
z	$z \cdot z \cdot z =$	z^3	$\sqrt[3]{z^3} = z$
$3x^2$	$3x^2 \cdot 3x^2 \cdot 3x^2 =$	$27x^6$	$\sqrt[3]{27x^6} = 3x^2$
x^n	$x^n \cdot x^n \cdot x^n =$	x^{3n}	$\sqrt[3]{x^{3n}} = x^n$

The following factoring formulas are used to factor the sum or difference of two perfect cubes.

Study tip

Pay attention to the pattern of the signs when factoring the sum or the difference of two perfect cubes.

Same signs

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Opposite signs

Same signs

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Opposite signs

Factors of the Sum or Difference of Two Perfect Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

EXAMPLE 7 Factor the Sum or Difference of Cubes

Factor.

a. $8a^3 + b^3$ **b.** $a^3 - 64$

Solution

a. $8a^3 + b^3 = (2a)^3 + b^3$ • Recognize the sum-of-cubes form.

$= (2a + b)(4a^2 - 2ab + b^2)$ • Factor.

b. $a^3 - 64 = a^3 - 4^3$ • Recognize the difference-of-cubes form.

$= (a - 4)(a^2 + 4a + 16)$ • Factor.

▶ Try Exercise 62, page 48

Certain trinomials can be expressed as quadratic trinomials by making suitable variable substitutions. A trinomial is **quadratic in form** if it can be written as

$$au^2 + bu + c$$

If we let $x^2 = u$, the trinomial $x^4 + 5x^2 + 6$ can be written as shown at the right. The trinomial is quadratic in form.

$$\begin{aligned} &x^4 + 5x^2 + 6 \\ &\quad = (x^2)^2 + 5(x^2) + 6 \\ &\quad = u^2 + 5u + 6 \end{aligned}$$

If we let $xy = u$, the trinomial $2x^2y^2 + 3xy - 9$ can be written as shown at the right. The trinomial is quadratic in form.

$$\begin{aligned} &2x^2y^2 + 3xy - 9 \\ &\quad = 2(xy)^2 + 3(xy) - 9 \\ &\quad = 2u^2 + 3u - 9 \end{aligned}$$

When a trinomial that is quadratic in form is factored, the variable part of the first term in each binomial factor will be u. For example, because $x^4 + 5x^2 + 6$ is quadratic in form when $x^2 = u$, the first term in each binomial factor will be x^2.

$$\begin{aligned} x^4 + 5x^2 + 6 &= (x^2)^2 + 5(x^2) + 6 \\ &= (x^2 + 2)(x^2 + 3) \end{aligned}$$

The trinomial $x^2y^2 - 2xy - 15$ is quadratic in form when $xy = u$. The first term in each binomial factor will be xy.

$$\begin{aligned} x^2y^2 - 2xy - 15 &= (xy)^2 - 2(xy) - 15 \\ &= (xy + 3)(xy - 5) \end{aligned}$$

EXAMPLE 8 Factor a Polynomial That Is Quadratic in Form

Factor.

a. $6x^2y^2 - xy - 12$

b. $x^4 + 5x^2 - 36$

Solution

a. $6x^2y^2 - xy - 12$

$= 6u^2 - u - 12$ • The trinomial is quadratic in form when $xy = u$. Then $x^2y^2 = u^2$.

$= (3u + 4)(2u - 3)$ • Factor.

$= (3xy + 4)(2xy - 3)$ • Replace u with xy.

(continued)

b. $x^4 + 5x^2 - 36$

$= u^2 + 5u - 36$ • The trinomial is quadratic in form when $x^2 = u$. Then $x^4 = u^2$.

$= (u - 4)(u + 9)$ • Factor.

$= (x^2 - 4)(x^2 + 9)$ • Replace u with x^2.

$= (x - 2)(x + 2)(x^2 + 9)$ • Factor the difference of squares. The sum of squares does not factor.

▶ Try Exercise 74, page 48

Note

$-a + b = -(a - b)$. Thus $-4y + 14 = -(4y - 14)$.

Factor by Grouping

Some polynomials can be **factored by grouping**. Pairs of terms that have a common factor are first grouped together. The process makes repeated use of the distributive property, as shown in the following factorization of $6y^3 - 21y^2 - 4y + 14$.

$6y^3 - 21y^2 - 4y + 14$

$= (6y^3 - 21y^2) - (4y - 14)$ • Group the first two terms and the last two terms.

$= 3y^2(2y - 7) - 2(2y - 7)$ • Factor out the GCF from each of the groups.

$= (2y - 7)(3y^2 - 2)$ • Factor out the common binomial factor.

When you factor by grouping, some experimentation may be necessary to find a grouping that fits the form of one of the special factoring formulas.

EXAMPLE 9 Factor by Grouping

Factor by grouping.

a. $a^2 + 10ab + 25b^2 - c^2$

b. $p^2 + p - q - q^2$

Solution

a. $a^2 + 10ab + 25b^2 - c^2$

$= (a^2 + 10ab + 25b^2) - c^2$ • Group the terms of the perfect-square trinomial.

$= (a + 5b)^2 - c^2$ • Factor the trinomial.

$= [(a + 5b) + c][(a + 5b) - c]$ • Factor the difference of squares.

$= (a + 5b + c)(a + 5b - c)$ • Simplify.

b. $p^2 + p - q - q^2$

$= p^2 - q^2 + p - q$ • Rearrange the terms.

$= (p^2 - q^2) + (p - q)$ • Regroup.

$= (p + q)(p - q) + (p - q)$ • Factor the difference of squares.

$= (p - q)(p + q + 1)$ • Factor out the common factor $(p - q)$.

▶ Try Exercise 80, page 48

General Factoring

A general factoring strategy for polynomials is shown below.

General Factoring Strategy

1. Factor out the GCF of all terms.
2. Try to factor a binomial as
   a. the difference of two squares
   b. the sum or difference of two cubes
3. Try to factor a trinomial
   a. as a perfect-square trinomial
   b. using the trial method
4. Try to factor a polynomial with more than three terms by grouping.
5. After each factorization, examine the new factors to see whether they can be factored.

EXAMPLE 10 Factor Using the General Factoring Strategy

Factor: $2vx^6 + 14vx^3 - 16v$

Solution

$2vx^6 + 14vx^3 - 16v$

$= 2v(x^6 + 7x^3 - 8)$ • The GCF is $2v$.

$= 2v(u^2 + 7u - 8)$ • $x^6 + 7x^3 - 8$ is quadratic in form. Let $u = x^3$. Then $u^2 = x^6$.

$= 2v(u + 8)(u - 1)$ • Factor.

$= 2v(x^3 + 8)(x^3 - 1)$ • Replace u with x^3. $x^3 + 8$ in the sum of cubes. $x^3 - 1$ is the difference of cubes.

$= 2v(x + 2)(x^2 - 2x + 4)(x - 1)(x^2 + x + 1)$ • Factor the sum and difference of cubes.

Try Exercise 86, page 48

EXERCISE SET P.4

Concept Check

In Exercises 1 to 6, find the GCF of the expressions.

1. $9x^2y^3$ and $12xy^4$
2. $16a^4b$ and $25a^2c^2$
3. $12x^3$ and $35y^3$
4. $9x^5yz^2$, $18x^2z^4$, and $3xyz^3$
5. $2v^3(w - 4)$ and $12v(w - 4)^2$
6. $15a^2b(c - 9)$ and $12a^4b^3(9 - c)$

Indicates Try It Exercises

In Exercises 7 to 10, state whether the polynomial is factored completely over the integers.

7. $x^2 + 4x$
8. $a^2 + 4$
9. $x^3 + 8$
10. $a^3 - 18$

In Exercises 11 to 18, factor out the GCF from each polynomial.

11. $5x + 20$

12. $8x^2 + 12x - 40$

13. $-15x^2 - 12x$

14. $-6y^2 - 54y$

15. $10x^2y + 6xy - 14xy^2$

16. $6a^3b^2 - 12a^2b + 72ab^3$

17. $(x - 3)(a + b) + (x - 3)(a + 2b)$

18. $(x - 4)(2a - b) + (x + 4)(2a - b)$

In Exercises 19 to 32, factor each trinomial over the integers.

19. $x^2 + 7x + 12$

20. $x^2 + 9x + 20$

21. $a^2 - 10a - 24$

22. $b^2 + 12b - 28$

23. $x^2 + 6x + 5$

24. $x^2 + 11x + 18$

25. $6x^2 + 25x + 4$

26. $8a^2 - 26a + 15$

27. $51x^2 - 5x - 4$

28. $57y^2 + y - 6$

29. $6x^2 + xy - 40y^2$

30. $8x^2 + 10xy - 25y^2$

31. $6x^2 + 23x + 15$

32. $9x^2 + 10x + 1$

In Exercises 33 to 38, use the factorization theorem to determine whether each trinomial is factorable over the integers.

33. $8x^2 + 26x + 15$

34. $16x^2 + 8x - 35$

35. $4x^2 - 5x + 6$

36. $6x^2 + 8x - 3$

37. $6x^2 - 14x + 5$

38. $10x^2 - 4x - 5$

In Exercises 39 to 52, factor each difference of squares over the integers.

39. $x^2 - 9$

40. $x^2 - 64$

41. $4a^2 - 49$

42. $81b^2 - 16c^2$

43. $1 - 100x^2$

44. $1 - 121y^2$

45. $(x + 1)^2 - 4$

46. $(5x + 3)^2 - 9$

47. $6x^2 - 216$

48. $-2z^3 + 2z$

49. $x^4 - 625$

50. $y^4 - 1$

51. $x^5 - 81x$

52. $3xy^6 - 48xy^2$

In Exercises 53 to 60, factor each perfect-square trinomial.

53. $x^2 + 10x + 25$

54. $y^2 + 6y + 9$

55. $a^2 - 14a + 49$

56. $b^2 - 24b + 144$

57. $4x^2 + 12x + 9$

58. $25y^2 + 40y + 16$

59. $z^4 + 4z^2w^2 + 4w^4$

60. $9x^4 - 30x^2y^2 + 25y^4$

In Exercises 61 to 68, factor each sum or difference of cubes over the integers.

61. $x^3 - 8$

62. $b^3 + 64$

63. $8x^3 - 27y^3$

64. $64u^3 - 27v^3$

65. $8 - x^6$

66. $1 + y^{12}$

67. $(x - 2)^3 - 1$

68. $(y + 3)^3 + 8$

In Exercises 69 to 76, factor over the integers the polynomials that are quadratic in form.

69. $x^4 - x^2 - 6$

70. $y^4 + 3y^2 + 2$

71. $x^2y^2 + 4xy - 5$

72. $x^2y^2 - 8xy + 12$

73. $4x^5 - 4x^3 - 8x$

74. $z^4 + 3z^2 - 4$

75. $z^4 + z^2 - 20$

76. $x^4 - 13x^2 + 36$

In Exercises 77 to 82, factor over the integers by grouping.

77. $3x^3 + x^2 + 6x + 2$

78. $18w^3 + 15w^2 + 12w + 10$

79. $ax^2 - ax + bx - b$

80. $a^2y^2 - ay^3 + ac - cy$

81. $6w^3 + 4w^2 - 15w - 10$

82. $10z^3 - 15z^2 - 4z + 6$

In Exercises 83 to 102, use the general factoring strategy to completely factor each polynomial. If the polynomial does not factor, then state that it is nonfactorable over the integers.

83. $18x^2 - 2$

84. $4bx^3 + 32b$

85. $16x^4 - 1$

86. $81y^4 - 16$

87. $12ax^2 - 23axy + 10ay^2$

88. $6ax^2 - 19axy - 20ay^2$

89. $3bx^3 + 4bx^2 - 3bx - 4b$

90. $2x^6 - 2$

91. $72bx^2 + 24bxy + 2by^2$

92. $64y^3 - 16y^2z + yz^2$

93. $(w - 5)^3 + 8$

94. $5xy + 20y - 15x - 60$

95. $x^2 + 6xy + 9y^2 - 1$

96. $4y^2 - 4yz + z^2 - 9$

97. $8x^2 + 3x - 4$

98. $16x^2 + 81$

99. $5x(2x - 5)^2 - (2x - 5)^3$

100. $6x(3x + 1)^3 - (3x + 1)^4$

101. $4x^2 + 2x - y - y^2$

102. $a^2 + a + b - b^2$

Enrichment Exercises

In Exercises 103 and 106, find all values of *k* such that the trinomial is a perfect-square trinomial.

103. $x^2 + kx + 16$

104. $36x^2 + kxy + 100y^2$

105. $x^2 + 16x + k$

106. $x^2 - 14xy + ky^2$

In Exercises 107 to 110, find all integer values of *k* for which the resulting trinomial will factor over the integers.

107. $x^2 + kx - 6$

108. $x^2 + kx + 12$

109. $2x^2 + kx - 10$

110. $3x^2 + kx + 5$

SECTION P.5

- Simplifying Rational Expressions
- Operations on Rational Expressions
- Determining the LCD of Rational Expressions
- Complex Fractions
- Application of Rational Expressions

Rational Expressions

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A3.

PS1. Simplify: $1 + \dfrac{1}{2 - \dfrac{1}{3}}$ [P.1]

PS2. Simplify: $\left(\dfrac{w}{x}\right)^{-1}\left(\dfrac{y}{z}\right)^{-1}$ [P.2]

PS3. What is the common binomial factor of $x^2 + 2x - 3$ and $x^2 + 7x + 12$? [P.4]

In Exercises PS4 to PS6, factor completely over the integers.

PS4. $(2x - 3)(3x + 2) - (2x - 3)(x + 2)$ [P.4]

PS5. $x^2 - 5x - 6$ [P.4]

PS6. $x^3 - 64$ [P.4]

Math Matters

Evidence from work left by early Egyptians more than 3600 years ago shows that they used, with one exception, unit fractions—that is, fractions whose numerators are 1. The one exception was 2/3. A unit fraction was represented by placing an oval over the symbol for the number in the denominator. For instance, 1/4 = .

A **rational expression** is a fraction in which the numerator and denominator are polynomials. For example, the expressions below are rational expressions.

$$\frac{3}{x + 1} \quad \text{and} \quad \frac{x^2 - 4x - 21}{x^2 - 9}$$

The **domain of a rational expression** is the set of all real numbers that can be used as replacements for the variable. Any value of the variable that causes division by zero is excluded from the domain of the rational expression. For example, the domain of

$$\frac{x + 3}{x^2 - 5x}, \quad x \neq 0, x \neq 5$$

is the set of all real numbers except 0 and 5. Both 0 and 5 are excluded values because the denominator $x^2 - 5x$ equals zero when $x = 0$ and also when $x = 5$. Sometimes the excluded values are specified to the right of a rational expression, as shown here. However, a rational expression is meaningful only for those real numbers that are not excluded values, regardless of whether the excluded values are specifically stated.

Question • What value of x must be excluded from the domain of $\frac{x-2}{x+1}$?

Rational expressions have properties similar to the properties of rational numbers.

Properties of Rational Expressions

For all rational expressions $\frac{P}{Q}$ and $\frac{R}{S}$, where $Q \neq 0$ and $S \neq 0$,

Equality $\quad \frac{P}{Q} = \frac{R}{S}$ if and only if $PS = QR$

Equivalent expressions $\quad \frac{P}{Q} = \frac{PR}{QR}, \quad R \neq 0$

Sign $\quad -\frac{P}{Q} = \frac{-P}{Q} = \frac{P}{-Q}$

Simplifying Rational Expressions

To **simplify a rational expression,** factor the numerator and denominator. Then use the equivalent expressions property to eliminate factors common to both the numerator and the denominator. A rational expression is *simplified* when 1 is the only common factor of both the numerator and the denominator.

EXAMPLE 1 Simplify a Rational Expression

Simplify: $\frac{7 + 20x - 3x^2}{2x^2 - 11x - 21}$

Solution

$$\frac{7 + 20x - 3x^2}{2x^2 - 11x - 21} = \frac{(7 - x)(1 + 3x)}{(x - 7)(2x + 3)} \quad \text{• Factor.}$$

$$= \frac{-(x - 7)(1 + 3x)}{(x - 7)(2x + 3)} \quad \text{• Use } (7 - x) = -(x - 7).$$

$$= \frac{-\cancel{(x - 7)}(1 + 3x)}{\cancel{(x - 7)}(2x + 3)} \quad \text{• } x \neq 7.$$

$$= \frac{-(1 + 3x)}{2x + 3}$$

$$= -\frac{3x + 1}{2x + 3}, \; x \neq -\frac{3}{2}$$

Try Exercise 10, page 56

Answer • When $x = -1$, $x + 1 = 0$. Therefore, -1 must be excluded from the domain. When $x = 2$, the value of $\frac{x-2}{x+1}$ is $\frac{2-2}{2+1} = \frac{0}{3} = 0$. The value of the numerator can equal zero; the value of the denominator cannot equal zero.