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CHAPTER P Preliminary Concepts

Sabermetrics

The film *Moneyball* was based on the book *Moneyball: The Art of Winning an Unfair Game* by Michael Lewis. It recounts the true story of how the Oakland Athletics baseball team used mathematics to select players for its team. They used what has become known as **sabermetrics,** introduced by Bill James, to objectively evaluate a player's performance using mathematics.

> Bill James defined sabermetrics as "the search for objective knowledge about baseball." Thus, sabermetrics attempts to answer objective questions about baseball, such as "which player on the Red Sox contributed the most to the team's offense?" or "How many home runs will Miguel Cabrera hit next year?" It cannot deal with the subjective judgments which are also important to the game, such as "Who is your favorite player?" or "That was a great game."[1]

In sabermetrics, a SLOB is not a bad thing. Instead, a SLOB is one of the measures of a player's performance. SLOB stands for "slugging times on base average." A SLOB value of 0.3 is considered very good. For instance, Lou Gehrig had a SLOB value of 0.283. Many of the sabermetric measures are based on ratios such as the expressions given in Exercises 129 and 130 on page 16.

[1] David J. Grabiner, "The Sabermetric Manifesto," *The Baseball Archive*. Available online at http://remarque.org/~grabiner/manifesto.txt

SECTION P.1

Sets
Union and Intersection of Sets
Interval Notation
Absolute Value and Distance
Exponential Expressions
Order of Operations Agreement
Simplifying Variable Expressions

The Real Number System

Sets

Human beings share the desire to organize and classify. Astronomers classify stars by such characteristics as color, mass, size, temperature, and distance from Earth. Mathematicians likewise place objects with similar properties in *sets*. A **set** is a collection of objects. The objects are called **elements** of the set. Sets are denoted by placing braces around the elements in the set.

The numbers that we use to count things, such as the number of books in a library or the number of songs in a music collection, are called the **natural numbers.**

$$\textbf{Natural numbers} = \{1, 2, 3, 4, 5, 6, \ldots\}$$

Each natural number greater than 1 is a *prime* number or a *composite* number. A **prime number** is a natural number greater than 1 that is divisible (evenly) only by itself and 1. For example, 2, 3, 5, 7, 11, and 13 are the first six prime numbers. A natural number, other than 1, that is not a prime number is a **composite number.** The numbers 4, 6, 8, and 9 are the first four composite numbers. Note that each of these numbers is divisible by a number other than itself and 1. For instance, 8 is divisible by 1, 2, 4, and 8.

The whole numbers include zero and the natural numbers.

$$\textbf{Whole numbers} = \{0, 1, 2, 3, 4, 5, 6, \ldots\}$$

We also need numbers to measure temperature below zero or, in accounting, when a company incurs a loss.

$$\textbf{Integers} = \{\ldots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots\}$$

The integers $\ldots, -6, -5, -4, -3, -2, -1$ are **negative integers.** The integers 1, 2, 3, 4, 5, 6, . . . are **positive integers** (or natural numbers). The integer 0 is neither a positive nor a negative integer.

Still other numbers are needed to designate part of a whole, such as a screw that is three-fourths inch long.

$$\textbf{Rational numbers} = \left\{\frac{p}{q}, \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\right\}$$

The numbers $\frac{3}{4}$, $-\frac{9}{2}$, and $\frac{7}{1}$ are examples of rational numbers. Note that $\frac{7}{1} = 7$. Because any integer n can be written with a denominator of $1\left(n = \frac{n}{1}\right)$, **all integers are rational numbers.**

A rational number written as a fraction can be written as a decimal by dividing the numerator by the denominator. As shown below, the result is either a **terminating decimal** such as 0.45 or a **repeating decimal** such as 0.218181818 . . . , where the digits 18 are continually repeated. In this case, we frequently place a bar over the repeating digits and write $0.2181818\ldots = 0.2\overline{18}$.

$$\begin{array}{r} 0.45 \\ 20\overline{)\,9.00} \\ -8\,0 \\ \hline 1\,00 \\ -1\,00 \\ \hline 0 \end{array}$$

This is a terminating decimal. The remainder is zero.

$$\frac{9}{20} = 0.45$$

$$\begin{array}{r} 0.21818 \\ 55\overline{)\,12.00000} \\ -11\,0 \\ \hline 1\,00 \\ -55 \\ \hline 450 \\ -440 \\ \hline 100 \\ -55 \\ \hline 450 \\ -440 \\ \hline 10 \end{array}$$

This is a repeating decimal. Note that the remainders 10 and 45 are repeating. The remainder is never zero.

$$\frac{12}{55} = 0.2\overline{18}$$

Math Matters

Archimedes (c. 287–212 B.C.E.) was the first to calculate π with any degree of precision. He was able to show that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

from which we get the approximation

$$3\frac{1}{7} = \frac{22}{7} \approx \pi$$

The use of the symbol π for this quantity was introduced by Leonhard Euler (1707–1783) in 1739, approximately 2000 years after Archimedes.

Math Matters

Sophie Germain (1776–1831) was born in Paris, France. Because enrollment in the university she wanted to attend was available only to men, Germain attended under the name of Antoine-August Le Blanc. Eventually her ruse was discovered, but not before she came to the attention of Pierre Lagrange, one of the best mathematicians of the time. He encouraged her work and became a mentor to her. A certain type of prime number is named after her, called a *Germain prime number.* It is a number p such that p and $2p + 1$ are both prime. For instance, 11 is a Germain prime because $2(11) + 1 = 23$, and 11 and 23 are both prime numbers. Germain primes are used in public key cryptography, a method used to send secure communications over the Internet.

Numbers that are not rational numbers are called **irrational numbers.** In decimal form, an irrational number has a decimal representation that never terminates nor repeats. One of the best known irrational numbers is pi, denoted by the Greek letter π. An approximate value of π is 3.14592654. . . . Other examples of irrational numbers are 2.13113111311113. . . and the square root of any prime number such as $\sqrt{11} \approx 3.31662479$. . . . The rational numbers and irrational numbers taken together are the **real numbers.**

The relationships among the various sets of numbers are shown in Figure P.1.

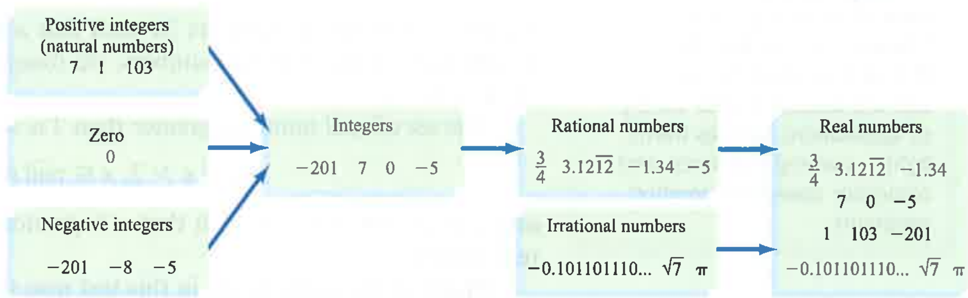

Figure P.1

EXAMPLE 1 Classify Real Numbers

Determine which of the following numbers are

a. integers **b.** rational numbers **c.** irrational numbers

d. real numbers **e.** prime numbers **f.** composite numbers

$$-0.2, \quad 0, \quad 0.\overline{3}, \quad 0.71771777177771\ldots, \quad \pi, \quad 6, \quad 7, \quad 41, \quad 51$$

Solution

a. Integers: 0, 6, 7, 41, 51

b. Rational numbers: $-0.2, 0, 0.\overline{3}, 6, 7, 41, 51$

c. Irrational numbers: $0.71771777177771\ldots, \pi$

d. Real numbers: $-0.2, 0, 0.\overline{3}, 0.71771777177771\ldots, \pi, 6, 7, 41, 51$

e. Prime numbers: 7, 41

f. Composite numbers: 6, 51

Try Exercise 8, page 14

Each member of a set is called an **element** of the set. For instance, if $C = \{2, 3, 5\}$, then the elements of C are 2, 3, and 5. The notation $2 \in C$ is read "2 is an element of C." Set A is a **subset** of set B if every element of A is also an element of B, and we write $A \subseteq B$. For instance, the set of negative integers $\{-1, -2, -3, -4, \ldots\}$ is a subset of the set of integers. The set of positive integers $\{1, 2, 3, 4, \ldots\}$ (the natural numbers) is also a subset of the set of integers.

Question • Are the integers a subset of the rational numbers?

The **empty set**, or **null set**, is the set that contains no elements. The symbol $\varnothing$ is used to represent the empty set. The set of people who have run a 2-minute mile is the empty set.

Note

The order of the elements of a set is not important. For instance, the set of natural numbers less than 6 given at the right could have been written {3, 5, 2, 1, 4}. It is customary, however, to list elements of a set in numerical order.

Answer • Yes.

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The set of natural numbers less than 6 is $\{1, 2, 3, 4, 5\}$. This is an example of a **finite set**; all the elements of the set can be listed. The set of all natural numbers is an example of an **infinite set**. There is no largest natural number, so all the elements of the set of natural numbers cannot be listed.

Sets are often written using **set-builder notation**. Set-builder notation can be used to describe almost any set, but it is especially useful when writing infinite sets. For instance, the set

$$\{2n \mid n \in \text{natural numbers}\}$$

is read as "the set of elements $2n$ such that n is a natural number." By replacing n with each of the natural numbers, we obtain the set of positive even integers: $\{2, 4, 6, 8, \ldots\}$.

The set of real numbers greater than 2 is written

$$\{x \mid x > 2, x \in \text{real numbers}\}$$

and is read "the set of x such that x is greater than 2 and x is an element of the real numbers."

Much of the work we do in this text uses the real numbers. With this in mind, we will frequently write, for instance, $\{x \mid x > 2, x \in \text{real numbers}\}$ in a shortened form as $\{x \mid x > 2\}$, where we assume that x is a real number.

Math Matters

A **fuzzy set** is one in which each element is given a "degree" of membership. The concepts behind fuzzy sets are used in a wide variety of applications such as traffic lights, washing machines, and computer speech recognition programs.

EXAMPLE 2 Use Set-Builder Notation

List the four smallest elements in $\{n^3 \mid n \in \text{natural numbers}\}$.

Solution

Because we want the four *smallest* elements, we choose the four smallest natural numbers. Thus $n = 1, 2, 3$, and 4. Therefore, the four smallest elements of the set $\{n^3 \mid n \in \text{natural numbers}\}$ are 1, 8, 27, and 64.

▶ Try Exercise 12, page 14

Union and Intersection of Sets

Just as operations such as addition and multiplication are performed on real numbers, operations are performed on sets. Two operations performed on sets are union and intersection. The union of two sets A and B is the set of elements that belong to A or to B, or to both A and B.

Definition of the Union of Two Sets

The **union** of two sets, written $A \cup B$, is the set of all elements that belong to either A or B. In set-builder notation, this is written

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

EXAMPLE

Given $A = \{2, 3, 4, 5\}$ and $B = \{0, 1, 2, 3, 4\}$, find $A \cup B$.

$A \cup B = \{0, 1, 2, 3, 4, 5\}$ • Note that an element that belongs to both sets is listed only once.

The intersection of the two sets A and B is the set of elements that belong to both A and B.

Definition of the Intersection of Two Sets

The **intersection** of two sets, written $A \cap B$, is the set of all elements that are common to both A and B. In set-builder notation, this is written

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

EXAMPLE

Given $A = \{2, 3, 4, 5\}$ and $B = \{0, 1, 2, 3, 4\}$, find $A \cap B$.

$A \cap B = \{2, 3, 4\}$ • The intersection of two sets contains the elements common to both sets.

If the intersection of two sets is the empty set, the two sets are said to be **disjoint.** For example, if $A = \{2, 3, 4\}$ and $B = \{7, 8\}$, then $A \cap B = \varnothing$ and A and B are disjoint sets.

EXAMPLE 3 Find the Union and Intersection of Sets

Find each intersection or union given $A = \{0, 2, 4, 6, 10, 12\}$, $B = \{0, 3, 6, 12, 15\}$, and $C = \{1, 2, 3, 4, 5, 6, 7\}$.

a. $A \cup C$ **b.** $B \cap C$ **c.** $A \cap (B \cup C)$ **d.** $B \cup (A \cap C)$

Solution

a. $A \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 10, 12\}$ • The elements that belong to A or C

b. $B \cap C = \{3, 6\}$ • The elements that belong to B and C

c. First, determine $B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 12, 15\}$. Then
$A \cap (B \cup C) = \{0, 2, 4, 6, 12\}$ • The elements that belong to A and $(B \cup C)$

d. First, determine $A \cap C = \{2, 4, 6\}$. Then
$B \cup (A \cap C) = \{0, 2, 3, 4, 6, 12, 15\}$ • The elements that belong to B or $(A \cap C)$

Try Exercise 22, page 14

Interval Notation

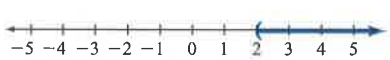

Figure P.2

The graph of $\{x \mid x > 2\}$ is shown in Figure P.2. The set is the real numbers greater than 2. The parenthesis at 2 indicates that 2 is not included in the set. Rather than write this set of real numbers using set-builder notation, we can write the set in **interval notation** as $(2, \infty)$.

In general, the interval notation

(a, b) represents all real numbers between a and b, not including a and b. This is an **open interval.** In set-builder notation, we write $\{x \mid a < x < b\}$. The graph of $(-4, 2)$ is shown in Figure P.3.

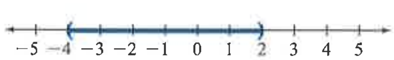

Figure P.3

$[a, b]$ represents all real numbers between a and b, including a and b. This is a **closed interval.** In set-builder notation, we write $\{x \mid a \le x \le b\}$. The graph of $[0, 4]$ is shown in Figure P.4. The brackets at 0 and 4 indicate that those numbers are included in the graph.

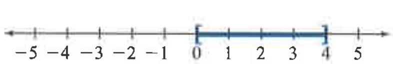

Figure P.4

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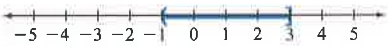


Figure P.5

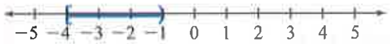


Figure P.6

$(a, b]$ represents all real numbers between a and b, not including a but including b. This is a **half-open interval.** In set-builder notation, we write $\{x \mid a < x \leq b\}$. The graph of $(-1, 3]$ is shown in Figure P.5.

$[a, b)$ represents all real numbers between a and b, including a but not including b. This is a **half-open interval.** In set-builder notation, we write $\{x \mid a \leq x < b\}$. The graph of $[-4, -1)$ is shown in Figure P.6.

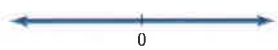
Subsets of the real numbers whose graphs extend forever in one or both directions can be represented by interval notation using the **infinity symbol** ∞ or the **negative infinity symbol** $-\infty$.

$(-\infty, a)$ represents all real numbers less than a.

(b, ∞) represents all real numbers greater than b.

$(-\infty, a]$ represents all real numbers less than or equal to a.

$[b, \infty)$ represents all real numbers greater than or equal to b.



$(-\infty, \infty)$ represents all real numbers.

EXAMPLE 4 Graph a Set Given in Interval Notation

Graph $(-\infty, 3]$. Write the interval in set-builder notation.

Solution

The set is the real numbers less than or equal to 3. In set-builder notation, this is the set $\{x \mid x \leq 3\}$. Draw a right bracket at 3, and darken the number line to the left of 3, as shown in Figure P.7.

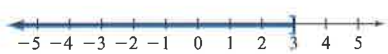


Figure P.7

▸ **Try Exercise 40, page 14**

> ***Caution***
>
> It is *never* correct to use a bracket when using the infinity symbol. For instance, $[-\infty, 3]$ is not correct. Nor is $[2, \infty]$ correct. Neither negative infinity nor positive infinity is a real number and therefore cannot be contained in a closed interval.

The set $\{x \mid x \leq -2\} \cup \{x \mid x > 3\}$ is the set of real numbers that are either less than or equal to -2 *or* greater than 3. We also could write this in interval notation as $(-\infty, -2] \cup (3, \infty)$. The graph of the set is shown in Figure P.8.

The set $\{x \mid x > -4\} \cap \{x \mid x < 1\}$ is the set of real numbers that are greater than -4 *and* less than 1. Note from Figure P.9 that this set is the interval $(-4, 1)$, which can be written in set-builder notation as $\{x \mid -4 < x < 1\}$.

Figure P.8

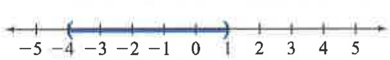


Figure P.9

EXAMPLE 5 Graph Intervals

Graph the following. Write **a.** and **b.** using interval notation. Write **c.** and **d.** using set-builder notation.

a. $\{x \mid x \leq -1\} \cup \{x \mid x \geq 2\}$ **b.** $\{x \mid x \geq -1\} \cap \{x \mid x < 5\}$

c. $(-\infty, 0) \cup [1, 3]$ **d.** $[-1, 3] \cap (1, 5)$

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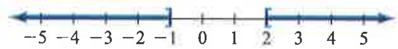
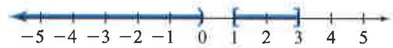
Solution

a. 

$(-\infty, -1] \cup [2, \infty)$

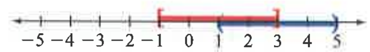
b. 

$[-1, 5)$

c. 

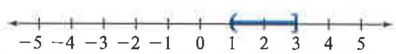
$\{x \mid x < 0\} \cup \{x \mid 1 \le x \le 3\}$

d. The graphs of $[-1, 3]$, in red, and $(1, 5)$, in blue, are shown below.



Note that the intersection of the sets occurs where the graphs intersect. Although $1 \in [-1, 3]$, $1 \notin (1, 5)$. Therefore, 1 does not belong to the intersection of the sets. On the other hand, $3 \in [-1, 3]$ and $3 \in (1, 5)$. Therefore, 3 belongs to the intersection of the sets.

Thus we have the following.



$\{x \mid 1 < x \le 3\}$

Try Exercise 50, page 14

Absolute Value and Distance

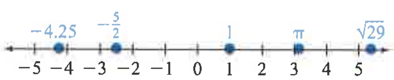


Figure P.10

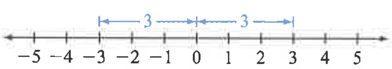


Figure P.11

The real numbers can be represented geometrically by a **coordinate axis** called a **real number line**. Figure P.10 shows a portion of a real number line. The number associated with a point on a real number line is called the **coordinate** of the point. The point corresponding to zero is called the **origin**. Every real number corresponds to a point on the number line, and every point on the number line corresponds to a real number.

The *absolute value* of a real number a, denoted $|a|$, is the distance between a and 0 on the number line. For instance, $|3| = 3$ and $|-3| = 3$ because both 3 and -3 are 3 units from zero. See Figure P.11.

In general, if $a \ge 0$, then $|a| = a$; however, if $a < 0$, then $|a| = -a$ because $-a$ is positive when $a < 0$. This leads to the following definition.

Note

The second part of the definition of absolute value states that if $a < 0$, then $|a| = -a$. For instance, if $a = -4$, then $|a| = |-4| = -(-4) = 4$.

Definition of Absolute Value

The **absolute value** of the real number a is defined by

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

EXAMPLE

$|5| = 5$ $\quad$ $|-4| = 4$ $\quad$ $|0| = 0$

EXAMPLE 6 Simplify an Absolute Value Expression

Simplify $|x + 4| - |2x - 6|$ given that $-3 \le x \le 2$.

Solution

Recall that $|a| = -a$ when $a < 0$ and $|a| = a$ when $a \ge 0$.

(continued)

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When $-3 \le x \le 2$, $x + 4 > 0$ and $2x - 6 < 0$. Therefore, $|x + 4| = x + 4$ and

$|2x - 6| = -(2x - 6)$. Thus

$$\begin{aligned} |x + 4| - |2x - 6| &= (x + 4) - [-(2x - 6)] \\ &= (x + 4) + (2x - 6) \\ &= 3x - 2 \end{aligned}$$

▶ **Try Exercise 60, page 14**

The definition of *distance* between two points on a real number line makes use of absolute value.

Definition of the Distance Between Points on a Real Number Line

If a and b are the coordinates of two points on a real number line, the **distance** between the graph of a and the graph of b, denoted by $d(a, b)$, is given by $d(a, b) = |a - b|$.

EXAMPLE

Find the distance between the point whose coordinate on the real number line is -2 and the point whose coordinate is 5.

$$d(-2, 5) = |-2 - 5| = |-7| = 7$$

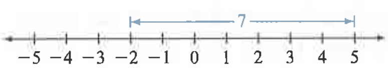

Figure P.12

Note in Figure P.12 that there are 7 units between -2 and 5. Also note that the order of the coordinates in the formula does not matter.

$$d(5, -2) = |5 - (-2)| = |7| = 7$$

EXAMPLE 7 Use Absolute Value to Express the Distance Between Two Points

Express the distance between a and -3 on the number line using absolute value notation.

Solution

$$d(a, -3) = |a - (-3)| = |a + 3|$$

▶ **Try Exercise 70, page 15**

> ***Math Matters***
>
> The expression 10^{100} is called a *googol*. The term was coined by the 9-year-old nephew of the American mathematician Edward Kasner. Many calculators do not provide for numbers of this magnitude, but it is no serious loss. To appreciate the magnitude of a googol, consider that if all the atoms in the known universe were counted, the number would not even be close to a googol. But if a googol is too small for you, try 10^{googol}, which is called a *googolplex*. As a final note, the name of the Internet site Google.com is a takeoff on the word *googol*.

Exponential Expressions

A compact method of writing $5 \cdot 5 \cdot 5 \cdot 5$ is 5^4. The expression 5^4 is written in **exponential notation.** Similarly, we can write

$$\frac{2x}{3} \cdot \frac{2x}{3} \cdot \frac{2x}{3} \quad \text{as} \quad \left(\frac{2x}{3}\right)^3$$

Exponential notation can be used to express the product of any expression that is used repeatedly as a factor.

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Definition of Natural Number Exponents

If b is any real number and n is a natural number, then

$$b^n = \overbrace{b \cdot b \cdot b \cdot \cdots \cdot b}^{b \text{ is a factor } n \text{ times}}$$

where b is the **base** and n is the **exponent**.

EXAMPLE

$$\left(\frac{3}{4}\right)^3 = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$$

$$-5^4 = -(5 \cdot 5 \cdot 5 \cdot 5) = -625$$

$$(-5)^4 = (-5)(-5)(-5)(-5) = 625$$

Pay close attention to the difference between -5^4 (the base is 5) and $(-5)^4$ (the base is -5).

EXAMPLE 8 Evaluate an Exponential Expression

Evaluate.

a. $(-3^4)(-4)^2$ **b.** $\dfrac{-4^4}{(-4)^4}$

Solution

a. $(-3^4)(-4)^2 = -(3 \cdot 3 \cdot 3 \cdot 3) \cdot (-4)(-4) = -81 \cdot 16 = -1296$

b. $\dfrac{-4^4}{(-4)^4} = \dfrac{-(4 \cdot 4 \cdot 4 \cdot 4)}{(-4)(-4)(-4)(-4)} = \dfrac{-256}{256} = -1$

Try Exercise 76, page 15

Order of Operations Agreement

The approximate pressure p, in pounds per square inch, on a scuba diver x feet below the water's surface is given by

$$p = 15 + 0.5x$$

The pressure on the diver at various depths is given below.

10 feet	$15 + 0.5(10) = 15 + 5 = 20$ pounds
20 feet	$15 + 0.5(20) = 15 + 10 = 25$ pounds
40 feet	$15 + 0.5(40) = 15 + 20 = 35$ pounds
70 feet	$15 + 0.5(70) = 15 + 35 = 50$ pounds

Note that the expression $15 + 0.5(70)$ has two operations, addition and multiplication. When an expression contains more than one operation, the operations must be performed in a specified order, as given by the Order of Operations Agreement.

The Order of Operations Agreement

If grouping symbols are present, evaluate by first performing the operations within the grouping symbols, innermost grouping symbols first, while observing the order given in steps 1 to 3.

Step 1 Evaluate exponential expressions.
Step 2 Do multiplication and division as they occur from left to right.
Step 3 Do addition and subtraction as they occur from left to right.

EXAMPLE

$5 - 7(23 - 5^2) - 16 \div 2^3$

$= 5 - 7(23 - 25) - 16 \div 2^3$ • Begin inside the parentheses and evaluate $5^2 = 25$.

$= 5 - 7(-2) - 16 \div 2^3$ • Continue inside the parentheses and evaluate $23 - 25 = -2$.

$= 5 - 7(-2) - 16 \div 8$ • Evaluate $2^3 = 8$.

$= 5 - (-14) - 2$ • Perform multiplication and division from left to right.

$= 17$ • Perform addition and subtraction from left to right.

EXAMPLE 9 Use the Order of Operations Agreement

Evaluate: $3 \cdot 5^2 - 6(-3^2 - 4^2) \div (-15)$

Solution

$3 \cdot 5^2 - 6(-3^2 - 4^2) \div (-15)$

$= 3 \cdot 5^2 - 6(-9 - 16) \div (-15)$ • Begin inside the parentheses.

$= 3 \cdot 5^2 - 6(-25) \div (-15)$ • Simplify $-9 - 16$.

$= 3 \cdot 25 - 6(-25) \div (-15)$ • Evaluate 5^2.

$= 75 + 150 \div (-15)$ • Do muliplitcation and division from left to right.

$= 75 + (-10)$

$= 65$ • Do addition.

▸ Try Exercise 80, page 15

Recall

Subtraction can be rewritten as addition of the opposite. Therefore,
$3x^2 - 4xy + 5x - y - 7$
$= 3x^2 + (-4xy) + 5x + (-y) + (-7)$
In this form, we can see that the terms (addends) are $3x^2$, $-4xy$, $5x$, $-y$, and -7.

$$3x^2 - 4xy + 5x - y - 7$$

One of the ways in which the Order of Operations Agreement is used is to evaluate variable expressions. The addends of a variable expression are called **terms**. The terms for the expression at the right are $3x^2$, $-4xy$, $5x$, $-y$, and -7. Observe that the sign of a term is the sign that immediately precedes it.

The terms $3x^2$, $-4xy$, $5x$, and $-y$ are **variable terms**. The term -7 is a **constant term**. Each variable term has a **numerical coefficient** and a **variable part**. The numerical coefficient for the term $3x^2$ is 3; the numerical coefficient for the term $-4xy$ is -4; the numerical coefficient for the term $5x$ is 5; and the numerical coefficient for the term $-y$ is -1. When the numerical coefficient is 1 or -1 (as in x and $-x$), the 1 is usually not written.

To **evaluate** a variable expression, replace the variables by their given values and then use the Order of Operations Agreement to simplify the result.

EXAMPLE 10 Evaluate a Variable Expression

a. Evaluate $\dfrac{x^3 - y^3}{x^2 + xy + y^2}$ when $x = 2$ and $y = -3$.

b. Evaluate $(x + 2y)^2 - 4z$ when $x = 3$, $y = -2$, and $z = -4$.

Solution

a. $\dfrac{x^3 - y^3}{x^2 + xy + y^2}$

$$\frac{2^3 - (-3)^3}{2^2 + 2(-3) + (-3)^2} = \frac{8 - (-27)}{4 - 6 + 9} = \frac{35}{7} = 5$$

b. $(x + 2y)^2 - 4z$

$$\begin{aligned}[3 + 2(-2)]^2 - 4(-4) &= [3 + (-4)]^2 - 4(-4) \\ &= (-1)^2 - 4(-4) \\ &= 1 - 4(-4) \\ &= 1 + 16 = 17\end{aligned}$$

Try Exercise 90, page 15

Simplifying Variable Expressions

Addition, multiplication, subtraction, and *division* are the operations of arithmetic. **Addition** of the two real numbers a and b is designated by $a + b$. If $a + b = c$, then c is the **sum** and the real numbers a and b are called **terms.**

Multiplication of the real numbers a and b is designated by ab or $a \cdot b$. If $ab = c$, then c is the **product** and the real numbers a and b are called **factors** of c.

The number $-b$ is referred to as the **additive inverse** of b. **Subtraction** of the real numbers a and b is designated by $a - b$ and is defined as the sum of a and the additive inverse of b. That is,

$$a - b = a + (-b)$$

If $a - b = c$, then c is called the **difference** of a and b.

The **multiplicative inverse** or **reciprocal** of the nonzero number b is $1/b$. The **division** of a and b, designated by $a \div b$ with $b \neq 0$, is defined as the product of a and the reciprocal of b. That is,

$$a \div b = a\left(\frac{1}{b}\right) \quad \text{provided that } b \neq 0$$

If $a \div b = c$, then c is called the **quotient** of a and b.

The notation $a \div b$ is often represented by the fractional notation a/b or $\dfrac{a}{b}$. The real number a is the **numerator,** and the nonzero real number b is the **denominator** of the fraction.

Properties of Real Numbers

Let a, b, and c be real numbers.

	Addition Properties	Multiplication Properties
Closure	$a + b$ is a unique real number.	ab is a unique real number.
Commutative	$a + b = b + a$	$ab = ba$

(continued)

	Addition Properties	Multiplication Properties
Associative	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Identity	There exists a unique real number 0 such that $a + 0 = 0 + a = a$.	There exists a unique real number 1 such that $a \cdot 1 = 1 \cdot a = a$.
Inverse	For each real number a, there is a unique real number $-a$ such that $a + (-a) = (-a) + a = 0$.	For each *nonzero* real number a, there is a unique real number $1/a$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.
Distributive	$a(b + c) = ab + ac$	

EXAMPLE 11 Identify Properties of Real Numbers

Identify the property of real numbers illustrated in each statement.

a. $(2a)b = 2(ab)$ **b.** $\left(\frac{1}{5}\right)11$ is a real number.

c. $4(x + 3) = 4x + 12$ **d.** $(a + 5b) + 7c = (5b + a) + 7c$

e. $\left(\frac{1}{2} \cdot 2\right)a = 1 \cdot a$ **f.** $1 \cdot a = a$

Solution

a. Associative property of multiplication

b. Closure property of multiplication

c. Distributive property

d. Commutative property of addition

e. Inverse property of multiplication

f. Identity property of multiplication

▶ Try Exercise 102, page 15

Note

Normally, we will not show, as we did at the right, all the steps involved in the simplification of a variable expression. For instance, we will just write $(6x)2 = 12x$, $3(4p + 5) = 12p + 15$, and $3x^2 + 9x^2 = 12x^2$. It is important to know, however, that every step in the simplification process depends on one of the properties of real numbers.

We can identify which properties of real numbers have been used to rewrite an expression by closely comparing the original and final expressions and noting any changes. For instance, to simplify $(6x)2$, both the commutative property and associative property of multiplication are used.

$$\begin{aligned} (6x)2 &= 2(6x) && \text{• Commutative property of multiplication} \\ &= (2 \cdot 6)x && \text{• Associative property of multiplication} \\ &= 12x \end{aligned}$$

To simplify $3(4p + 5)$, use the distributive property.

$$\begin{aligned} 3(4p + 5) &= 3(4p) + 3(5) && \text{• Distributive property} \\ &= 12p + 15 \end{aligned}$$

Terms that have the same variable part are called **like terms**. The distributive property is also used to simplify an expression with like terms such as $3x^2 + 9x^2$.

$$\begin{aligned} 3x^2 + 9x^2 &= (3 + 9)x^2 && \text{• Distributive property} \\ &= 12x^2 \end{aligned}$$

Note from this example that like terms are combined by adding the coefficients of the like terms.

Question • Are the terms $2x^2$ and $3x$ like terms?

EXAMPLE 12 Simplify Variable Expressions

Simplify.

a. $5 + 3(2x - 6)$

b. $4x - 2[7 - 5(2x - 3)]$

Solution

a. $5 + 3(2x - 6) = 5 + 6x - 18$ • Use the distributive property.

$= 6x - 13$ • Add the constant terms.

b. $4x - 2[7 - 5(2x - 3)]$

$= 4x - 2[7 - 10x + 15]$ • Use the distributive property to remove the inner parentheses.

$= 4x - 2[-10x + 22]$ • Simplify.

$= 4x + 20x - 44$ • Use the distributive property to remove the brackets.

$= 24x - 44$ • Simplify.

▶ Try Exercise 120, page 15

An **equation** is a statement of equality between two numbers or two expressions. There are four basic properties of equality that relate to equations.

Properties of Equality

Let a, b, and c be real numbers.

Reflexive	$a = a$
Symmetric	If $a = b$, then $b = a$.
Transitive	If $a = b$ and $b = c$, then $a = c$.
Substitution	If $a = b$, then a may be replaced by b in any expression that involves a.

EXAMPLE 13 Identify Properties of Equality

Identify the property of equality illustrated in each statement.

a. If $3a + b = c$, then $c = 3a + b$.

b. $5(x + y) = 5(x + y)$

c. If $4a - 1 = 7b$ and $7b = 5c + 2$, then $4a - 1 = 5c + 2$.

d. If $a = 5$ and $b(a + c) = 72$, then $b(5 + c) = 72$.

Solution

a. Symmetric **b.** Reflexive **c.** Transitive **d.** Substitution

▶ Try Exercise 106, page 15

Answer • No. The variable parts are not the same. The variable part of $2x^2$ is $x \cdot x$. The variable part of $3x$ is x.

EXERCISE SET P.1

Concept Check

1. Which of the following numbers are prime numbers?
   i. 39 **ii.** 53 **iii.** 102 **iv.** 97

2. Give an example of a rational number that is not an integer.

3. If $A = \{-7, -3, 0, 2, 5, 8\}$ and $B = \{-3, -1, 0, 1, 3, 5, 7\}$, what numbers are common to both A and B?

4. Use the numbers $-12, -5, 0, 3, 6$, and 9.
   a. Which number has the greatest absolute value?
   b. Which number has the least absolute value?

5. If $a < 0$, is a^2 positive or negative?

6. **a.** Name the endpoints of the interval $[-2, 5)$.
   b. Is $0 \in [-2, 5)$?
   c. Is $-2 \in [-2, 5)$?
   d. Is $5 \in [-2, 5)$?

In Exercises 7 and 8, determine whether each number is an integer, a rational number, an irrational number, a prime number, or a real number.

7. $-\frac{1}{5}, 0, -44, \pi, 3.14, 5.050050005000005\ldots, \sqrt{81}, 53$

8. $\frac{5}{\sqrt{7}}, \frac{5}{7}, 31, -2\frac{1}{2}, 4.235653907493, 51, 0.888\ldots$

In Exercises 9 to 14, list the four smallest elements of each set.

9. $\{2x \mid x \in \text{positive integers}\}$

10. $\{|x| \mid x \in \text{integers}\}$

11. $\{y \mid y = 2x + 1, x \in \text{natural numbers}\}$

12. $\{y \mid y = x^2 - 1, x \in \text{integers}\}$

13. $\{z \mid z = |x|, x \in \text{integers}\}$

14. $\{z \mid z = |x| - x, x \in \text{negative integers}\}$

In Exercises 15 to 24, perform the operations given that $A = \{-3, -2, -1, 0, 1, 2, 3\}$, $B = \{-2, 0, 2, 4, 6\}$, $C = \{0, 1, 2, 3, 4, 5, 6\}$, and $D = \{-3, -1, 1, 3\}$.

15. $A \cup B$
16. $C \cup D$
17. $A \cap C$
18. $C \cap D$
19. $B \cap D$
20. $B \cup (A \cap C)$
21. $D \cap (B \cup C)$
22. $(A \cap B) \cup (A \cap C)$
23. $(B \cup C) \cap (B \cup D)$
24. $(A \cap C) \cup (B \cap D)$

Indicates Try It Exercises

In Exercises 25 to 36, graph each set. Write sets given in interval notation in set-builder notation, and write sets given in set-builder notation in interval notation.

25. $(-2, 3)$
26. $[1, 5]$
27. $[-5, -1]$
28. $(-3, 3)$
29. $[2, \infty)$
30. $(-\infty, 4)$
31. $\{x \mid 3 < x < 5\}$
32. $\{x \mid x < -1\}$
33. $\{x \mid x \geq -2\}$
34. $\{x \mid -1 \leq x < 5\}$
35. $\{x \mid 0 \leq x \leq 1\}$
36. $\{x \mid -4 < x \leq 5\}$

In Exercises 37 to 52, graph each set.

37. $(-\infty, 0) \cup [2, 4]$
38. $(-3, 1) \cup (3, 5)$
39. $(-4, 0) \cap [-2, 5]$
40. $(-\infty, 3] \cap (2, 6)$
41. $(1, \infty) \cup (-2, \infty)$
42. $(-4, \infty) \cup (0, \infty)$
43. $(1, \infty) \cap (-2, \infty)$
44. $(-4, \infty) \cap (0, \infty)$
45. $[-2, 4] \cap [4, 5]$
46. $(-\infty, 1] \cap [1, \infty)$
47. $(-2, 4) \cap (4, 5)$
48. $(-\infty, 1) \cap (1, \infty)$
49. $\{x \mid x < -3\} \cup \{x \mid 1 < x < 2\}$
50. $\{x \mid -3 \leq x < 0\} \cup \{x \mid x \geq 2\}$
51. $\{x \mid x < -3\} \cup \{x \mid x < 2\}$
52. $\{x \mid x < -3\} \cap \{x \mid x < 2\}$

In Exercises 53 to 62, write each expression without absolute value symbols.

53. $-|-5|$
54. $-|-4|^2$
55. $|3| \cdot |-4|$
56. $|3| - |-7|$
57. $|\pi^2 + 10|$
58. $|\pi^2 - 10|$
59. $|x - 4| + |x + 5|$, given $0 < x < 1$
60. $|x + 6| + |x - 2|$, given $0 < x < 2$
61. $|2x| - |x - 1|$, given $0 < x < 1$
62. $|x + 1| + |x - 3|$, given $x > 3$

In Exercises 63 to 74, use absolute value notation to describe the given situation.

63. $d(m, n)$

64. $d(p, 8)$

65. The distance between x and 3

66. The distance between a and -2

67. The distance between x and -2 is 4.

68. The distance between z and 5 is 1.

69. The distance between a and 4 is less than 5.

70. The distance between z and 5 is greater than 7.

71. The distance between x and -2 is greater than 4.

72. The distance between y and -3 is greater than 6.

73. The distance between x and 4 is greater than 0 and less than 1.

74. The distance between y and -3 is greater than 0 and less than 0.5.

In Exercises 75 to 82, evaluate the expression.

75. $-5^3(-4)^2$

76. $-\dfrac{-6^3}{(-3)^4}$

77. $4 + (3 - 8)^2$

78. $-2 \cdot 3^4 - (6 - 7)^6$

79. $28 \div (-7 + 5)^2$

80. $(3 - 5)^2(3^2 - 5^2)$

81. $7 + 2[3(-2)^3 - 4^2 \div 8]$

82. $5 - 4[3 - 6(2 \cdot 3^2 - 12 \div 4)]$

In Exercises 83 to 94, evaluate the variable expression for $x = 3$, $y = -2$, and $z = -1$.

83. $-y^3$

84. $-y^2$

85. $2xyz$

86. $-3xz$

87. $-2x^2y^2$

88. $2y^3z^2$

89. $xy - z(x - y)^2$

90. $(z - 2y)^2 - 3z^3$

91. $\dfrac{x^2 + y^2}{x + y}$

92. $\dfrac{2xy^2z^4}{(y - z)^4}$

93. $\dfrac{3y}{x} - \dfrac{2z}{y}$

94. $(x - z)^2(x + z)^2$

In Exercises 95 to 108, state the property of real numbers or the property of equality that is used.

95. $(ab^2)c = a(b^2c)$

96. $2x - 3y = -3y + 2x$

97. $4(2a - b) = 8a - 4b$

98. $6 + (7 + a) = 6 + (a + 7)$

99. $(3x)y = y(3x)$

100. $4ab + 0 = 4ab$

101. $1 \cdot (4x) = 4x$

102. $7(a + b) = 7(b + a)$

103. $x^2 + 1 = x^2 + 1$

104. If $a + b = 2$, then $2 = a + b$.

105. If $2x + 1 = y$ and $y = 3x - 2$, then $2x + 1 = 3x - 2$.

106. If $4x + 2y = 7$ and $x = 3$, then $4(3) + 2y = 7$.

107. $4 \cdot \dfrac{1}{4} = 1$

108. $ab + (-ab) = 0$

109. Is division of real numbers an associative operation? Give a reason for your answer.

110. Is subtraction of real numbers a commutative operation? Give a reason for your answer.

111. Which of the properties of real numbers are satisfied by the integers?

112. Which of the properties of real numbers are satisfied by the rational numbers?

In Exercises 113 to 122, simplify the variable expression.

113. $2 + 3(2x - 5)$

114. $4 + 2(2a - 3)$

115. $5 - 3(4x - 2y)$

116. $7 - 2(5n - 8m)$

117. $3(2a - 4b) - 4(a - 3b)$

118. $5(4r - 7t) - 2(10r + 3t)$

119. $5a - 2[3 - 2(4a + 3)]$

120. $6 + 3[2x - 4(3x - 2)]$

121. $\dfrac{3}{4}(5a + 2) - \dfrac{1}{2}(3a - 5)$

122. $-\dfrac{2}{5}(2x + 3) + \dfrac{3}{4}(3x - 7)$

123. Area of a Triangle The area of a triangle is given by

$$\text{Area} = \frac{1}{2}bh$$

where b is the base of the triangle and h is its height. Find the area of a triangle whose base is 3 inches and whose height is 4 inches.

124. Volume of a Box The volume of a rectangular box is given by

$$\text{Volume} = lwh$$

where l is the length, w is the width, and h is the height of the box. Find the volume of a classroom that has a length of 40 feet, a width of 30 feet, and a height of 12 feet.

125. Heart Rate The heart rate, in beats per minute, of a certain runner during a cool-down period can be approximated by

$$\text{Heart rate} = 65 + \frac{53}{4t + 1}$$

where t is the number of minutes after the start of cool-down. Find the runner's heart rate after 10 minutes. Round to the nearest natural number.

126. Body Mass Index According to the National Institutes of Health, body mass index (BMI) is a measure of body fat based on height and weight that applies to both adult men and women, with values between 18.5 and 24.9 considered healthy. BMI is calculated as $\text{BMI} = \frac{705w}{h^2}$, where w is the person's weight in pounds and h is the person's height in inches. Find the BMI for a person who weighs 160 pounds and is 5 feet 10 inches tall. Round to the nearest natural number.

127. Physics The height, in feet, of a ball t seconds after it is thrown upward is given by

$$\text{Height} = -16t^2 + 80t + 4$$

Find the height of the ball 2 seconds after it has been thrown upward.

128. Chemistry Salt is being added to water in such a way that the concentration of salt, in grams per liter, is given by $\text{concentration} = \frac{50t}{t + 1}$, where t is the time in minutes after the introduction of the salt. Find the concentration of salt after 24 minutes.

129. Sabermetrics *Slugging percentage* (SLG) is one of the measurements of a baseball player's performance. It is given by the ratio $\frac{\text{singles} + 2 \cdot 2B + 3 \cdot 3B + 4 \cdot 4B}{AB}$, where singles is the number of singles, $2B$ is the number of doubles, $3B$ is the number of triples, and $4B$ is the number of home runs hit by a player. The abbreviation AB is the number of at bats the player had. In 2011, Miguel Cabrera had 197 singles, 48 doubles, 0 triples, 30 home runs, and 572 at bats. Find his SLG. Round to the nearest thousandth.

130. Sabermetrics *Pythagorean expectation* is a formula that tries to determine how many games a team "should have" won during a season. It is based on the number of runs scored by a team in one season and the number of runs allowed by the team for the season. Pythagorean expectation is given by the ratio $\frac{(\text{runs scored})^2}{(\text{runs scored})^2 + (\text{runs allowed})^2}$. Multiplying this ratio by the number of games played in a season (162) gives the number of games the team "should have" won. In 2011, the Boston Red Sox won 90 games, scored 875 runs, and allowed 757 runs. According to the Pythagorean expectation, how many games should the Red Sox have won? Round to the nearest whole number.

Enrichment Exercises

In Exercises 131 and 132, let A and B be any two sets.

131. If $A \cap B = B$, what can be said about B?

132. If $A \cup B = B$, what can be said about A?

In Exercises 133 to 136, let A be any set. Perform the given operation.

133. $A \cup A$

134. $A \cap A$

135. $A \cup \varnothing$

136. $A \cap \varnothing$

137. If a and b are the coordinates of two points on a number line, give an example of a point whose coordinates are between a and b.

138. Define an operation denoted by $\oplus$ and given by $a \oplus b = a^2 + b^2$. Does $\oplus$ satisfy the commutative property? Does $\oplus$ satisfy the associative property?

139. A *deleted delta neighborhood* of a number a on a number line is the set of all points x that are within δ (the Greek letter delta) units of a but not including a. Write the deleted delta neighborhood of a using absolute value notation.

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