SECTION P.2

Integer and Rational Number Exponents

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A1.

PS1. Simplify: $2^2 \cdot 2^3$ [P.1]

PS2. Simplify: $\frac{4^3}{4^5}$ [P.1]

PS3. Simplify: $(2^3)^2$ [P.1]

PS4. Simplify: $3.14(10^5)$ [P.1]

PS5. True or false: $3^4 \cdot 3^2 = 9^6$ [P.1]

PS6. True or false: $(3 + 4)^2 = 3^2 + 4^2$ [P.1]

Integer Exponents

Recall that if n is a natural number, then $b^n = \overbrace{b \cdot b \cdot b \cdot \cdots \cdot b}^{b \text{ is a factor } n \text{ times}}$. We can extend the definition of exponent to all integers. We begin with the case of zero as an exponent.

Definition of b^0

For any nonzero real number b, $b^0 = 1$.

EXAMPLE

$3^0 = 1$ $\quad \left(\frac{3}{4}\right)^0 = 1$ $\quad -7^0 = -1$ $\quad (a^2 + 1)^0 = 1$

Note

Note that $-7^0 = -(7^0) = -1$.

Now we extend the definition to include negative integers.

Definition of b^{-n}

If $b \neq 0$ and n is a natural number, then $b^{-n} = \frac{1}{b^n}$ and $\frac{1}{b^{-n}} = b^n$.

EXAMPLE

$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ $\quad \frac{1}{4^{-3}} = 4^3 = 64$ $\quad \frac{5^{-2}}{7^{-1}} = \frac{7}{5^2} = \frac{7}{25}$

EXAMPLE 1 Evaluate an Exponential Expression

Evaluate.

a. $(-2^4)(-3)^2$

b. $\frac{(-4)^{-3}}{(-2)^{-5}}$

c. $-\pi^0$

(continued)

Solution

a. $(-2^4)(-3)^2 = -(2 \cdot 2 \cdot 2 \cdot 2)(-3)(-3) = -(16)(9) = -144$

b. $\dfrac{(-4)^{-3}}{(-2)^{-5}} = \dfrac{(-2)(-2)(-2)(-2)(-2)}{(-4)(-4)(-4)} = \dfrac{-32}{-64} = \dfrac{1}{2}$

c. $-\pi^0 = -(\pi^0) = -1$

Try Exercise 24, page 28

When working with exponential expressions containing variables, we must ensure that a value of the variable does not result in an undefined expression. Take, for instance, $x^{-2} = \dfrac{1}{x^2}$. Because division by zero is not allowed, for the expression x^{-2}, we must assume that $x \neq 0$. Therefore, to avoid problems with undefined expressions, we will use the following restriction agreement.

Restriction Agreement

The expressions 0^0, 0^n (where n is a negative integer), and $\dfrac{a}{0}$ are all undefined expressions. Therefore, all values of variables in this text are restricted to avoid any one of these expressions.

EXAMPLE

In the expression $\dfrac{x^0y^{-3}}{z-4}$, $x \neq 0$, $y \neq 0$, and $z \neq 4$.

In the expression $\dfrac{(a-1)^0}{b+2}$, $a \neq 1$ and $b \neq -2$.

Exponential expressions containing variables are simplified using the following properties of exponents.

Properties of Exponents

If m, n, and p are integers and a and b are real numbers, then

Product $b^m \cdot b^n = b^{m+n}$

Quotient $\dfrac{b^m}{b^n} = b^{m-n}, \quad b \neq 0$

Power $(b^m)^n = b^{mn}$

$(a^mb^n)^p = a^{mp}b^{np}$

$\left(\dfrac{a^m}{b^n}\right)^p = \dfrac{a^{mp}}{b^{np}}, \quad b \neq 0$

EXAMPLE

$a^4 \cdot a \cdot a^3 = a^{4+1+3} = a^8$
- Add the exponents of the like bases. Recall that $a = a^1$.

$(x^4y^3)(xy^5z^2) = x^{4+1}y^{3+5}z^2 = x^5y^8z^2$
- Add the exponents of the like bases.

$\dfrac{a^7b}{a^2b^5} = a^{7-2}b^{1-5} = a^5b^{-4} = \dfrac{a^5}{b^4}$
- Subtract the exponents of the like bases.

$(uv^3)^5 = u^{1\cdot5}v^{3\cdot5} = u^5v^{15}$ • Multiply the exponents.

$\left(\frac{2x^5}{5y^4}\right)^3 = \frac{2^{1\cdot3}x^{5\cdot3}}{5^{1\cdot3}y^{4\cdot3}} = \frac{2^3x^{15}}{5^3y^{12}} = \frac{8x^{15}}{125y^{12}}$ • Multiply the exponents.

Question • Can the exponential expression x^5y^3 be simplified using the properties of exponents?

Integrating Technology

Exponential expressions such as a^{b^c} can be confusing. The generally accepted meaning of a^{b^c} is $a^{(b^c)}$. However, some graphing calculators do not evaluate exponential expressions in this way. Enter 2^3^4 in a graphing calculator. If the result is approximately 2.42×10^{24}, then the calculator evaluated $2^{(3^4)}$. If the result is 4096, then the calculator evaluated $(2^3)^4$. To ensure that you calculate the value you intend, we strongly urge you to use parentheses. For instance, entering 2^(3^4) will produce 2.42×10^{24} and entering (2^3)^4 will produce 4096.

To simplify an expression involving exponents, write the expression in a form in which *each base occurs at most once* and *no powers of powers or negative exponents occur.*

EXAMPLE 2 Simplify Exponential Expressions

Simplify.

a. $(5x^2y)(-4x^3y^5)$ **b.** $(3x^2yz^{-4})^3$ **c.** $\frac{-12x^5y}{18x^2y^6}$ **d.** $\left(\frac{4p^2q}{6pq^4}\right)^{-2}$

Solution

a. $(5x^2y)(-4x^3y^5) = [5(-4)]x^{2+3}y^{1+5}$ • Multiply the coefficients. Multiply the variables by adding the exponents of the like bases.

$= -20x^5y^6$

b. $(3x^2yz^{-4})^3 = 3^{1\cdot3}x^{2\cdot3}y^{1\cdot3}z^{-4\cdot3}$ • Use the power property of exponents.

$= 3^3x^6y^3z^{-12} = \frac{27x^6y^3}{z^{12}}$

c. $\frac{-12x^5y}{18x^2y^6} = -\frac{2}{3}x^{5-2}y^{1-6}$ • Simplify $\frac{-12}{18} = -\frac{2}{3}$. Divide the variables by subtracting the exponents of the like bases.

$= -\frac{2}{3}x^3y^{-5}$

$= -\frac{2x^3}{3y^5}$

d. $\left(\frac{4p^2q}{6pq^4}\right)^{-2} = \left(\frac{2p^{2-1}q^{1-4}}{3}\right)^{-2} = \left(\frac{2pq^{-3}}{3}\right)^{-2}$ • Use the quotient property of exponents.

$= \frac{2^{1(-2)}p^{1(-2)}q^{-3(-2)}}{3^{1(-2)}} = \frac{2^{-2}p^{-2}q^6}{3^{-2}}$ • Use the power property of exponents.

$= \frac{9q^6}{4p^2}$ • Write the answer in simplest form.

▸ Try Exercise 50, page 29

Answer • No. The bases are not the same.

Math Matters

- Approximately 3.1×10^6 orchid seeds weigh 1 ounce.
- Computer scientists measure an operation in nanoseconds. 1 nanosecond = 1×10^{-9} second
- If a spaceship traveled at 25,000 mph, it would require approximately 2.7×10^9 years to travel from one end of the universe to the other.

Scientific Notation

The exponent theorems provide a compact method of writing very large or very small numbers. The method is called *scientific notation.* A number written in **scientific notation** has the form $a \cdot 10^n$, where n is an integer and $1 \le a < 10$. The following procedure is used to change a number from its decimal form to scientific notation.

For numbers greater than 10, move the decimal point to the position to the right of the first digit. The exponent n will equal the number of places the decimal point has been moved. For example,

$$7{,}430{,}000 = 7.43 \times 10^6$$

6 places

For numbers less than 1, move the decimal point to the right of the first nonzero digit. The exponent n will be negative, and its absolute value will equal the number of places the decimal point has been moved. For example,

$$0.00000078 = 7.8 \times 10^{-7}$$

7 places

To change a number from scientific notation to its decimal form, reverse the procedure. That is, if the exponent is positive, move the decimal point to the right the same number of places as the exponent. For example,

$$3.5 \times 10^5 = 350{,}000$$

5 places

If the exponent is negative, move the decimal point to the left the same number of places as the absolute value of the exponent. For example,

$$2.51 \times 10^{-8} = 0.0000000251$$

8 places

Most calculators display very large and very small numbers in scientific notation. The number $450{,}000^2$ is displayed as `2.025 E 11`. This means $450{,}000^2 = 2.025 \times 10^{11}$.

EXAMPLE 3 Simplify an Expression Using Scientific Notation

One of the purposes of the Apollo 15 mission was to place a lunar Laser Ranging RetroReflector (LRRR) on the moon. The purpose of the LRRR is to precisely measure the distance from Earth to the moon. A laser beam is sent from a station on Earth to the LRRR, which then reflects the laser beam back to Earth.

Assuming the laser beam travels at 3.0×10^8 meters per second and the distance to the moon is 3.8×10^8 meters, find the round-trip time for the laser beam to reach the moon and the reflected beam to return to Earth. Round to the nearest hundredth of a second.

Solution

To find the time, divide the distance to the moon by the speed of the laser beam. Then multiply that result by 2 to obtain the round-trip time.

$$t = \frac{3.8 \times 10^8}{3.0 \times 10^8} = \frac{3.8}{3.0} \times 10^{8-8} \approx 1.267 \times 10^0 = 1.267 \times 1 = 1.267$$

We multiply 1.267 by 2 and see that the round-trip time for the laser beam is approximately 2.53 seconds.

Try Exercise 58, page 29

Rational Exponents and Radicals

To this point, the expression b^n has been defined for real number b and integers n. Now we wish to extend the definition of exponents to include rational numbers so that expressions such as $2^{1/2}$ will be meaningful. Not just any definition will do. We want a definition of rational exponents for which the properties of integer exponents are true. The following example shows the direction we can take to accomplish our goal.

If the product property for exponential expressions is to hold for rational exponents, then for rational numbers p and q, $b^p b^q = b^{p+q}$. For example,

$$9^{1/2} \cdot 9^{1/2} \quad \text{must equal} \quad 9^{1/2+1/2} = 9^1 = 9$$

Thus $9^{1/2}$ must be a square root of 9. That is, $9^{1/2} = 3$.

The example suggests that $b^{1/n}$ can be defined in terms of roots according to the following definition.

Definition of $b^{1/n}$

If n is an even positive integer and $b \geq 0$, then $b^{1/n}$ is the nonnegative real number such that $(b^{1/n})^n = b$.

If n is an odd positive integer, then $b^{1/n}$ is the real number such that $(b^{1/n})^n = b$.

EXAMPLE

- $25^{1/2} = 5$ because $5^2 = 25$.
- $(-64)^{1/3} = -4$ because $(-4)^3 = -64$.
- $16^{1/2} = 4$ because $4^2 = 16$.
- $-16^{1/2} = -(16^{1/2}) = -4$.
- $(-16)^{1/2}$ is not a real number.
- $(-32)^{1/5} = -2$ because $(-2)^5 = -32$.

If n is an even positive integer and $b < 0$, then $b^{1/n}$ is a *complex number*. Complex numbers are discussed in Section P.6.

To define expressions such as $8^{2/3}$, we will extend our definition of exponents even further. Because we want the power property $(b^p)^q = b^{pq}$ to be true for rational exponents also, we must have $(b^{1/n})^m = b^{m/n}$. With this in mind, we make the following definition.

Definition of $b^{m/n}$

For all positive integers m and n such that m/n is in simplest form, and for all real numbers b for which $b^{1/n}$ is a real number,

$$b^{m/n} = (b^{1/n})^m = (b^m)^{1/n}$$

Because $b^{m/n}$ is defined as $(b^{1/n})^m$ and as $(b^m)^{1/n}$, we can evaluate expressions such as $8^{4/3}$ in more than one way. For example, because $8^{1/3}$ is a real number, $8^{4/3}$ can be evaluated in either of the following ways.

$$8^{4/3} = (8^{1/3})^4 = 2^4 = 16$$

$$8^{4/3} = (8^4)^{1/3} = 4096^{1/3} = 16$$

Of the two methods, the $b^{m/n} = (b^{1/n})^m$ method is usually easier to apply, provided you can evaluate $b^{1/n}$.

EXAMPLE 4 Evaluate a Number with a Rational Exponent

Simplify.

a. $64^{2/3}$ **b.** $32^{-3/5}$ **c.** $\left(\dfrac{16}{81}\right)^{-3/4}$

Solution

a. $64^{2/3} = (64^{1/3})^2 = 4^2 = 16$

b. $32^{-3/5} = (32^{1/5})^{-3} = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

c. $\left(\dfrac{16}{81}\right)^{-3/4} = \left(\dfrac{81}{16}\right)^{3/4} = \left[\left(\dfrac{81}{16}\right)^{1/4}\right]^3 = \left(\dfrac{3}{2}\right)^3 = \dfrac{27}{8}$

▶ Try Exercise 62, page 29

The following exponent properties were stated earlier, but they are restated here to remind you that they have now been extended to apply to rational exponents.

Properties of Rational Exponents

If p, q, and r represent rational numbers and a and b are positive real numbers, then

Product $b^p \cdot b^q = b^{p+q}$

Quotient $\dfrac{b^p}{b^q} = b^{p-q}$

Power $(b^p)^q = b^{pq}$ $(a^p b^q)^r = a^{pr} b^{qr}$

$\left(\dfrac{a^p}{b^q}\right)^r = \dfrac{a^{pr}}{b^{qr}}$ $b^{-p} = \dfrac{1}{b^p}$

Recall that an exponential expression is in simplest form when no powers of powers or negative exponents occur and each base occurs at most once.

EXAMPLE 5 Simplify Exponential Expressions

Simplify.

a. $(2x^{1/3}y^{3/5})^2\,(9x^3y^{3/2})^{1/2}$ **b.** $\dfrac{(a^{3/4}b^{1/2})^2}{(a^{2/3}b^{3/4})^3}$

Solution

a.
$$(2x^{1/3}y^{3/5})^2(9x^3y^{3/2})^{1/2} = (2^2x^{2/3}y^{6/5})(9^{1/2}x^{3/2}y^{3/4})$$
• Use the power property.
$$= (4x^{2/3}y^{6/5})(3x^{3/2}y^{3/4})$$
$$= 12x^{\frac{2}{3}+\frac{3}{2}}y^{\frac{6}{5}+\frac{3}{4}} = 12x^{\frac{4}{6}+\frac{9}{6}}y^{\frac{24}{20}+\frac{15}{20}}$$
• Add the exponents on like bases.
$$= 12x^{13/6}y^{39/20}$$

b. $\dfrac{(a^{3/4}b^{1/2})^2}{(a^{2/3}b^{3/4})^3} = \dfrac{a^{3/2}b}{a^2b^{9/4}}$ • Use the power property.

$= a^{\frac{3}{2}-2}b^{1-\frac{9}{4}}$ • Subtract the exponents on like bases.

$= a^{\frac{3}{2}-\frac{4}{2}}b^{\frac{4}{4}-\frac{9}{4}} = a^{-1/2}b^{-5/4}$

$= \dfrac{1}{a^{1/2}b^{5/4}}$

Try Exercise 68, page 29

Math Matters

The formula for kinetic energy (energy of motion) that is used in Einstein's Theory of Relativity involves a radical,

$$\text{K.E}_r = mc^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)$$

where m is the mass of the object at rest, v is the speed of the object, and c is the speed of light.

Simplifying Radical Expressions

Radicals, expressed by the notation $\sqrt[n]{b}$, are also used to denote roots. The number b is the **radicand,** and the positive integer n is the **index** of the radical.

Definition of $\sqrt[n]{b}$

If n is a positive integer and b is a real number such that $b^{1/n}$ is a real number, then $\sqrt[n]{b} = b^{1/n}$.

If the index n equals 2, then the radical $\sqrt[2]{b}$ is written as simply $\sqrt{b}$, and it is referred to as the **principal square root of *b***, or simply the **square root of *b***.

The symbol $\sqrt{b}$ is reserved to represent the nonnegative square root of b. To represent the negative square root of b, write $-\sqrt{b}$. For example, $\sqrt{25} = 5$, whereas $-\sqrt{25} = -5$.

Definition of $(\sqrt[n]{b})^m$

For all positive integers n, all integers m, and all real numbers b such that $\sqrt[n]{b}$ is a real number, $(\sqrt[n]{b})^m = \sqrt[n]{b^m} = b^{m/n}$.

When $\sqrt[n]{b}$ is a real number, the equations

$$b^{m/n} = \sqrt[n]{b^m} \quad \text{and} \quad b^{m/n} = (\sqrt[n]{b})^m$$

can be used to write exponential expressions such as $b^{m/n}$ in radical form. Use the denominator n as the index of the radical and the numerator m as the power of the radicand or as the power of the radical. For example,

$(5xy)^{2/3} = (\sqrt[3]{5xy})^2 = \sqrt[3]{25x^2y^2}$ • Use the denominator 3 as the index of the radical and the numerator 2 as the power of the radical.

The equations

$$b^{m/n} = \sqrt[n]{b^m} \quad \text{and} \quad b^{m/n} = (\sqrt[n]{b})^m$$

also can be used to write radical expressions in exponential form. For example,

$\sqrt{(2ab)^3} = (2ab)^{3/2}$ • Use the index 2 as the denominator of the power and the exponent 3 as the numerator of the power.

The definition of $(\sqrt[n]{b})^m$ often can be used to evaluate radical expressions. For instance,

$$(\sqrt[3]{8})^4 = 8^{4/3} = (8^{1/3})^4 = 2^4 = 16$$

Care must be exercised when simplifying even roots (square roots, fourth roots, sixth roots, and so on) of variable expressions. Consider $\sqrt{x^2}$ when $x = 5$ and when $x = -5$.

Case 1 If $x = 5$, then $\sqrt{x^2} = \sqrt{5^2} = \sqrt{25} = 5 = x$.

Case 2 If $x = -5$, then $\sqrt{x^2} = \sqrt{(-5)^2} = \sqrt{25} = 5 = -x$.

These two cases suggest that

$$\sqrt{x^2} = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}$$

TO REVIEW

Absolute Value
See pages 7–8.

Recalling the definition of absolute value, we can write this more compactly as $\sqrt{x^2} = |x|$.

Simplifying odd roots of a variable expression does not require using the absolute value symbol. Consider $\sqrt[3]{x^3}$ when $x = 5$ and when $x = -5$.

Case 1 If $x = 5$, then $\sqrt[3]{x^3} = \sqrt[3]{5^3} = \sqrt[3]{125} = 5 = x$.

Case 2 If $x = -5$, then $\sqrt[3]{x^3} = \sqrt[3]{(-5)^3} = \sqrt[3]{-125} = -5 = x$.

Thus $\sqrt[3]{x^3} = x$.

Although we have illustrated this principle only for square roots and cube roots, the same reasoning can be applied to other cases. The general result is given below.

Definition of $\sqrt[n]{b^n}$

If n is an even natural number and b is a real number, then

$$\sqrt[n]{b^n} = |b|$$

If n is an odd natural number and b is a real number, then

$$\sqrt[n]{b^n} = b$$

EXAMPLE

$\sqrt[4]{16z^4} = 2|z|$ $\qquad$ $\sqrt[5]{32a^5} = 2a$

Because radicals are defined in terms of rational powers, the properties of radicals are similar to those of exponential expressions.

Properties of Radicals

If m and n are natural numbers and a and b are positive real numbers, then

Product $\quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

Quotient $\quad \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

Index $\quad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

A radical is in **simplest form** if it meets all of the following criteria:

1. The radicand contains only powers less than the index. ($\sqrt{x^5}$ does not satisfy this requirement because 5, the exponent, is greater than 2, the index.)
2. The index of the radical is as small as possible. ($\sqrt[9]{x^3}$ does not satisfy this requirement because $\sqrt[9]{x^3} = x^{3/9} = x^{1/3} = \sqrt[3]{x}$.)
3. The denominator has been rationalized. That is, no radicals occur in the denominator. ($1/\sqrt{2}$ does not satisfy this requirement.)
4. No fractions occur under the radical sign. ($\sqrt[4]{2/x^3}$ does not satisfy this requirement.)

Radical expressions are simplified by using the properties of radicals. Here are some examples.

EXAMPLE 6 Simplify Radical Expressions

Simplify.

a. $\sqrt{48x^7y^2}$ **b.** $\sqrt[3]{162x^4y^6}$ **c.** $\sqrt[4]{32x^3y^4}$

Solution

a. $\sqrt{48x^7y^2} = \sqrt{(2^4 \cdot 3)x^7y^2} = \sqrt{(2^2x^3y)^2 \cdot 3x}$ • Factor and group factors that can be written as a power of the index, 2.

$= \sqrt{(2^2x^3y)^2} \cdot \sqrt{3x}$ • Use the product property of radicals.

$= 4|x^3y|\sqrt{3x}$ • Recall that for n even, $\sqrt[n]{b^n} = |b|$.

b. $\sqrt[3]{162x^4x^6} = \sqrt[3]{(2 \cdot 3^4)x^4y^6}$ • Factor and group factors that can be written as a power of the index.

$= \sqrt[3]{(3xy^2)^3 \cdot (2 \cdot 3x)}$

$= \sqrt[3]{(3xy^2)^3} \cdot \sqrt[3]{6x}$ • Use the product property of radicals.

$= 3xy^2\sqrt[3]{6x}$ • Recall that for n odd, $\sqrt[n]{b^n} = b$.

c. $\sqrt[4]{32x^3y^4} = \sqrt[4]{2^5x^3y^4} = \sqrt[4]{(2^4y^4) \cdot (2x^3)}$ • Factor and group factors that can be written as a power of the index.

$= \sqrt[4]{2^4y^4} \cdot \sqrt[4]{2x^3}$ • Use the product property of radicals.

$= 2|y|\sqrt[4]{2x^3}$ • Recall that for n even, $\sqrt[n]{b^n} = |b|$.

Try Exercise 84, page 29

Like radicals have the same radicand and the same index. For instance,

$$3\sqrt[3]{5xy^2} \quad \text{and} \quad -4\sqrt[3]{5xy^2}$$

are like radicals. Addition and subtraction of like radicals are accomplished by using the distributive property. For example,

$$4\sqrt{3x} - 9\sqrt{3x} = (4 - 9)\sqrt{3x} = -5\sqrt{3x}$$

$$2\sqrt[3]{y^2} + 4\sqrt[3]{y^2} - \sqrt[3]{y^2} = (2 + 4 - 1)\sqrt[3]{y^2} = 5\sqrt[3]{y^2}$$

The sum $2\sqrt{3} + 6\sqrt{5}$ cannot be simplified further because the radicands are not the same. The sum $3\sqrt[3]{x} + 5\sqrt[4]{x}$ cannot be simplified because the indices are not the same.

Sometimes it is possible to simplify radical expressions that do not appear to be like radicals by simplifying each radical expression.

EXAMPLE 7 Combine Radical Expressions

Simplify: $5x\sqrt[3]{16x^4} - \sqrt[3]{128x^7}$

Solution

$5x\sqrt[3]{16x^4} - \sqrt[3]{128x^7}$

$= 5x\sqrt[3]{2^4x^4} - \sqrt[3]{2^7x^7}$ • Factor.

$= 5x\sqrt[3]{2^3x^3} \cdot \sqrt[3]{2x} - \sqrt[3]{2^6x^6} \cdot \sqrt[3]{2x}$ • Group factors that can be written as a power of the index.

$= 5x(2x\sqrt[3]{2x}) - 2^2x^2 \cdot \sqrt[3]{2x}$ • Use the product property of radicals.

$= 10x^2\sqrt[3]{2x} - 4x^2\sqrt[3]{2x}$ • Simplify.

$= 6x^2\sqrt[3]{2x}$

Try Exercise 92, page 29

Multiplication of radical expressions is accomplished by using the distributive property. For instance,

$\sqrt{5}(\sqrt{20} - 3\sqrt{15}) = \sqrt{5}(\sqrt{20}) - \sqrt{5}(3\sqrt{15})$ • Use the distributive property.

$= \sqrt{100} - 3\sqrt{75}$ • Multiply the radicals.

$= 10 - 3 \cdot 5\sqrt{3}$ • Simplify.

$= 10 - 15\sqrt{3}$

Finding the product of more complicated radical expressions may require repeated use of the distributive property.

EXAMPLE 8 Multiply Radical Expressions

Perform the indicated operation.

a. $(5\sqrt{6} - 7)(3\sqrt{6} + 2)$

b. $(3 - \sqrt{x - 7})^2, x \geq 7$

Solution

a. $(5\sqrt{6} - 7)(3\sqrt{6} + 2)$

$= 5\sqrt{6}(3\sqrt{6} + 2) - 7(3\sqrt{6} + 2)$ • Use the distributive property.

$= (15 \cdot 6 + 10\sqrt{6}) - (21\sqrt{6} + 14)$ • Use the distributive property.

$= 90 + 10\sqrt{6} - 21\sqrt{6} - 14$ • Simplify.

$= 76 - 11\sqrt{6}$

b. $(3 - \sqrt{x - 7})^2$

$= (3 - \sqrt{x - 7})(3 - \sqrt{x - 7})$

$= 9 - 3\sqrt{x - 7} - 3\sqrt{x - 7} + (\sqrt{x - 7})^2$ • Use the distributive property.

$= 9 - 6\sqrt{x - 7} + (x - 7)$ • $(\sqrt{x - 7})^2 = x - 7$, since $x \geq 7$.

$= 2 - 6\sqrt{x - 7} + x$

Try Exercise 102, page 29

To **rationalize the denominator** of a fraction means to write the fraction in an equivalent form that does not involve any radicals in the denominator. This is accomplished by multiplying the numerator and denominator of the radical expression by an expression that will cause the radicand in the denominator to be a perfect root of the index.

$$\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{3^2}} = \frac{5\sqrt{3}}{3}$$

• Recall that $\sqrt{3}$ means $\sqrt[2]{3}$. Multiply numerator and denominator by $\sqrt{3}$ so that the radicand is a perfect root of the index of the radical.

$$\frac{2}{\sqrt[3]{7}} = \frac{2}{\sqrt[3]{7}} \cdot \frac{\sqrt[3]{7^2}}{\sqrt[3]{7^2}} = \frac{2\sqrt[3]{7^2}}{\sqrt[3]{7^3}} = \frac{2\sqrt[3]{49}}{7}$$

• Multiply numerator and denominator by $\sqrt[3]{7^2}$ so that the radicand is a perfect root of the index of the radical.

$$\frac{5}{\sqrt[4]{x^5}} = \frac{5}{\sqrt[4]{x^5}} \cdot \frac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}} = \frac{5\sqrt[4]{x^3}}{\sqrt[4]{x^8}} = \frac{5\sqrt[4]{x^3}}{x^2}$$

• Multiply numerator and denominator by $\sqrt[4]{x^3}$ so that the radicand is a perfect root of the index of the radical.

EXAMPLE 9 Rationalize the Denominator

Rationalize the denominator.

a. $\dfrac{5}{\sqrt[3]{a}}$ **b.** $\sqrt{\dfrac{3}{32y}}, y > 0$

Solution

a. $\dfrac{5}{\sqrt[3]{a}} = \dfrac{5}{\sqrt[3]{a}} \cdot \dfrac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}} = \dfrac{5\sqrt[3]{a^2}}{\sqrt[3]{a^3}} = \dfrac{5\sqrt[3]{a^2}}{a}$

• Use $\sqrt[3]{a} \cdot \sqrt[3]{a^2} = \sqrt[3]{a^3} = a$.

b. $\sqrt{\dfrac{3}{32y}} = \dfrac{\sqrt{3}}{\sqrt{32y}} = \dfrac{\sqrt{3}}{4\sqrt{2y}} = \dfrac{\sqrt{3}}{4\sqrt{2y}} \cdot \dfrac{\sqrt{2y}}{\sqrt{2y}} = \dfrac{\sqrt{6y}}{8y}$

▶ Try Exercise 112, page 29

To rationalize the denominator of a fractional expression such as

$$\frac{1}{\sqrt{m} + \sqrt{n}}$$

we use the conjugate of $\sqrt{m} + \sqrt{n}$, which is $\sqrt{m} - \sqrt{n}$. The product of these conjugate pairs does not involve a radical.

$$(\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n}) = m - n$$

EXAMPLE 10 Rationalize the Denominator

Rationalize the denominator.

a. $\dfrac{3 + 2\sqrt{5}}{1 - 4\sqrt{5}}$

b. $\dfrac{2 + 4\sqrt{x}}{3 - 5\sqrt{x}}, x > 0$

Solution

a. $\dfrac{3 + 2\sqrt{5}}{1 - 4\sqrt{5}} = \dfrac{3 + 2\sqrt{5}}{1 - 4\sqrt{5}} \cdot \dfrac{1 + 4\sqrt{5}}{1 + 4\sqrt{5}}$

• Multiply numerator and denominator by the conjugate of the denominator.

$$= \frac{3(1 + 4\sqrt{5}) + 2\sqrt{5}(1 + 4\sqrt{5})}{1^2 - (4\sqrt{5})^2}$$

(continued)

$$= \frac{3 + 12\sqrt{5} + 2\sqrt{5} + 8 \cdot 5}{1 - 16 \cdot 5}$$ • Simplify.

$$= \frac{43 + 14\sqrt{5}}{-79}$$

$$= -\frac{43 + 14\sqrt{5}}{79}$$

b. $$\frac{2 + 4\sqrt{x}}{3 - 5\sqrt{x}} = \frac{2 + 4\sqrt{x}}{3 - 5\sqrt{x}} \cdot \frac{3 + 5\sqrt{x}}{3 + 5\sqrt{x}}$$ • Multiply numerator and denominator by the conjugate of the denominator.

$$= \frac{2(3 + 5\sqrt{x}) + 4\sqrt{x}(3 + 5\sqrt{x})}{3^2 - (5\sqrt{x})^2}$$

$$= \frac{6 + 10\sqrt{x} + 12\sqrt{x} + 20x}{9 - 25x}$$

$$= \frac{6 + 22\sqrt{x} + 20x}{9 - 25x}$$

Try Exercise 116, page 29

EXERCISE SET P.2

Concept Check

In Exercises 1 to 8, evaluate each expression.

1. -5^3 **2.** $(-5)^3$

3. $\left(\frac{2}{3}\right)^0$ **4.** -6^0

5. 4^{-2} **6.** 3^{-4}

7. $\frac{1}{2^{-5}}$ **8.** $\frac{1}{3^{-3}}$

In Exercises 9 to 12, write the number in scientific notation.

9. 2,011,000,000,000 **10.** 49,100,000,000

11. 0.000000000562 **12.** 0.000000402

In Exercises 13 to 16, change the number from scientific notation to decimal notation.

13. 3.14×10^7 **14.** 4.03×10^9

15. -2.3×10^{-6} **16.** 6.14×10^{-8}

In Exercises 17 to 22, evaluate each exponential expression.

17. $4^{3/2}$ **18.** $-16^{3/2}$

19. $-64^{2/3}$ **20.** $125^{4/3}$

21. $9^{-3/2}$ **22.** $32^{-4/5}$

In Exercises 23 to 52, write the exponential expression in simplest form.

23. $\frac{2^{-3}}{6^{-3}}$ **24.** $\frac{4^{-2}}{2^{-3}}$

25. $-2x^0$ **26.** $\frac{x^0}{4}$

27. $2x^{-4}$ **28.** $3y^{-2}$

29. $\frac{5}{z^{-6}}$ **30.** $\frac{8}{x^{-5}}$

31. $(x^3y^2)(xy^5)$ **32.** $(uv^6)(u^2v)$

33. $(-2ab^4)(-3a^2b^5)$ **34.** $(9xy^2)(-2x^2y^5)$

35. $(-4x^{-3}y)(7x^5y^{-2})$ **36.** $(-6x^4y)(7x^{-3}y^{-5})$

37. $\frac{6a^4}{8a^8}$ **38.** $\frac{12x^3}{16x^4}$

39. $\frac{12x^3y^4}{18x^5y^2}$ **40.** $\frac{5v^4w^{-3}}{10v^8}$

41. $\frac{36a^{-2}b^3}{3ab^4}$ **42.** $\frac{-48ab^{10}}{32a^4b^3}$

43. $(-2m^3n^2)(-3mn^2)^2$ **44.** $(2a^3b^2)^3(-4a^4b^2)$

Indicates Try It Exercises

45. $(x^{-2}y)^2(xy)^{-2}$

46. $(x^{-1}y^2)^{-3}(x^2y^{-4})^{-3}$

47. $\left(\frac{3a^2b^3}{6a^4b^4}\right)^2$

48. $\left(\frac{2ab^2c^3}{5ab^2}\right)^3$

49. $\frac{(-4x^2y^3)^2}{(2xy^2)^3}$

50. $\frac{(-3a^2b^3)^2}{(-2ab^4)^3}$

51. $\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^2$

52. $\left(\frac{x^{-3}y^{-4}}{x^{-2}y}\right)^{-2}$

In Exercises 53 to 60, perform the indicated operation and write the answer in scientific notation.

53. $(3 \times 10^{12})(9 \times 10^{-5})$

54. $(8.9 \times 10^{-5})(3.4 \times 10^{-6})$

55. $\frac{9 \times 10^{-3}}{6 \times 10^8}$

56. $\frac{2.5 \times 10^8}{5 \times 10^{10}}$

57. $\frac{(3.2 \times 10^{-11})(2.7 \times 10^{18})}{1.2 \times 10^{-5}}$

58. $\frac{(6.9 \times 10^{27})(8.2 \times 10^{-13})}{4.1 \times 10^{15}}$

59. $\frac{(4.0 \times 10^{-9})(8.4 \times 10^5)}{(3.0 \times 10^{-6})(1.4 \times 10^{18})}$

60. $\frac{(7.2 \times 10^8)(3.9 \times 10^{-7})}{(2.6 \times 10^{-10})(1.8 \times 10^{-8})}$

In Exercises 61 to 76, evaluate each exponential expression.

61. $\left(\frac{4}{9}\right)^{1/2}$

62. $\left(\frac{16}{25}\right)^{3/2}$

63. $\left(\frac{1}{8}\right)^{-4/3}$

64. $\left(\frac{8}{27}\right)^{-2/3}$

65. $(4a^{2/3}b^{1/2})(2a^{1/3}b^{3/2})$

66. $(6a^{3/5}b^{1/4})(-3a^{1/5}b^{3/4})$

67. $(-3x^{2/3})(4x^{1/4})$

68. $(-5x^{1/3})(-4x^{1/2})$

69. $(81x^8y^{12})^{1/4}$

70. $(27x^3y^6)^{2/3}$

71. $\frac{16z^{3/5}}{12z^{1/5}}$

72. $\frac{6a^{2/3}}{9a^{1/3}}$

73. $(2x^{2/3}y^{1/2})(3x^{1/6}y^{1/3})$

74. $\frac{x^{1/3}y^{5/6}}{x^{2/3}y^{1/6}}$

75. $\frac{9a^{3/4}b}{3a^{2/3}b^2}$

76. $\frac{12x^{1/6}y^{1/4}}{16x^{3/4}y^{1/2}}$

In Exercises 77 to 86, simplify each radical expression.

77. $\sqrt{45}$

78. $\sqrt{75}$

79. $\sqrt[3]{24}$

80. $\sqrt[3]{135}$

81. $\sqrt[3]{-135}$

82. $\sqrt[3]{-250}$

83. $\sqrt{24x^2y^3}$

84. $\sqrt{18x^2y^5}$

85. $\sqrt[3]{16a^3y^7}$

86. $\sqrt[3]{54m^2n^7}$

In Exercises 87 to 94, simplify each radical and then combine like radicals.

87. $2\sqrt{32} - 3\sqrt{98}$

88. $5\sqrt[3]{32} + 2\sqrt[3]{108}$

89. $-8\sqrt[4]{48} + 2\sqrt[4]{243}$

90. $2\sqrt[3]{40} - 3\sqrt[3]{135}$

91. $4\sqrt[3]{32y^4} + 3y\sqrt[3]{108y}$

92. $-3x\sqrt[3]{54x^4} + 2\sqrt[3]{16x^7}$

93. $x\sqrt[3]{8x^3y^4} - 4y\sqrt[3]{64x^6y}$

94. $4\sqrt{a^5b} - a^2\sqrt{ab}$

In Exercises 95 to 104, find the indicated product and express each result in simplest form.

95. $(\sqrt{5} + 3)(\sqrt{5} + 4)$

96. $(\sqrt{7} + 2)(\sqrt{7} - 5)$

97. $(\sqrt{2} - 3)(\sqrt{2} + 3)$

98. $(2\sqrt{7} + 3)(2\sqrt{7} - 3)$

99. $(3\sqrt{z} - 2)(4\sqrt{z} + 3)$

100. $(4\sqrt{a} - \sqrt{b})(3\sqrt{a} + 2\sqrt{b})$

101. $(\sqrt{x} + 2)^2$

102. $(3\sqrt{5y} - 4)^2$

103. $(\sqrt{x - 3} + 2)^2$

104. $(\sqrt{2x + 1} - 3)^2$

In Exercises 105 to 126, simplify each expression by rationalizing the denominator. Write the result in simplest form. Assume $x > 0$ and $y > 0$.

105. $\frac{2}{\sqrt{2}}$

106. $\frac{3x}{\sqrt{3}}$

107. $\sqrt{\frac{5}{18}}$

108. $\sqrt{\frac{7}{40}}$

109. $\frac{3}{\sqrt[3]{2}}$

110. $\frac{2}{\sqrt[3]{4}}$

111. $\frac{4}{\sqrt[3]{8x^2}}$

112. $\frac{2}{\sqrt[4]{4y}}$

113. $\frac{3}{\sqrt{3} + 4}$

114. $\frac{2}{\sqrt{5} - 2}$

115. $\frac{6}{2\sqrt{5} + 2}$

116. $\frac{-7}{3\sqrt{2} - 5}$

117. $\dfrac{3 + 2\sqrt{5}}{5 - 3\sqrt{5}}$

118. $\dfrac{6 - 3\sqrt{2}}{5 - \sqrt{2}}$

119. $\dfrac{6\sqrt{3} - 11}{4\sqrt{3} - 7}$

120. $\dfrac{2\sqrt{7} + 8}{12\sqrt{7} - 6}$

121. $\dfrac{2 + \sqrt{x}}{3 - 2\sqrt{x}}$

122. $\dfrac{4 - 2\sqrt{x}}{5 + 3\sqrt{x}}$

123. $\dfrac{x - \sqrt{5}}{x + 2\sqrt{5}}$

124. $\dfrac{x + 3\sqrt{7}}{x + 2\sqrt{7}}$

125. $\dfrac{3}{\sqrt{5} + \sqrt{x}}$

126. $\dfrac{5}{\sqrt{y} - \sqrt{3}}$

127. **Weight of an Orchid Seed** An orchid seed weighs approximately 3.2×10^{-8} ounce. If a package of seeds contains 1 ounce of orchid seeds, how many seeds are in the package?

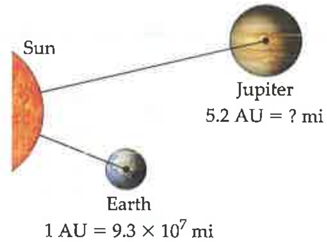
128. **Biology** The weight of one *E. coli* bacterium is approximately 670 femtograms, where 1 femtogram $= 1 \times 10^{-15}$ gram. If one *E. coli* bacterium can divide into two bacteria every 20 minutes, then after 24 hours there would be (assuming all bacteria survived) approximately 4.7×10^{21} bacteria. What is the weight, in grams, of these bacteria?

129. **Doppler Effect** Astronomers can approximate the distance to a galaxy by measuring its *red shift,* which is a shift in the wavelength of light due to the velocity of the galaxy. This is similar to the way the sound of a siren coming toward you seems to have a higher pitch than the sound of the siren moving away from you. A formula for red shift is $\dfrac{\lambda_r - \lambda_s}{\lambda_s}$, where λ_r and λ_s are wavelengths of a certain frequency of light. Calculate the red shift for a galaxy for which $\lambda_r = 5.13 \times 10^{-7}$ meter and $\lambda_s = 5.06 \times 10^{-7}$ meter.

130. **Laser Wavelength** The wavelength of a certain helium-neon laser is 800 nanometers. (1 nanometer is 1×10^{-9} meter.) The frequency, in cycles per second, of this wave is $\dfrac{1}{\text{wavelength}}$. What is the frequency of this laser?

131. **Astronomy** The Sun is approximately 1.44×10^{11} meters from Earth. If light travels 3×10^{8} meters per second, how many minutes does it take light from the sun to reach Earth?

132. **Astronomical Unit** Earth's mean distance from the Sun is 9.3×10^{7} miles. This distance is called the *astronomical unit* (AU). Jupiter is 5.2 AU from the Sun. Find the distance in miles from Jupiter to the Sun.



133. **Medicine** *Body surface area* (BSA) is a measure of the surface area of an adult human. A calculation of this number is important in prescribing medications for patients. One formula given by E. A. Gehan and S. L. George is $\text{BSA} = 0.0235h^{0.3964} \cdot w^{0.51456}$, where BSA is measured in meter^2, h is the height of a person in centimeters, and w is the weight of a person in kilograms. Find the BSA of a person who is 178 cm tall and weighs 73 kg. Round to the nearest hundredth.

134. **Drug Potency** The amount A (in milligrams) of digoxin, a drug taken by cardiac patients, remaining in the blood t hours after a patient takes a 2-milligram dose is given by $A = 2(10^{-0.0078t})$.

a. How much digoxin remains in the blood of a patient 4 hours after taking a 2-milligram dose?

b. Suppose that a patient takes a 2-milligram dose of digoxin at 1:00 P.M. and another 2-milligram dose at 5:00 P.M. How much digoxin remains in the patient's blood at 6:00 P.M.?

135. **Oceanography** The percent P of light that will pass to a depth d, in meters, at a certain place in the ocean is given by $P = 10^{2-(d/40)}$. Find, to the nearest percent, the amount of light that will pass to a depth of **a.** 10 meters and **b.** 25 meters below the surface of the ocean.

136. **Learning Theory** In a psychology experiment, students were given a nine-digit number to memorize. The percent P of students who remembered the number t minutes after it was read to them can be given by $P = 90 - 3t^{2/3}$. What percent of the students remembered the number after 1 hour?

Enrichment Exercises

In Exercises 137 to 140, rationalize the numerator, a technique that is occasionally used in calculus. For Exercises 137 and 138, begin by writing the expression with a 1 in the denominator.

137. $\sqrt{n^2 + 4} - n$

138. $\sqrt{n^2 + 3n} - n$

139. $\dfrac{\sqrt{4 + h} - 2}{h}$

140. $\dfrac{\sqrt{9 + h} - 3}{h}$