SECTION P.3

Operations on Polynomials
Applications of Polynomials

Polynomials

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A2.

PS1. Simplify: $-3(2a - 4b)$ [P.1]

PS2. Simplify: $5 - 2(2x - 7)$ [P.1]

PS3. Simplify: $2x^2 + 3x - 5 + x^2 - 6x - 1$ [P.1]

PS4. Simplify: $4x^2 - 6x - 1 - 5x^2 + x$ [P.1]

PS5. True or false: $4 - 3x - 2x^2 = 2x^2 - 3x + 4$ [P.1]

PS6. True or false: $\dfrac{12 + 15}{4} = \dfrac{\overset{3}{\cancel{12}} + 15}{\cancel{4}} = 18$ [P.1]

Operations on Polynomials

A **monomial** is a constant, a variable, or the product of a constant and one or more variables, with the variables having only *nonnegative* integer exponents.

-8	z	$7y$	$-12a^2bc^3$
A number	A variable	The product of a constant and one variable	The product of a constant and several variables

The expression $3x^{-2}$ is *not* a monomial because it is the product of a constant and a variable with a *negative* integer exponent.

The constant multiplying the variables is called the **numerical coefficient** or **coefficient**. For $7y$, the coefficient is 7; for $-12a^2bc^3$, the coefficient is -12. The coefficient of z is 1 because $z = 1 \cdot z$. Similarly, the coefficient of $-x$ is -1 because $-x = -1 \cdot x$.

The **degree of a monomial** is the sum of the exponents of the variables. The degree of a nonzero constant is 0. The constant 0 has no degree.

$7y$	$-12a^2bc^3$	-8
Degree is 1 because $y = y^1$.	Degree is $2 + 1 + 3 = 6$.	Degree is 0.

TO REVIEW

Terms
See page 10.

Note

The sign of a term is the sign that precedes the term.

A **polynomial** is the sum of a finite number of monomials. Each monomial is called a **term** of the polynomial. The **degree of a polynomial** is the greatest of the degrees of the terms. See Table P.1.

Table P.1 Terms and Degree of a Polynomial

Polynomial	Terms	Degree
$5x^4 - 6x^3 + 5x^2 - 7x - 8$	$5x^4, -6x^3, 5x^2, -7x, -8$	4
$-3xy^2 - 8xy + 6x$	$-3xy^2, -8xy, 6x$	3

Terms that have exactly the same variables raised to the same powers are called **like terms**. For example, $14x^2$ and $-x^2$ are like terms. $7x^2y$ and $5yx^2$ are like terms; the order of the variables is not important. The terms $6xy^2$ and $6x^2y$ are not like terms; the exponents on the variables are different.

A polynomial is said to be in simplest form if all its like terms have been combined. For example, the simplified form of $4x^2 + 3x + 5x - x^2$ is $3x^2 + 8x$. A **binomial** is a simplified polynomial with two terms; $3x^4 - 7$, $2xy - y^2$, and $x + 1$ are binomials. A **trinomial** is a simplified polynomial with three terms; $3x^2 + 6x - 1$, $2x^2 - 3xy + 7y^2$, and $x + y + 2$ are trinomials. A nonzero constant, such as 5, is a **constant polynomial**.

Definition of the Standard Form of a Polynomial

The **standard form of a polynomial** of degree n in the variable x is

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where $a_n \neq 0$ and n is a nonnegative integer. The coefficient a_n is the **leading coefficient**, and a_0 is the **constant term.**

EXAMPLE

Polynomial	Standard Form	Leading Coefficient
$6x - 7 + 2x^3$	$2x^3 + 6x - 7$	2
$4z^3 - 2z^4 + 3z - 9$	$-2z^4 + 4z^3 + 3z - 9$	-2
$y^5 - 3y^3 + 1 - 2y - y^2$	$y^5 - 3y^3 - y^2 - 2y + 1$	1

EXAMPLE 1 Identify Terms Related to a Polynomial

Write the polynomial $6x^3 - x + 5 - 2x^4$ in standard form. Identify the degree, terms, constant term, leading coefficient, and coefficients of the polynomial.

Solution

A polynomial is in standard form when the terms are written in decreasing powers of the variable. The standard form of the polynomial is $-2x^4 + 6x^3 - x + 5$. In this form, the degree is 4; the terms are $-2x^4$, $6x^3$, $-x$, and 5; the constant term is 5. The leading coefficient is -2; the coefficients are -2, 6, -1, and 5.

▸ Try Exercise 16, page 36

To add polynomials, add the coefficients of the like terms.

EXAMPLE 2 Add Polynomials

Add: $(3x^3 - 2x^2 - 6) + (4x^2 - 6x - 7)$

Solution

$$(3x^3 - 2x^2 - 6) + (4x^2 - 6x - 7)$$
$$= 3x^3 + (-2x^2 + 4x^2) + (-6x) + [(-6) + (-7)]$$
$$= 3x^3 + 2x^2 - 6x - 13$$

▸ Try Exercise 28, page 36

The **additive inverse of the polynomial** $3x - 7$ is

$$-(3x - 7) = -3x + 7$$

Question • What is the additive inverse of $3x^2 - 8x + 7$?

Answer • The additive inverse is $-3x^2 + 8x - 7$.

To subtract a polynomial, we add its additive inverse. For example,

$$\begin{aligned}(2x - 5) - (3x - 7) &= (2x - 5) + (-3x + 7)\\ &= [2x + (-3x)] + [(-5) + 7]\\ &= -x + 2\end{aligned}$$

The distributive property is used to multiply polynomials. For instance,

$$\begin{aligned}(2x^2 - 5x + 3)(3x + 4) &= (2x^2 - 5x + 3)(3x) + (2x^2 - 5x + 3)4\\ &= (6x^3 - 15x^2 + 9x) + (8x^2 - 20x + 12)\\ &= 6x^3 - 7x^2 - 11x + 12\end{aligned}$$

Although we could always multiply polynomials using the preceding procedure, we frequently use a vertical format. Here is the same product as shown previously using that format.

$$\begin{array}{rrrrl} & 2x^2 & -\ 5x & +\ 3 & \\ & & 3x & +\ 4 & \\ \hline & 8x^2 & -\ 20x & +\ 12 & = (2x^2 - 5x + 3)4\\ 6x^3 & -\ 15x^2 & +\ 9x & & = (2x^2 - 5x + 3)(3x)\\ \hline 6x^3 & -\ 7x^2 & -\ 11x & +\ 12 & \end{array}$$

EXAMPLE 3 Multiply Polynomials

Multiply: $(2x - 5)(x^3 - 4x + 2)$

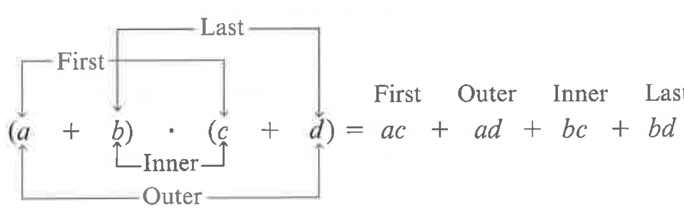
Solution

Note in the following solution how like terms are placed in columns.

$$\begin{array}{rrrrr} & x^3 & & -\ 4x & +\ 2\\ & & & 2x & -\ 5\\ \hline & -\ 5x^3 & & +\ 20x & -\ 10\\ 2x^4 & & -\ 8x^2 & +\ 4x & \\ \hline 2x^4 & -\ 5x^3 & -\ 8x^2 & +\ 24x & -\ 10\end{array}$$

▶ **Try Exercise 42, page 37**

If the terms of the binomials $(a + b)$ and $(c + d)$ are labeled as shown below, then the product of the two binomials can be computed mentally by the **FOIL method.**



In the following illustration, we find the product of $(7x - 2)$ and $(5x + 4)$ by the FOIL method.

$$\begin{aligned}& \qquad\ \ \text{First}\qquad\ \text{Outer}\qquad\ \text{Inner}\qquad\quad \text{Last}\\ (7x - 2)(5x + 4) &= (7x)(5x) + (7x)(4) + (-2)(5x) + (-2)(4)\\ &= \ \ 35x^2 \ +\ 28x \ -\ 10x \ -\ 8\\ &= \ \ 35x^2 + 18x - 8\end{aligned}$$

EXAMPLE 4 Multiply Binomials

Multiply.

a. $(4x + 5)(3x - 7)$

b. $(2x - 3y)(4x - 5y)$

Solution

a.
$$\begin{aligned}(4x + 5)(3x - 7) &= (4x)(3x) - (4x)7 + 5(3x) - 5(7)\\ &= 12x^2 - 28x + 15x - 35\\ &= 12x^2 - 13x - 35\end{aligned}$$

b.
$$\begin{aligned}(2x - 3y)(4x - 5y) &= (2x)(4x) - (2x)(5y) - (3y)(4x) + (3y)(5y)\\ &= 8x^2 - 10xy - 12xy + 15y^2\\ &= 8x^2 - 22xy + 15y^2\end{aligned}$$

▸ Try Exercise 54, page 37

Certain products occur so frequently in algebra that they deserve special attention. See Table P.2.

Table P.2 Special Product Formulas

Special Form	Formula(s)
(Sum)(Difference)	$(x + y)(x - y) = x^2 - y^2$
$(\text{Binomial})^2$	$(x + y)^2 = x^2 + 2xy + y^2$ $(x - y)^2 = x^2 - 2xy + y^2$

The variables x and y in these special product formulas can be replaced by other algebraic expressions, as shown in Example 5.

EXAMPLE 5 Use the Special Product Formulas

Find each special product.

a. $(7x + 10)(7x - 10)$ **b.** $(2y^2 + 11z)^2$

Solution

a. $(7x + 10)(7x - 10) = (7x)^2 - (10)^2 = 49x^2 - 100$

b. $(2y^2 + 11z)^2 = (2y^2)^2 + 2[(2y^2)(11z)] + (11z)^2 = 4y^4 + 44y^2z + 121z^2$

Try Exercise 60, page 37

Many application problems require you to *evaluate polynomials*. To **evaluate a polynomial**, substitute the given value or values for the variable or variables and then perform the indicated operations using the Order of Operations Agreement.

EXAMPLE 6 Evaluate a Polynomial

Evaluate the polynomial $2x^3 - 6x^2 + 7$ for $x = -4$.

Solution

$2x^3 - 6x^2 + 7$

$2(-4)^3 - 6(-4)^2 + 7 = 2(-64) - 6(16) + 7$ • Substitute -4 for x. Evaluate the powers.

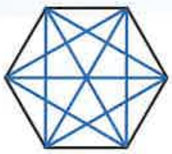
$= -128 - 96 + 7$ • Perform the multiplications.

$= -217$ • Perform the additions and subtractions.

Try Exercise 70, page 37

Applications of Polynomials

EXAMPLE 7 Solve an Application



A diagonal of a polygon is a line segment from one vertex to any other nonadjacent vertex. The diagonals of a regular hexagon (one whose sides are equal) are shown at the left. The number of distinct diagonals of a polygon is given by $\frac{1}{2}n^2 - \frac{3}{2}n$, where n is the number of sides of the polygon. Just as an artist or musician may view a painting or composition as elegant, mathematicians view regular polygons that can be constructed with a straightedge and compass as elegant. In 1796, Carl Friedrich Gauss, one of the greatest mathematicians who ever lived, proved that it was possible to draw a regular 17-sided polygon with just a straightedge and compass. How many distinct diagonals are in a 17-gon?

Solution

$\frac{1}{2}n^2 - \frac{3}{2}n$

$\frac{1}{2}(17)^2 - \frac{3}{2}(17) = \frac{1}{2}(289) - \frac{3}{2}(17) = 119$ • Substitute 17 for n. Then simplify.

There are 119 diagonals in a 17-gon.

Try Exercise 80, page 37

Math Matters

The procedure used by the computer to determine whether a number is prime or composite is a *polynomial time algorithm*, because the time required can be estimated using a polynomial. The procedure used to factor a number is an *exponential time algorithm*. In the field of *computational complexity*, it is important to distinguish between polynomial time algorithms and exponential time algorithms. Example 8 illustrates that the polynomial time algorithm can be run in about 2 seconds, whereas the exponential time algorithm requires about 44 minutes!

EXAMPLE 8 Solve an Application

A scientist determines that the average time in seconds that it takes a particular computer to determine whether an n-digit natural number is prime or composite is given by

$$0.002n^2 + 0.002n + 0.009, \quad 20 \le n \le 40$$

The average time in seconds that it takes the computer to factor an n-digit number is given by

$$0.00032(1.7)^n, \quad 20 \le n \le 40$$

Estimate the average time it takes the computer to

a. determine whether a 30-digit number is prime or composite

b. factor a 30-digit number

(continued)

Solution

a. $0.002n^2 + 0.002n + 0.009$

$0.002(30)^2 + 0.002(30) + 0.009 = 1.8 + 0.06 + 0.009 = 1.869 \approx 2$ seconds

b. $0.00032(1.7)^n$

$0.00032(1.7)^{30} \approx 0.00032(8{,}193{,}465.726)$

≈ 2600 seconds

▶ Try Exercise 82, page 38

EXERCISE SET P.3

Concept Check

In Exercises 1 to 10, match the descriptions, labeled A to J, with the appropriate examples.

A. $x^3y + xy$

B. $7x^2 + 5x - 11$

C. $\frac{1}{2}x^2 + xy + y^2$

D. $4xy$

E. $8x^3 - 1$

F. $3 - 4x^2$

G. 8

H. $3x^5 - 4x^2 + 7x - 11$

I. $8x^4 - \sqrt{5}x^3 + 7$

J. 0

1. A monomial of degree 2
2. A binomial of degree 3
3. A polynomial of degree 5
4. A binomial with a leading coefficient of -4
5. A zero-degree polynomial
6. A fourth-degree polynomial that has a third-degree term
7. A trinomial with integer coefficients
8. A trinomial in x and y
9. A polynomial with no degree
10. A fourth-degree binomial

In Exercises 11 to 14, use the special product formulas to perform the indicated operation.

11. $(x - 4)(x + 4)$
12. $(y + 5)^2$
13. $(z - 4)^2$
14. $(a + 2)(a - 2)$

In Exercises 15 to 20, for each polynomial, determine its *a.* standard form, *b.* degree, *c.* coefficients, *d.* leading coefficient, and *e.* terms.

15. $2x + x^2 - 7$
16. $-3x^2 - 11 - 12x^4$
17. $x^3 - 1$
18. $4x^2 - 2x + 7$
19. $2x^4 + 3x^3 + 5 + 4x^2$
20. $3x^2 - 5x^3 + 7x - 1$

In Exercises 21 to 26, determine the degree of the given polynomial.

21. $3xy^2 - 2xy + 7x$
22. $x^3 + 3x^2y + 3xy^2 + y^3$
23. $4x^2y^2 - 5x^3y^2 + 17xy^3$
24. $-9x^5y + 10xy^4 - 11x^2y^2$
25. xy
26. $5x^2y - y^4 + 6xy$

In Exercises 27 to 44, perform the indicated operation and simplify if possible by combining like terms. Write the result in standard form.

27. $(3x^2 + 4x + 5) + (2x^2 + 7x - 2)$
28. $(5y^2 - 7y + 3) + (2y^2 + 8y + 1)$
29. $(4w^3 - 2w + 7) + (5w^3 + 8w^2 - 1)$
30. $(5x^4 - 3x^2 + 9) + (3x^3 - 2x^2 - 7x + 3)$
31. $(r^2 - 2r - 5) - (3r^2 - 5r + 7)$
32. $(7s^2 - 4s + 11) - (-2s^2 + 11s - 9)$
33. $(u^3 - 3u^2 - 4u + 8) - (u^3 - 2u + 4)$
34. $(5v^4 - 3v^2 + 9) - (6v^4 + 11v^2 - 10)$

Indicates Try It Exercises

35. $(4x - 5)(2x^2 + 7x - 8)$

36. $(5x - 7)(3x^2 - 8x - 5)$

37. $(3x^2 - 5x + 6)(3x - 1)$

38. $(3x - 4)(x^2 - 6x - 9)$

39. $(2x + 6)(5x^3 - 6x^2 + 4)$

40. $(2x^3 - 7x - 1)(6x - 3)$

41. $(x^3 - 4x^2 + 9x - 6)(2x + 5)$

42. $(3x^3 + 4x^2 - x + 7)(3x - 2)$

43. $(3x^2 - 2x + 5)(2x^2 - 5x + 2)$

44. $(2y^3 - 3y + 4)(2y^2 - 5y + 7)$

In Exercises 45 to 58, use the FOIL method to find the indicated product.

45. $(y + 2)(y + 1)$ **46.** $(y - 5)(y + 3)$

47. $(2x + 4)(5x + 1)$ **48.** $(5x - 3)(2x + 7)$

49. $(4z - 3)(z - 4)$ **50.** $(5z - 6)(z - 1)$

51. $(a + 6)(a - 3)$ **52.** $(a - 10)(a + 4)$

53. $(5x - 11y)(2x - 7y)$ **54.** $(3a - 5b)(4a - 7b)$

55. $(9x + 5y)(2x + 5y)$ **56.** $(3x - 7z)(5x - 7z)$

57. $(3p + 5q)(2p - 7q)$ **58.** $(2r - 11s)(5r + 8s)$

In Exercises 59 to 66, use the special product formulas to perform the indicated operation.

59. $(3x + 5)(3x - 5)$ **60.** $(4x^2 - 3y)(4x^2 + 3y)$

61. $(3x^2 - y)^2$ **62.** $(6x + 7y)^2$

63. $(4w + z)^2$ **64.** $(3x - 5y^2)^2$

65. $[(x + 5) + y][(x + 5) - y]$

66. $[(x - 2y) + 7][(x - 2y) - 7]$

In Exercises 67 to 74, evaluate the given polynomial for the indicated value of the variable.

67. $x^2 + 7x - 1$, for $x = 3$

68. $x^2 - 8x + 2$, for $x = 4$

69. $-x^2 + 5x - 3$, for $x = -2$

70. $-x^2 - 5x + 4$, for $x = -5$

71. $3x^3 - 2x^2 - x + 3$, for $x = -1$

72. $5x^3 - x^2 + 5x - 3$, for $x = -1$

73. $1 - x^5$, for $x = -2$

74. $1 - x^3 - x^5$, for $x = 2$

75. **Recreation** The air resistance (in pounds) on a cyclist riding a bicycle in an upright position can be given by $0.016v^2$, where v is the speed of the cyclist in miles per hour (mph). Find the air resistance on a cyclist when
a. $v = 10$ mph **b.** $v = 15$ mph

76. **Highway Engineering** On an expressway, the recommended *safe distance* between cars in feet is given by $0.015v^2 + v + 10$, where v is the speed of the car in miles per hour. Find the safe distance when
a. $v = 30$ mph **b.** $v = 55$ mph

77. **Geometry** The volume of a right circular cylinder (as shown below) is given by $\pi r^2 h$, where r is the radius of the base and h is the height of the cylinder. Find the volume when
a. $r = 3$ inches, $h = 8$ inches
b. $r = 5$ centimeters, $h = 12$ centimeters

78. **Automotive Engineering** The fuel efficiency (in miles per gallon of gas) of a car is given by $-0.02v^2 + 1.5v + 2$, where v is the speed of the car in miles per hour. Find the fuel efficiency when
a. $v = 45$ mph **b.** $v = 60$ mph

79. **Psychology** Based on data from one experiment, the reaction time, in hundredths of a second, of a person to a visual stimulus varies according to age and is given by the expression $0.005x^2 - 0.32x + 12$, where x is the age of the person. Find the reaction time to the stimulus for a person who is
a. $x = 20$ years old **b.** $x = 50$ years old

80. **Committee Membership** The number of committees consisting of exactly 3 people that can be formed from a group of n people is given by the polynomial

$$\frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n$$

Find the number of committees consisting of exactly 3 people that can be formed from a group of 21 people.

81. **Chess Matches** The number of chess matches that can be played among n chess players is given by the polynomial $\frac{1}{2}n^2 - \frac{1}{2}n$. Find the number of chess matches that can be played among the members of a group of 150 people.

82. **Computer Science** A computer scientist determines that the time in seconds it takes a particular computer to calculate n digits of π is given by the polynomial

$$4.3 \times 10^{-6}n^2 - 2.1 \times 10^{-4}n$$

where $1000 \le n \le 10{,}000$. Estimate the time it takes the computer to calculate π to

a. 1000 digits **b.** 5000 digits **c.** 10,000 digits

83. **Computer Science** If n is a positive integer, then $n!$, which is read "n factorial," is given by

$$n(n-1)(n-2)\cdots 2 \cdot 1$$

For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. A computer scientist determines that each time a program is run on a particular computer, the time in seconds required to compute $n!$ is given by the polynomial

$$1.9 \times 10^{-6}n^2 - 3.9 \times 10^{-3}n$$

where $1000 \le n \le 10{,}000$. Using this polynomial, estimate the time it takes this computer to calculate 4000! and 8000!.

84. **Air Velocity of a Cough** The velocity, in meters per second, of the air that is expelled during a cough is given by velocity $= 6r^2 - 10r^3$, where r is the radius of the trachea in centimeters.

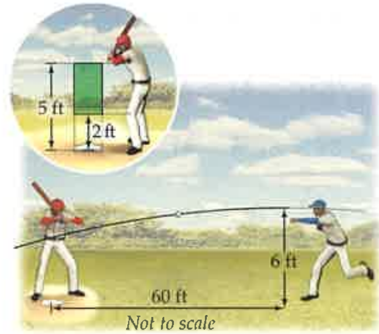
a. Find the velocity as a polynomial in standard form.

b. Find the velocity of the air in a cough when the radius of the trachea is 0.35 cm. Round to the nearest hundredth.

85. **Sports** The height, in feet, of a baseball t seconds after it is released by a pitcher is given by (ignoring air resistance)

$$\text{Height} = -16t^2 + 4.7881t + 6$$

For the pitch to be a strike, it must be at least 2 feet high and no more than 5 feet high when it crosses home plate. If it takes 0.5 second for the ball to reach home plate, will the ball be high enough to be a strike?



86. **Medicine** The temperature, in degrees Fahrenheit, of a patient after receiving a certain medication is given by

$$\text{Temperature} = 0.0002t^3 - 0.0114t^2 + 0.0158t + 104$$

where t is the number of minutes after receiving the medication.

a. What was the patient's temperature just before the medication was given?

b. What was the patient's temperature 25 minutes after the medication was given?

Enrichment Exercises

In Exercises 87 to 90, perform the indicated operation or operations and simplify.

87. $(4d-1)^2 - (2d-3)^2$

88. $(2x+3)^2 - (2x+3)(2x-3)$

89. $(3c-2)(4c+1)(5c-2)$

90. $(3x-2)^4$

91. What is the degree of the product of a polynomial of degree 3 and a polynomial of degree 5?

92. What is the degree of the sum of a polynomial of degree 3 and a polynomial of degree 5?

MID-CHAPTER P QUIZ

1. Evaluate $2x^3 - 4(3xy - z^2)$ for $x = -2$, $y = 3$, and $z = -4$.

2. Simplify: $5 - 2[3x - 5(2x - 3) + 1]$

3. Simplify: $\dfrac{24x^{-3}y^4}{6x^2y^{-3}}$

4. Simplify: $(3a^{-1/2}b^{3/4})^2(-2a^{2/3}b^{5/6})^3$

5. Simplify: $\sqrt[3]{16a^4b^9c^8}$

6. Simplify: $\dfrac{2}{3 - 2\sqrt{5}}$

7. Multiply: $(3x - 4y)(2x + 5y)$

8. Multiply: $(2a + 7)^2$

9. Multiply: $(2x - 3)(4x^2 + 5x - 7)$