General Factoring

A general factoring strategy for polynomials is shown below.

General Factoring Strategy

1. Factor out the GCF of all terms.
2. Try to factor a binomial as
   a. the difference of two squares
   b. the sum or difference of two cubes
3. Try to factor a trinomial
   a. as a perfect-square trinomial
   b. using the trial method
4. Try to factor a polynomial with more than three terms by grouping.
5. After each factorization, examine the new factors to see whether they can be factored.

Alternative to Example 10

Factor.

a. $x^3 - 2x^2 - x + 2$

■ $(x - 2)(x + 1)(x - 1)$

b. $4x^2 + 4x + 1 - y^2$

■ $(2x + 1 - y)(2x + 1 + y)$

EXAMPLE 10 Factor Using the General Factoring Strategy

Factor: $2vx^6 + 14vx^3 - 16v$

Solution

$2vx^6 + 14vx^3 - 16v$

$= 2v(x^6 + 7x^3 - 8)$ • The GCF is $2v$.

$= 2v(u^2 + 7u - 8)$ • $x^6 + 7x^3 - 8$ is quadratic in form. Let $u = x^3$. Then $u^2 = x^6$.

$= 2v(u + 8)(u - 1)$ • Factor.

$= 2v(x^3 + 8)(x^3 - 1)$ • Replace u with x^3. $x^3 + 8$ in the sum of cubes. $x^3 - 1$ is the difference of cubes.

$= 2v(x + 2)(x^2 - 2x + 4)(x - 1)(x^2 + x + 1)$ • Factor the sum and difference of cubes.

Try Exercise 86, page 48

EXERCISE SET P.4

Concept Check

In Exercises 1 to 6, find the GCF of the expressions.

1. $9x^2y^3$ and $12xy^4$
   $3xy^3$
2. $16a^4b$ and $25a^2c^2$
   a^2
3. $12x^3$ and $35y^3$
   1
4. $9x^5yz^2$, $18x^2z^4$, and $3xyz^3$
   $3xz^2$
5. $2y^3(w - 4)$ and $12y(w - 4)^2$
   $2y(w - 4)$
6. $15a^2b(c - 9)$ and $12a^4b^3(9 - c)$
   $3a^2b(c - 9)$

In Exercises 7 to 10, state whether the polynomial is factored completely over the integers.

7. $x^2 + 4x$
   No
8. $a^2 + 4$
   Yes
9. $x^3 + 8$
   No
10. $a^3 - 18$
    Yes

Indicates Try It Exercises

In Exercises 11 to 18, factor out the GCF from each polynomial.

11. $5x + 20$
$5(x + 4)$

12. $8x^2 + 12x - 40$
$4(2x^2 + 3x - 10)$

13. $-15x^2 - 12x$
$-3x(5x + 4)$

14. $-6y^2 - 54y$
$-6y(y + 9)$

15. $10x^2y + 6xy - 14xy^2$
$2xy(5x + 3 - 7y)$

16. $6a^3b^2 - 12a^2b + 72ab^3$
$6ab(a^2b - 2a + 12b^2)$

17. $(x - 3)(a + b) + (x - 3)(a + 2b)$ $(x - 3)(2a + 3b)$

18. $(x - 4)(2a - b) + (x + 4)(2a - b)$ $(2a - b)(2x)$

In Exercises 19 to 32, factor each trinomial over the integers.

19. $x^2 + 7x + 12$
$(x + 3)(x + 4)$

20. $x^2 + 9x + 20$
$(x + 4)(x + 5)$

21. $a^2 - 10a - 24$
$(a - 12)(a + 2)$

22. $b^2 + 12b - 28$
$(b + 14)(b - 2)$

23. $x^2 + 6x + 5$
$(x + 5)(x + 1)$

24. $x^2 + 11x + 18$
$(x + 9)(x + 2)$

25. $6x^2 + 25x + 4$
$(6x + 1)(x + 4)$

26. $8a^2 - 26a + 15$
$(4a - 3)(2a - 5)$

27. $51x^2 - 5x - 4$
$(17x + 4)(3x - 1)$

28. $57y^2 + y - 6$
$(19y - 6)(3y + 1)$

29. $6x^2 + xy - 40y^2$
$(3x + 8y)(2x - 5y)$

30. $8x^2 + 10xy - 25y^2$
$(4x - 5y)(2x + 5y)$

31. $6x^2 + 23x + 15$
$(6x + 5)(x + 3)$

32. $9x^2 + 10x + 1$
$(9x + 1)(x + 1)$

In Exercises 33 to 38, use the factorization theorem to determine whether each trinomial is factorable over the integers.

33. $8x^2 + 26x + 15$
Factorable over the integers

34. $16x^2 + 8x - 35$
Factorable over the integers

35. $4x^2 - 5x + 6$
Not factorable over the integers

36. $6x^2 + 8x - 3$
Not factorable over the integers

37. $6x^2 - 14x + 5$
Not factorable over the integers

38. $10x^2 - 4x - 5$
Not factorable over the integers

In Exercises 39 to 52, factor each difference of squares over the integers.

39. $x^2 - 9$
$(x + 3)(x - 3)$

40. $x^2 - 64$
$(x + 8)(x - 8)$

41. $4a^2 - 49$
$(2a + 7)(2a - 7)$

42. $81b^2 - 16c^2$
$(9b + 4c)(9b - 4c)$

43. $1 - 100x^2$
$(1 + 10x)(1 - 10x)$

44. $1 - 121y^2$
$(1 + 11y)(1 - 11y)$

45. $(x + 1)^2 - 4$
$(x + 3)(x - 1)$

46. $(5x + 3)^2 - 9$
$5x(5x + 6)$

47. $6x^2 - 216$
$6(x + 6)(x - 6)$

48. $-2z^3 + 2z$
$-2z(z + 1)(z - 1)$

49. $x^4 - 625$
$(x + 5)(x - 5)(x^2 + 25)$

50. $y^4 - 1$
$(y + 1)(y - 1)(y^2 + 1)$

51. $x^5 - 81x$
$x(x + 3)(x - 3)(x^2 + 9)$

52. $3xy^6 - 48xy^2$
$3xy^2(y + 2)(y - 2)(y^2 + 4)$

In Exercises 53 to 60, factor each perfect-square trinomial.

53. $x^2 + 10x + 25$
$(x + 5)^2$

54. $y^2 + 6y + 9$
$(y + 3)^2$

55. $a^2 - 14a + 49$
$(a - 7)^2$

56. $b^2 - 24b + 144$
$(b - 12)^2$

57. $4x^2 + 12x + 9$
$(2x + 3)^2$

58. $25y^2 + 40y + 16$
$(5y + 4)^2$

59. $z^4 + 4z^2w^2 + 4w^4$
$(z^2 + 2w^2)^2$

60. $9x^4 - 30x^2y^2 + 25y^4$
$(3x^2 - 5y^2)^2$

In Exercises 61 to 68, factor each sum or difference of cubes over the integers.

61. $x^3 - 8$
$(x - 2)(x^2 + 2x + 4)$

62. $b^3 + 64$
$(b + 4)(b^2 - 4b + 16)$

63. $8x^3 - 27y^3$
$(2x - 3y)(4x^2 + 6xy + 9y^2)$

64. $64u^3 - 27v^3$
$(4u - 3v)(16u^2 + 12uv + 9v^2)$

65. $8 - x^6$
$(2 - x^2)(4 + 2x^2 + x^4)$

66. $1 + y^{12}$
$(1 + y^4)(1 - y^4 + y^8)$

67. $(x - 2)^3 - 1$
$(x - 3)(x^2 - 3x + 3)$

68. $(y + 3)^3 + 8$
$(y + 5)(y^2 + 4y + 7)$

In Exercises 69 to 76, factor over the integers the polynomials that are quadratic in form.

69. $x^4 - x^2 - 6$
$(x^2 - 3)(x^2 + 2)$

70. $y^4 + 3y^2 + 2$
$(y^2 + 1)(y^2 + 2)$

71. $x^2y^2 + 4xy - 5$
$(xy + 5)(xy - 1)$

72. $x^2y^2 - 8xy + 12$
$(xy - 6)(xy - 2)$

73. $4x^5 - 4x^3 - 8x$
$4x(x^2 - 2)(x^2 + 1)$

74. $z^4 + 3z^2 - 4$
$(z + 1)(z - 1)(z^2 + 4)$

75. $z^4 + z^2 - 20$
$(z + 2)(z - 2)(z^2 + 5)$

76. $x^4 - 13x^2 + 36$
$(x + 2)(x - 2)(x + 3)(x - 3)$

In Exercises 77 to 82, factor over the integers by grouping.

77. $3x^3 + x^2 + 6x + 2$
$(3x + 1)(x^2 + 2)$

78. $18w^3 + 15w^2 + 12w + 10$
$(6w + 5)(3w^2 + 2)$

79. $ax^2 - ax + bx - b$
$(x - 1)(ax + b)$

80. $a^2y^2 - ay^3 + ac - cy$
$(a - y)(ay^2 + c)$

81. $6w^3 + 4w^2 - 15w - 10$ $(3w + 2)(2w^2 - 5)$

82. $10z^3 - 15z^2 - 4z + 6$ $(2z - 3)(5z^2 - 2)$

In Exercises 83 to 102, use the general factoring strategy to completely factor each polynomial. If the polynomial does not factor, then state that it is nonfactorable over the integers.

83. $18x^2 - 2$
$2(3x - 1)(3x + 1)$

84. $4bx^3 + 32b$
$4b(x + 2)(x^2 - 2x + 4)$

85. $16x^4 - 1$
$(2x - 1)(2x + 1)(4x^2 + 1)$

86. $81y^4 - 16$
$(3y - 2)(3y + 2)(9y^2 + 4)$

87. $12ax^2 - 23axy + 10ay^2$
$a(3x - 2y)(4x - 5y)$

88. $6ax^2 - 19axy - 20ay^2$
$a(6x + 5y)(x - 4y)$

89. $3bx^3 + 4bx^2 - 3bx - 4b$
$b(3x + 4)(x - 1)(x + 1)$

90. $2x^6 - 2$
$2(x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)$

91. $72bx^2 + 24bxy + 2by^2$
$2b(6x + y)^2$

92. $64y^3 - 16y^2z + yz^2$
$y(8y - z)^2$

93. $(w - 5)^3 + 8$
$(w - 3)(w^2 - 12w + 39)$

94. $5xy + 20y - 15x - 60$
$5(x + 4)(y - 3)$

95. $x^2 + 6xy + 9y^2 - 1$
$(x + 3y - 1)(x + 3y + 1)$

96. $4y^2 - 4yz + z^2 - 9$
$(2y - z - 3)(2y - z + 3)$

97. $8x^2 + 3x - 4$
Not factorable over the integers

98. $16x^2 + 81$
Not factorable over the integers

99. $5x(2x - 5)^2 - (2x - 5)^3$
$(2x - 5)^2(3x + 5)$

100. $6x(3x + 1)^3 - (3x + 1)^4$
$(3x + 1)^3(3x - 1)$

101. $4x^2 + 2x - y - y^2$
$(2x - y)(2x + y + 1)$

102. $a^2 + a + b - b^2$
$(a + b)(a - b + 1)$

Enrichment Exercises

In Exercises 103 and 106, find all values of k such that the trinomial is a perfect-square trinomial.

103. $x^2 + kx + 16$
$-8, 8$

104. $36x^2 + kxy + 100y^2$
$-120, 120$

105. $x^2 + 16x + k$ 64

106. $x^2 - 14xy + ky^2$ 49

In Exercises 107 to 110, find all integer values of k for which the resulting trinomial will factor over the integers.

107. $x^2 + kx - 6$
$-5, -1, 1, 5$

108. $x^2 + kx + 12$
$-13, -8, -7, 7, 8, 13$

109. $2x^2 + kx - 10$
$-19, -8, -1, 1, 8, 19$

110. $3x^2 + kx + 5$
$-16, -8, 8, 16$

SECTION P.5

Rational Expressions

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A3.

PS1. Simplify: $1 + \dfrac{1}{2 - \dfrac{1}{3}}$ [P.1] $\frac{8}{5}$

PS2. Simplify: $\left(\dfrac{w}{x}\right)^{-1}\left(\dfrac{y}{z}\right)^{-1}$ [P.2] $\frac{xz}{wy}$

PS3. What is the common binomial factor of $x^2 + 2x - 3$ and $x^2 + 7x + 12$? [P.4] $x + 3$

In Exercises PS4 to PS6, factor completely over the integers.

PS4. $(2x - 3)(3x + 2) - (2x - 3)(x + 2)$ [P.4] $2x(2x - 3)$

PS5. $x^2 - 5x - 6$ [P.4] $(x - 6)(x + 1)$

PS6. $x^3 - 64$ [P.4] $(x - 4)(x^2 + 4x + 16)$

A **rational expression** is a fraction in which the numerator and denominator are polynomials. For example, the expressions below are rational expressions.

$$\frac{3}{x + 1} \quad \text{and} \quad \frac{x^2 - 4x - 21}{x^2 - 9}$$

The **domain of a rational expression** is the set of all real numbers that can be used as replacements for the variable. Any value of the variable that causes division by zero is excluded from the domain of the rational expression. For example, the domain of

$$\frac{x + 3}{x^2 - 5x}, \quad x \neq 0, x \neq 5$$

is the set of all real numbers except 0 and 5. Both 0 and 5 are excluded values because the denominator $x^2 - 5x$ equals zero when $x = 0$ and also when $x = 5$. Sometimes the excluded values are specified to the right of a rational expression,

Math Matters

Evidence from work left by early Egyptians more than 3600 years ago shows that they used, with one exception, unit fractions—that is, fractions whose numerators are 1. The one exception was 2/3. A unit fraction was represented by placing an oval over the symbol for the number in the denominator. For instance, 1/4 = .

TO REVIEW

Factors of the Sum or Difference of Two Perfect Cubes
See page 44.

Solution

$$\frac{x^2 + 6x + 9}{x^3 + 27} \div \frac{x^2 + 7x + 12}{x^3 - 3x^2 + 9x}$$

$$= \frac{(x + 3)^2}{(x + 3)(x^2 - 3x + 9)} \div \frac{(x + 4)(x + 3)}{x(x^2 - 3x + 9)} \qquad \text{• Factor.}$$

$$= \frac{(x + 3)^2}{(x + 3)(x^2 - 3x + 9)} \cdot \frac{x(x^2 - 3x + 9)}{(x + 4)(x + 3)} \qquad \text{• Multiply by the reciprocal.}$$

$$= \frac{\cancel{(x + 3)^2}x\cancel{(x^2 - 3x + 9)}}{\cancel{(x + 3)}\cancel{(x^2 - 3x + 9)}(x + 4)\cancel{(x + 3)}} \qquad \text{• Simplify.}$$

$$= \frac{x}{x + 4}$$

Try Exercise 30, page 57

Addition of rational expressions with a **common denominator** is accomplished by writing the sum of the numerators over the common denominator. For example,

$$\frac{5x}{18} + \frac{x}{18} = \frac{5x + x}{18} = \frac{6x}{18} = \frac{\cancel{6}x}{\cancel{6} \cdot 3} = \frac{x}{3}$$

If the rational expressions do not have a common denominator, then they can be written as equivalent expressions that have a common denominator by multiplying the numerator and denominator of each of the rational expressions by the required polynomials. The following procedure can be used to determine the least common denominator (LCD) of rational expressions. It is similar to the process used to find the LCD of rational numbers.

Determining the LCD of Rational Expressions

1. Factor each denominator completely and express repeated factors using exponential notation.
2. Identify the largest power of each factor in any single factorization. The LCD is the product of each factor raised to its largest power.

For example, the rational expressions

$$\frac{1}{x + 3} \quad \text{and} \quad \frac{5}{2x - 1}$$

have an LCD of $(x + 3)(2x - 1)$. The rational expressions

$$\frac{5x}{(x + 5)(x - 7)^3} \quad \text{and} \quad \frac{7}{x(x + 5)^2(x - 7)}$$

have an LCD of $x(x + 5)^2(x - 7)^3$.

Alternative to Example 4
Perform the indicated operation.

a. $\dfrac{2x + 1}{x - 3} + \dfrac{x - 5}{x - 4}$

■ $\dfrac{3x^2 - 15x + 11}{(x - 3)(x - 4)}$

b. $\dfrac{3x + 2}{x^2 - x - 12} - \dfrac{2x + 1}{x^2 + 8x + 15}$

■ $\dfrac{x^2 + 24x + 14}{(x - 4)(x + 3)(x + 5)}$

EXAMPLE 4 Add and Subtract Rational Expressions

Perform the indicated operation and then simplify, if possible.

a. $\dfrac{2x + 1}{x - 3} + \dfrac{x + 2}{x + 5}$

b. $\dfrac{39x + 36}{x^2 - 3x - 10} - \dfrac{23x - 16}{x^2 - 7x + 10}$

Solution

a. The LCD is $(x-3)(x+5)$. Write equivalent fractions in terms of the LCD, and then add.

$$\frac{2x+1}{x-3}+\frac{x+2}{x+5}=\frac{2x+1}{x-3}\cdot\frac{x+5}{x+5}+\frac{x+2}{x+5}\cdot\frac{x-3}{x-3}$$

$$=\frac{2x^2+11x+5}{(x-3)(x+5)}+\frac{x^2-x-6}{(x-3)(x+5)}$$

$$=\frac{(2x^2+11x+5)+(x^2-x-6)}{(x-3)(x+5)}$$ • Add.

$$=\frac{3x^2+10x-1}{(x-3)(x+5)}$$ • Simplify.

b. Factor the denominators:

$x^2-3x-10=(x-5)(x+2)$

$x^2-7x+10=(x-5)(x-2)$

The LCD is $(x-5)(x+2)(x-2)$. Write equivalent fractions in terms of the LCD, and then subtract.

$$\frac{39x+36}{x^2-3x-10}-\frac{23x-16}{x^2-7x+10}$$

$$=\frac{39x+36}{(x-5)(x+2)}\cdot\frac{x-2}{x-2}-\frac{23x-16}{(x-5)(x-2)}\cdot\frac{x+2}{x+2}$$

$$=\frac{39x^2-42x-72}{(x-5)(x+2)(x-2)}-\frac{23x^2+30x-32}{(x-5)(x+2)(x-2)}$$

$$=\frac{(39x^2-42x-72)-(23x^2+30x-32)}{(x-5)(x+2)(x-2)}$$

$$=\frac{16x^2-72x-40}{(x-5)(x+2)(x-2)}=\frac{8(2x^2-9x-5)}{(x-5)(x+2)(x-2)}$$

$$=\frac{8(2x+1)(x-5)}{(x-5)(x+2)(x-2)}=\frac{8(2x+1)}{(x+2)(x-2)}$$

▸ **Try Exercise 38, page 57**

Alternative to Example 5

Simplify:

$$\frac{x+4}{x-5}+\frac{x-3}{x-4}\cdot\frac{x^2-5x+4}{x^2-9}$$

■ $\frac{2x^2+x+17}{(x-5)(x+3)}$

EXAMPLE 5 Use the Order of Operations Agreement with Rational Expressions

Simplify: $\frac{x+3}{x-2}-\frac{x+4}{x-1}\div\frac{x^2+5x+4}{x^2+4x-5}$

Solution

The Order of Operations Agreement requires that division be completed before subtraction. To divide fractions, multiply by the reciprocal as shown below.

$$\frac{x+3}{x-2}-\frac{x+4}{x-1}\div\frac{x^2+5x+4}{x^2+4x-5}$$

$$=\frac{x+3}{x-2}-\frac{x+4}{x-1}\cdot\frac{x^2+4x-5}{x^2+5x+4}$$ • Multiply by the reciprocal.

(continued)