4.1 Graphing Polynomial Functions

Targets:

1. I can identify and evaluate polynomial functions.
2. I can graph polynomial functions.
3. I can describe end behavior of polynomial functions.

Explore It! Graphing Polynomial Functions

A. $f(x) = -x^2 - 1$

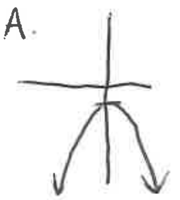
B. $f(x) = x^3 + 1$

C. $f(x) = -\frac{1}{4}x^4 - x^3$

D. $f(x) = \sqrt{x}$

1. Identify each function in the list at the left in which $f(x)$ is a polynomial. Graph each function you identified. For each function,
   a. describe the end behavior.
   b. Identify the term with the greatest exponent. How does the exponent affect the graph? How does the coefficient of the term affect the graph?

A.

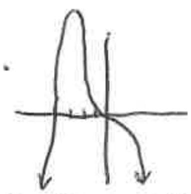


$y \to -\infty$ as $x \to -\infty$
$y \to -\infty$ as $x \to \infty$
Exp: parabola Coef: Reflection

B.

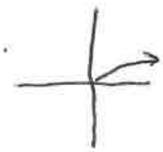
$y \to -\infty$ as $x \to -\infty$
$y \to \infty$ as $x \to \infty$
Ex: Cubic Coef: no effect

C.



$y \to -\infty$ as $x \to -\infty$
$y \to -\infty$ as $x \to \infty$
Ex: Quartic Coef: Ref, V. Shr

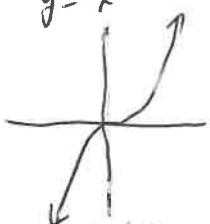
D.



$y = 0$ as $x = 0$
$y \to \infty$ as $x \to \infty$
Ex: Ø Coef: Ø

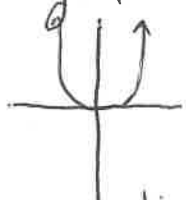
2. Use Desmos to graph $y = x^3$ and $y = x^4$. Compare the graphs. One of these graphs is *cubic* and the other is *quartic.* Which do you think is which? Explain.

$y = x^3$



Cubic

$y = x^4$

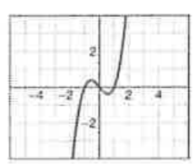


Quartic

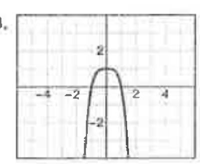
3. Identify each function as cubic or quartic. Then match each function with its graph. Explain your reasoning.

i. $f(x) = x^3 - x$ Cubic A
ii. $f(x) = -x^3 + x$ Cubic D
iii. $f(x) = -x^4 + 1$ Quart B
iv. $f(x) = x^4 - x^2$ Quart C

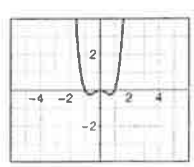
A.



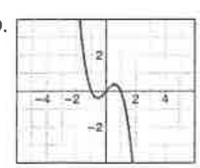
B.



C.



D.



4. What are some characteristics of the graphs of cubic polynomial functions? Quartic polynomial functions?

Cubic: Falls on one side and rises on the other and thru origin
Quartic: Either rises or falls on both sides and thru origin

Polynomial Functions

A **polynomial** is a monomial or sum a sum monomials. A **polynomial function** is a function of the form:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_n x + a_0$$

where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers. For this function, a_n, is the leading coefficient, n is the degree, and a_0 is the constant term. A polynomial function is in ***standard form*** when its terms are written in descending order of exponents form left to right.

Common Polynomial Functions			
Degree	**Type**	**Standard Form**	**Example**
0	Constant	$f(x) = a_0$	$f(x) = -14$
1	Linear	$f(x) = a_1x + a_0$	$f(x) = 5x - 7$
2	Quadratic	$f(x) = a_2x^2 + a_1x + a_0$	$f(x) = 2x^2 + x - 9$
3	Cubic	$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$	$f(x) = x^3 - x^2 + 3x$
4	Quartic	$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$	$f(x) = x^4 + 2x - 1$

Identify Polynomial Functions

Example 1

Determine whether the function of the polynomial is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

a. $f(x) = -2x^3 + 5x + 8$ $-2x^3 + 5x + 8$, 3, -2

b. $g(x) = -0.8x^3 + \sqrt{2}x^4 - 12$ $\sqrt{2}x^4 - 0.8x^3 - 12$, 4, $\sqrt{2}$

c. $h(x) = -x^2 + 7x^{-1} + 4x$ Not Polynomial

d. $k(x) = x^2 + 3^x$ Not Polynomial

Evaluate the Polynomial Function

Example 2

a. Evaluate $f(x) = x^4 - 8x^2 + 5x - 7$ when $x = 3$.

$(3)^4 - 8(3)^2 + 5(3) - 7$

17

b. Evaluate $f(x) = -2x^4 + 6x^3 - 3x + 11$ when $x = 4$.

$-2(4)^4 + 6(4)^3 - 3(4) + 11$

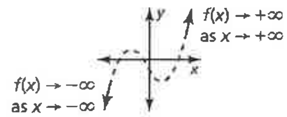
-129

End Behavior

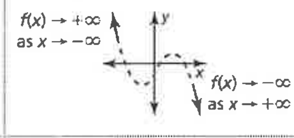
KEY IDEA

End Behavior of Polynomial Functions

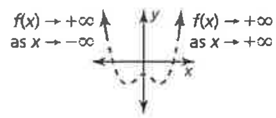
Degree: odd
Leading coefficient: positive



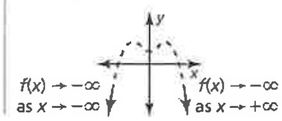
Degree: odd
Leading coefficient: negative



Degree: even
Leading coefficient: positive



Degree: even
Leading coefficient: negative

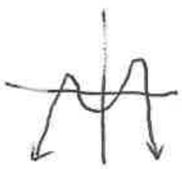


REMEMBER

The expression "$x \to +\infty$" is read as "x approaches positive infinity."

<u>**Example 3**</u>

a. Describe the end behavior of $f(x) = -0.5x^4 + 2.5x^2 + x - 1$.



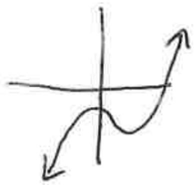
$y \to -\infty$ as $x \to -\infty$
$y \to \infty$ as $x \to \infty$

b. Describe the end behavior of $f(x) = -0.3x^3 + 1.7x^2 - 4x + 6$

$y \to \infty$ as $x \to -\infty$
$y \to -\infty$ as $x \to \infty$

Self-Assessment

8. Describe the end behavior of $f(x) = 0.25x^3 - x^2 - 1$.

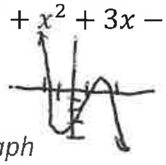


$y \to -\infty$ as $x \to -\infty$
$y \to \infty$ as $x \to \infty$

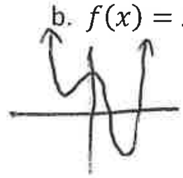
Graphing Polynomials Functions

Example 4: Graph.

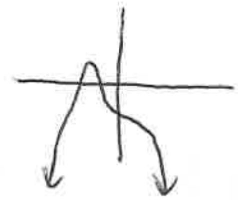
a. $f(x) = -x^3 + x^2 + 3x - 3$



b. $f(x) = x^4 - x^3 - 4x^2 + 4$



c. $f(x) = -x^4 - x^3 + 2x^2 - x - 3$

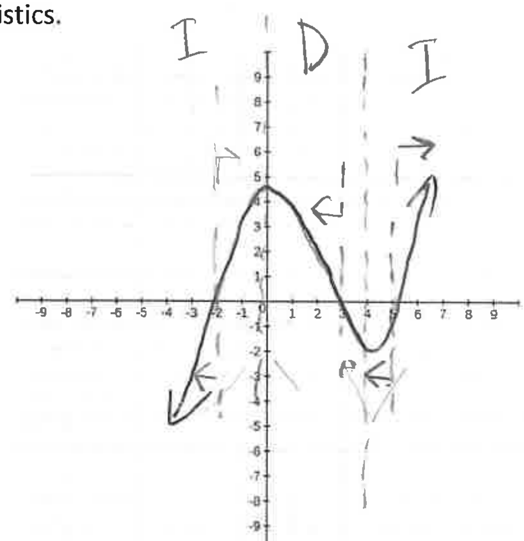


Sketching a Graph

Example 5:

a. Sketch a graph of a polynomial function f with the following characteristics.

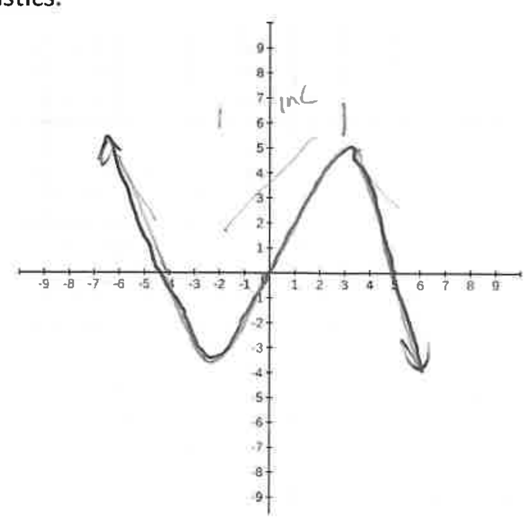
- f is increasing when $x < 0$ and $x > 4$.
- f is decreasing when $0 < x < 4$.
- $f(x) > 0$ when $-2 < x < 3$ and $x > 5$.
- $f(x) < 0$ when $x < -2$ and $3 < x < 5$



Use the graph to describe the degree and leading coefficient of f.

b. Sketch a graph of a polynomial function f with the following characteristics.

- f is increasing when $-2 < x < 3$.
- f is decreasing when $x < -2$ and $x > 3$.
- $f(x) > 0$ when $x < -4$ and $0 < x < 5$.
- $f(x) < 0$ when $-4 < x < 0$ and $x > 5$.



Use the graph to describe the degree and leading coefficient of f.

Modeling Real Life

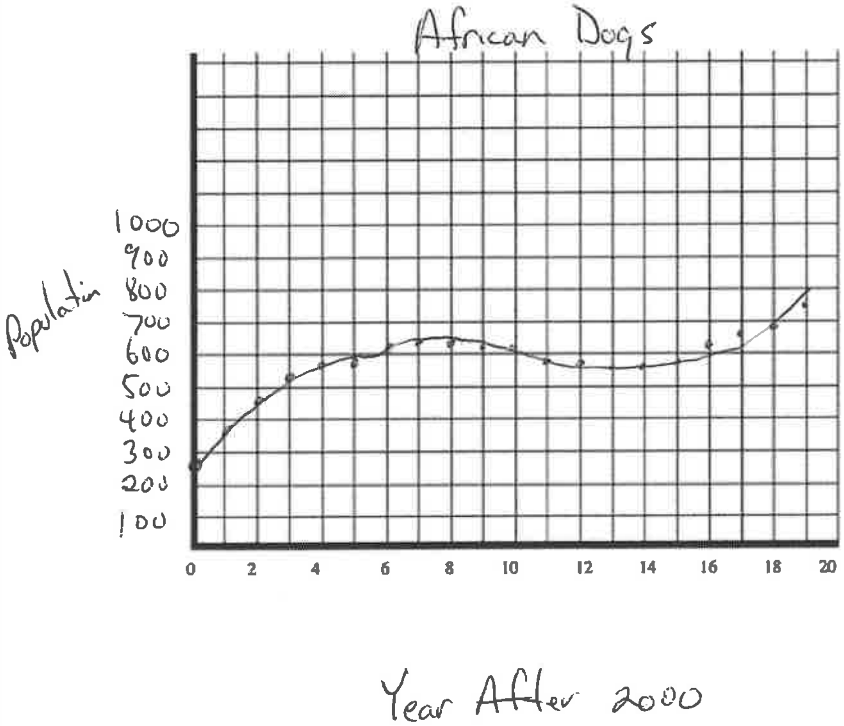
Example 6:

The African wild dog is one of the most endangered carnivores on Earth. The estimated population of African wild dogs under human care can be modeled by the polynomial function $p(t) = 0.368t^3 - 11.45t^2 + 109.5t + 286$, where t represents the number of years after 2000.

a. Use technology to graph the function for $1 \leq t \leq 18$. Describe the behavior of the graph on this interval.

b. What is the average rate of change in the number of dogs from 2001 to 2018?

c. Do you think this model can be used for years after 2018? Explain your reasoning.



a. Population increases from 2001 to 2006, decreases from 2006 to 2014, increases from 2014 to 2018

b. $t=1$ and $t=18$

$$\frac{P(18)-P(1)}{18-1} = \frac{693{,}376-384.418}{17} = 18{,}174$$

$\approx$ 18 dogs/year

c. For a few years, but unlimited growth is unreasonable

Pg 158,1-35 odds

4.2 Adding, Subtracting, and Multiplying Polynomials

Targets:

1. I can add and subtract polynomials.
2. I can multiply polynomials and use special product patterns.
3. I can use Pascal's Triangle to expand binomials.

Pascals Triangle

$11^0 \rightarrow$ 1

$11^1 \rightarrow$ 1 1

$11^2 \rightarrow$ 1 2 1

$11^3 \rightarrow$ 1 3 3 1

$11^4 \rightarrow$ 1 4 6 4 1

Explore It! Expanding Binomials

Work with a partner.

1. Use the diagram. Find the value of each expression. Write one digit of the value in each box. What pattern(s) do you notice?

Outside boxes all contain 1

Sum of 2 consecutive boxes is the number under.

2. Find each product. Explain your steps.

a. $(x+1)^2$

$x^2 + 2x + 1$

Same as 11^2

b. $(x+1)^3$

$x^3 + 3x^2 + 3x + 1$

11^3

What pattern do you notice between the values of 11^n and the terms of $(x+1)^n$ for $0 \leq n \leq 3$? Does this patten continue for $(x+1)^4$? Explain your reasoning.

$(x+1)^n$ same as 11^n

3. Find the each product. Explain your steps.

a. $(a+b)^3$

$a^3 + 3a^2b + 3ab^2 + b^3$

b. $(a-b)^3$

$a^3 - 3a^2b + 3ab^2 - b^3$

What other pattern(s) do you notice when cubing these binomials?

exponent of a decreases as the exponent of b increases

$(x+y)^4$

$1x^4y^0 \quad 4x^3y \quad 6x^2y^2 \quad 4xy^3 \quad 1x^0y^4$

Pascal's Triangle

1 $\quad 11^0$

1 1 $\quad 11^1$

1 2 1 $\quad 11^2$

1 3 3 1 $\leftarrow 11^3$

1 4 6 4 1 $\quad 11^4$

4. Explain how you can use Pascal's Triangle to find each product. Then find the product.

a. $(x+2)^3$ $\quad 11^3 \quad x^3+6x^2+12x+8$

- First cubed x^3 times second term raised to 0 power times coef 1
- Second is first term squared, times second term raised to 1st power times coef 3 $6x^2$
- third is first term raised to 1 times second term raised to 2nd power time coef of 3 $12x$ - Fourth is one 8

b. $(2x-3)^3$

$8x^3-36x^2+54x-27$

$1(2x)^3(-3)^0 + 3(2x)^2(-3)^1 + 3(2x)^1(-3)^2 \; 1$

$8x^3-36x^2+54x-27$

Adding and Subtracting Polynomials

Example 1: Find the sum.

a. $(3x^3+2x^2-x-7)+(x^3-10x^2+8)$

$4x^3-8x^2-x+1$

b. $(9y^3+3y^2-2y+1)+(-5y^2+y-4)$

$9y^3-2y^2-y-3$

Example 2: Find the difference.

a. $(2x^3+6x^2-x+1)-(8x^3-3x^2-2x+9)$

$-6x^3+9x^2+x+10$

b. $(3z^2+2z-4)-(2z^2+3z)$

$3z^2-z-4$

Multiplying Polynomials

Example 3: Find the product.

a. $(-x^2 + 2x + 4)(x - 3)$

$-x^3 + 3x^2 + 2x^2 - 6x + 4x - 12$

$-x^3 + 5x^2 - 2x - 12$

b. $(y + 5)(3y^2 - 2y + 2)$

$3y^3 - 2y^2 + 2y + 15y^2 - 10y + 10$

$3y^3 + 13y^2 - 8y + 10$

Multiplying Three Binomials

Example 4: Find the product.

a. $(x - 1)(x + 4)(x + 5)$

$(x^2 + 3x - 4)(x + 5)$

$x^3 + 5x^2 + 3x^2 + 15x - 4x - 20$

$x^3 + 8x^2 + 11x - 20$

b. $(x + 2)(x - 1)(x - 3)$

$(x^2 + x - 2)(x - 3)$

$x^3 - 3x^2 + x^2 - 3x - 2x + 6$

$x^3 - 2x^2 - 5x + 6$

COMMON ERROR
In general,

$(a \pm b)^2 \neq a^2 \pm b^2$

and

$(a \pm b)^3 \neq a^3 \pm b^3$.

KEY IDEA

Special Product Patterns

Sum and Difference	**Example**
$(a + b)(a - b) = a^2 - b^2$	$(x + 3)(x - 3) = x^2 - 9$

Square of a Binomial	**Example**
$(a + b)^2 = a^2 + 2ab + b^2$	$(y + 4)^2 = y^2 + 8y + 16$
$(a - b)^2 = a^2 - 2ab + b^2$	$(2t - 5)^2 = 4t^2 - 20t + 25$

Cube of a Binomial	**Example**
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	$(z + 3)^3 = z^3 + 9z^2 + 27z + 27$
$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$	$(m - 2)^3 = m^3 - 6m^2 + 12m - 8$

Proving a Polynomial Identity

Example 5:

a. Prove the polynomial identify for the cube of a binomial representing a sum: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Pascals

$a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^3 = (a+b)(a+b)(a+b)$

$(a^2 + 2ab + b^2)(a + b)$

$a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3$

$a^3 + 3a^2b + 3ab^2 + b^3$

Pascal's Triangle

When you arrange the coefficients of the variables in the expansion of $(a+b)^n$, you will see a special pattern called **Pascal's Triangle**. Pascal's Triangle is named after a French mathematician Blaise Pascal (1632-1662).

KEY IDEA

Pascal's Triangle

In Pascal's Triangle, the first and last numbers in each row are 1. Every number other than 1 is the sum of the closest two numbers in the row directly above it. The numbers in Pascal's Triangle are the same numbers that are the coefficients of binomial expansions, as shown in the first six rows.

	n	$(a+b)^n$	Binomial Expansion	Pascal's Triangle
0th row	0	$(a+b)^0 =$	1	1
1st row	1	$(a+b)^1 =$	$1a + 1b$	1 1
2nd row	2	$(a+b)^2 =$	$1a^2 + 2ab + 1b^2$	1 2 1
3rd row	3	$(a+b)^3 =$	$1a^3 + 3a^2b + 3ab^2 + 1b^3$	1 3 3 1
4th row	4	$(a+b)^4 =$	$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$	1 4 6 4 1
5th row	5	$(a+b)^5 =$	$1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$	1 5 10 10 5 1

Example 7: Use Pascal's Triangle to expand.

a. $(x-2)^5$

$1x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + 1(-2)^5$

$x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

b. $(3y+1)^3$

$1(3y)^3 + 3(3y)^2(1) + 3(3y)(1)^2 + 1(1)^3$

$27y^3 + 27y^2 + 9y + 1$

You Try!

c. $(x-3)^4$

$1(x)^4 + 4(x)^3(-3) + 6(x)^2(-3)^2 + 4(x)(-3)^3 + 1(-3)^4$

$x^4 - 12x^3 + 54x^2 - 108x + 81$

Pg 166,1-45 odds

4.3 Dividing Polynomials

Targets:

1. I can use long division to divide polynomials by other polynomials.
2. I can divide polynomials by binomials of the form $x - k$ using synthetic division.
3. I can explain the Remainder Theorem.

Explore It! Dividing Polynomials

Work with a partner.

a. Consider the polynomial $x^3 + 2x^2 - x - 2$. Use technology to explore the graph of the polynomial divided by the binomial $x + a$ for the given values of a. What do you notice? What can you conclude?

i. $a = 1$

ii. $a = 2$

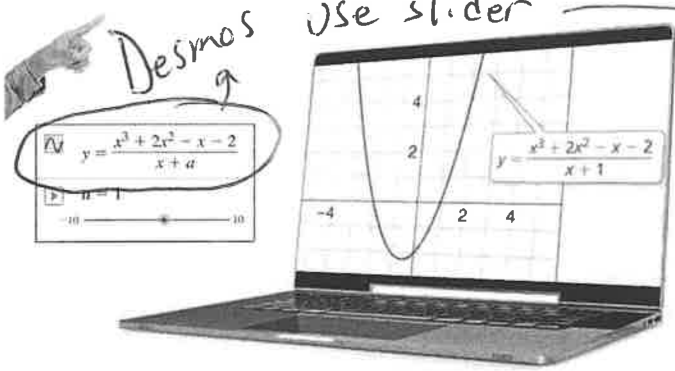
iii. $a = 3$

iv. $a = -1$

v. $a = -2$

vi. $a = 4$

Desmos use slider →



$$\frac{x^3 + 2x^2 - x - 2}{x + 1}$$

$-1, 1, 2$ Parabolas

$-2, 3, 4$ Cubic Functions

c. Use technology to explore the graph of the polynomial $x^4 + 7x^3 + 9x^2 - 7x - 10$ divided by the binomial $x + a$ for several values of a. Make several observations about the graph.

$-3, -2, 0, 3, 4$ Quartic

$-1, 1, 2, 5$ Cubic

Long Division of Numbers

Divide 1423 ÷ 4 using long division.

$355\ 3/4$

$4\overline{)1423}$

12

22

20

23

20

3

Divide $\frac{1424}{4}$ using long division.

356

$4\overline{)1424}$

12

22

20

24

24

0

Long Division of Polynomials

When you divide a polynomial f(x) by a nonzero polynomial divisor d(x), you get a quotient polynomial q(x) and a remainder polynomial r(x).

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

The degree of the division d(x) is less than or equal to the degree of the dividend f(x). Also, the degree of the remainder r(x) must be less than the degree of the divisor. When the remainder is 0, the divisor *divides evenly* into the dividend. One way to divide polynomials is called **polynomial long division.**

Example 1: Divide using polynomial long division.

a. $2x^4 + 3x^3 + 5x - 1$ by $x^2 + 3x + 2$.

$2x^2 - 3x + 5 + \frac{-4x-11}{x^2+3x+2}$

$x^2+3x+2\overline{)2x^4+3x^3+0x^2+5x-1}$

$-2x^4 \mp 6x^3 \mp 4x^2$

$-3x^3-4x^2+5x$

$+3x^3+9x^2+6x$

$5x^2+11x-1$

$-5x^2 \mp 15x \mp 10$

$-4x-11$

b. $(4x^2 + 3x - 11) \div (x + 1)$

$4x - 1 + \frac{-10}{x+1}$

$x+1\overline{)4x^2+3x-11}$

$-4x^2 \mp 4x$

$-x-11$

$+x+1$

-10

You Try! Divide using polynomial long division.

c. $2x^2 - 5x - 3$ by $x - 3$.

$$\begin{array}{r} 2x + 1 \\ x-3 \overline{) 2x^2 - 5x - 3} \\ \underline{-2x^2 + 6x} \\ x - 3 \\ \underline{-x + 3} \\ 0 \end{array}$$

d. $(x^4 + 2x^2 - x + 5) \div (x^2 - x + 1)$

$$x^2 + x + 2 + \frac{3}{x^2 - x + 1}$$

$$\begin{array}{r} x^2 - x + 1 \overline{) x^4 + 0x^3 + 2x^2 - x + 5} \\ \underline{-x^4 + x^3 \mp x^2} \\ x^3 + x^2 - x \\ \underline{-x^3 + x^2 \mp x} \\ 2x^2 - 2x + 5 \\ \underline{-2x^2 + 2x \mp 2} \\ 3 \end{array}$$

Synthetic Division

Synthetic division is a shortcut for dividing polynomials by binomials of the form $x - k$.

Example 2: Divide using synthetic division.

a. $-x^3 + 4x^2 + 9$ by $x - 3$.

$x - 3 = 0$

$x = 3$

$$\begin{array}{r|rrrr} 3 & -1 & 4 & 0 & 9 \\ & \downarrow & -3 & 3 & 9 \\ \hline & -1 & 1 & 3 & \boxed{18} \end{array}$$

$$-x^2 + x + 3 + \frac{18}{x-3}$$

b. $(3x^3 - 2x^2 + 2x - 5) \div (x + 1)$

$x + 1 = 0$

$x = -1$

$$\begin{array}{r|rrrr} -1 & 3 & -2 & 2 & -5 \\ & & -3 & 5 & -7 \\ \hline & 3 & -5 & 7 & \boxed{-12} \end{array}$$

$$3x^2 - 5x + 7 + \frac{-12}{x+1}$$

The Remainder Theorem

KEY IDEA

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

Example 4: Use synthetic division to evaluate the function for the indicated value of x. Replaces Direct Substituti

a. $f(x) = 5x^3 - x^2 + 13x + 29; x = -4$

$$\begin{array}{r|rrrr} -4 & 5 & -1 & 13 & 29 \\ & & -20 & 84 & -388 \\ \hline & 5 & -21 & 97 & -359 \end{array}$$

-359

b. $f(x) = 4x^3 - 2x^2 - 5x + 11; f(-2)$

$$\begin{array}{r|rrrr} -2 & 4 & -2 & -5 & 11 \\ & & -8 & 20 & -30 \\ \hline & 4 & -10 & 15 & -19 \end{array}$$

-19

Self Assessment

Use synthetic division to evaluate the function for the indicated value of x.

a. $f(x) = 4x^2 - 10x - 21; x = 5$ b. $f(x) = 5x^4 + 2x^3 - 20x - 6; f(2)$

a.
$$\begin{array}{r|rrr} 5 & 4 & -10 & -21 \\ & & 20 & 50 \\ \hline & 4 & 10 & -29 \end{array}$$

-29

b.
$$\begin{array}{r|rrrrr} 2 & 5 & 2 & 0 & -20 & -6 \\ & & 10 & 24 & 48 & 56 \\ \hline & 5 & 12 & 24 & 28 & 50 \end{array}$$

50

Pg 173,1-29 odds

4.4 Factoring Polynomials

Target: I can use the Factor Theorem.

Class Opener: Factoring Practice

Factor each polynomial completely.

a. $x^3 - 4x^2 - 5x$

$x(x^2 - 4x - 5)$

$x(x-5)(x+1)$

b. $3y^5 - 48y^3$

$3y^3(y^2 - 16)$

$3y^3(y-4)(y+4)$

c. $5z^4 + 30z^3 + 45z^2$

$5z^2(z^2 + 6z + 9)$

$5z^2(z+3)^2$

d. $x^3 - 125$

$(x-5)(x^2 + 5x + 25)$

e. $16s^5 + 54s^2$

$2s^2(8s^3 + 27)$

$2s^2(2s+3)(4s^2 - 6s + 9)$

f. $z^3 + 5z^2 - 4z - 20$

$z^2(z+5) - 4(z+5)$

$(z^2 - 4)(z+5)$

$(z+2)(z-2)(z+5)$

g. $16x^4 - 81$

$(4x^2 + 9)(4x^2 - 9)$

$(4x^2 + 9)(2x+3)(2x-3)$

h. $3p^8 + 15p^5 + 18p^2$

$3p^2(p^6 + 5p^3 + 6)$

$3p^2(p^3 + 2)(p^3 + 3)$

How do you know when a polynomial is factored completely?

Turns prime

When we did polynomial long division and synthetic division in Section 4.3, the remainder was often nonzero. If the remainder is zero when you divide by *x – k,* the polynomial divides evenly and *x - k* is a factor of the polynomial.

KEY IDEA

The Factor Theorem

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

Example 5: Determine whether the given binomial is a factor of f(x).

a. $x^2 + 2x - 4; x - 2$

$$\begin{array}{r|rrr} 2 & 1 & 2 & -4 \\ & & 2 & 8 \\ \hline & 1 & 4 & \boxed{4} \end{array}$$

No

b. $3x^4 + 15x^3 - x^2 + 25; x + 5$

$$\begin{array}{r|rrrrr} -5 & 3 & 15 & -1 & 0 & 25 \\ & & -15 & 0 & 5 & -25 \\ \hline & 3 & 0 & -1 & 5 & \boxed{0} \end{array}$$

Yes

Example 6:

a. Show that $x + 3$ is a factor of $f(x) = x^4 + 3x^3 - x - 3$. Then, factor $f(x)$ completely.

$$\begin{array}{r|rrrrr} -3 & 1 & 3 & 0 & -1 & -3 \\ & & -3 & 0 & 0 & 3 \\ \hline & 1 & 0 & 0 & -1 & \boxed{0} \end{array}$$

$x^3 - 1$

$(x-1)(x^2+x+1)$

$x^3(x+3) - 1(x+3)$

$(x^3-1)(x+3)$

$(x-1)(x^2+x+1)(x+3)$

b. Show that $x - 2$ is a factor of $f(x) = x^4 - 2x^3 + x - 2$. Then, factor $f(x)$ completely.

$$\begin{array}{r|rrrrr} 2 & 1 & -2 & 0 & 1 & -2 \\ & & 2 & 0 & 0 & 2 \\ \hline & 1 & 0 & 0 & 1 & \boxed{0} \end{array}$$

$x^3(x-2) + 1(x-2)$

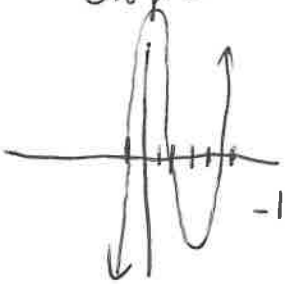
$(x^3+1)(x-2)$

$(x+1)(x^2 - x + 1)(x-2)$

Modeling Real Life

Example 7: A roller coaster starts at a height of 34 feet and then goes through an underground tunnel. The function $h(t) = 4t^3 - 21t^2 + 9t + 34$ represents the coaster height h (in feet) after t seconds, where $0 \leq t \leq 5$. How long is the coaster in the tunnel?

Graphed



so

$-1, 2?$

$$\begin{array}{r|rrrr} -1 & 4 & -21 & 9 & 34 \\ & & -4 & 25 & -34 \\ \hline & 4 & -25 & 34 & 0 \end{array}$$

$$\begin{array}{r|rrr} 2 & 4 & -25 & 34 \\ & & 8 & -34 \\ \hline & 4 & -17 & 0 \end{array}$$

$4t - 17 = 0$

$4t = 17$

$t = \frac{17}{4} = 4.25$

Pg 180,1-45 odds,55,57

$4.25 - 2 = 2.25$ seconds underground

4.5 Solving Polynomial Equations

Targets:

1. I can explain how solutions of equations and zeros of functions are related.
2. I can solve polynomial equations.
3. I can write a polynomial function when given information.

Explore It! Solving Polynomial Equations with Repeated Solutions

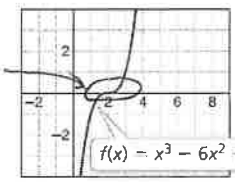
Work with a partner. Polynomial equations can have distinct solutions or repeated solutions. In parts (a)-(f), solve the equation algebraically. Then repeated use the graph to describe the behavior of the related function near any repeated zeros. What do you notice?

2| 1 -6 12 -8
2 -8 8
1 -4 4 |0

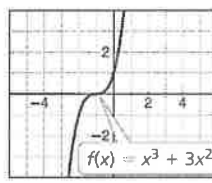
$x^2 - 4x + 4$ Triple root

$(x-2)^2$ $x = 2, 2, 2$

a. $x^3 - 6x^2 + 12x - 8 = 0$



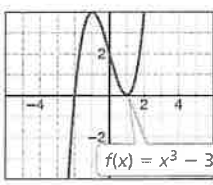
b. $x^3 + 3x^2 + 3x + 1 = 0$



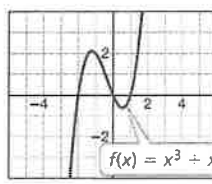
$x = -1, -1, -1$

c. $x^3 - 3x + 2 = 0$

$x = -2, 1, 1$



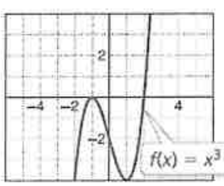
d. $x^3 + x^2 - 2x = 0$



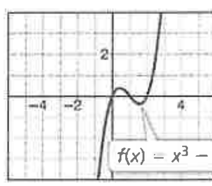
$x = -2, 0, 1$

e. $x^3 - 3x - 2 = 0$

$x = -1, -1, 2$



f. $x^3 - 3x^2 + 2x = 0$



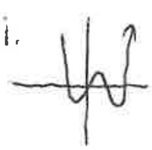
$x = 0, 1, 2$

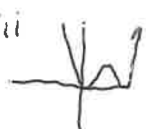
g. **Choose Tools:** Graph the related function for each quartic equation and describe the behavior of the function near its zeros.

i. $x^4 - 4x^3 + 5x^2 - 2x = 0$
ii. $x^4 - 4x^3 + 4x^2 = 0$
iii. $x^4 - 2x^3 - x^2 + 2x = 0$
iv. $x^4 + 3x^2 = 0$

i.

$x = 0, 1, 2$

ii.

$x = -1, 0, 1, 2$

iii

$x = 0, 2$

iv.

$x = -3, 0$ (flat)

h. Describe what it means when a polynomial equation has a repeated solution. How can you determine whether a polynomial equation has a solution?

The polynomial equation contains a factor raised to a power; A polynomial function has a repeated solution if its graph touches the x-axis at a point without crossing it, or the graph appears flat at an x-intercept

Solve the Polynomial Equation by Factoring

Example 1:

a. Solve. $2x^3 - 12x^2 + 18x = 0$

$2x(x^2 - 6x + 9)$

$2x(x-3)^2$

$2x = 0$ $\quad$ $x - 3 = 0$ $\quad$ double

$x = 0$ $\quad$ $x = 3$

b. Solve. $5x^3 - 40x^2 = -80x$

$5x(x^2 - 8x + 16)$

$5x = 0$ $\quad$ $x - 4 = 0$ $\quad$ double

$x = 0$ $\quad$ $x = 4$

Find the Zeros of a Polynomial Function

Example 2: Find the zeros of the function. Then sketch a graph of the function.

a. $f(x) = -2x^4 + 16x^2 - 32$

$0 = -2(x^4 - 8x^2 + 16)$

$0 = -2(x^2 - 4)(x^2 - 4)$

$0 = -2(x+2)(x-2)(x+2)(x-2)$

$x = 2, -2$

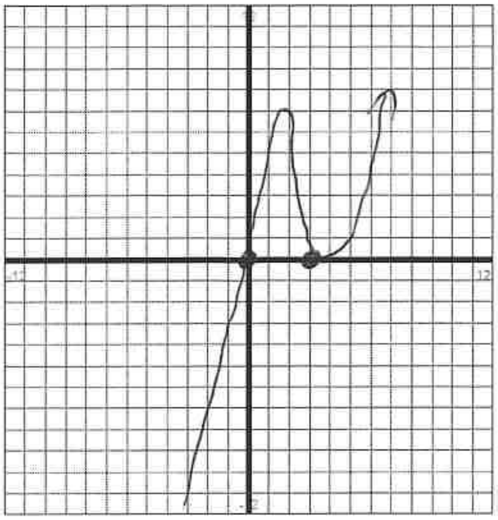
b. $f(x) = 3x^4 - 6x^2 + 3$

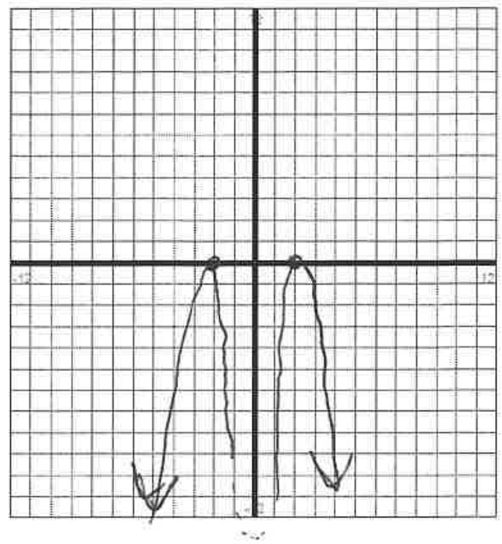
$0 = 3(x^4 - 2x^2 + 1)$

$0 = 3(x^2 - 1)(x^2 - 1)$

$0 = 3(x+1)(x-1)(x+1)(x-1)$

$x = 1, -1$





Example 3: Find all the real solutions.

a. $x^3 - 8x^2 + 11x + 20 = 0$

$\pm\frac{20}{1}$ $\frac{1}{1}$ $\frac{2}{1}$ $\frac{4}{1}$ $\frac{5}{1}$ $\frac{10}{1}$ $\frac{20}{1}$

$-1 \,|\; 1 \;\; -8 \;\; 11 \;\; 20$

$\quad\;\; -1 \;\; 9 \;\; -20$

$\quad 1 \;\; -9 \;\; 20 \;\; |\, 0$

$x^2 - 9x + 20$

$(x-4)(x-5)$

$-1, 4, 5$

b. $x^3 + x^2 - 24 = 14x$

—change

$\frac{24}{1}$ $\frac{1}{1}$ $\frac{2}{1}$ $\frac{3}{1}$ $\frac{4}{1}$ $\frac{6}{1}$ $\frac{8}{1}$ $\frac{12}{1}$ $\frac{24}{1}$ $\to 2, -3, -4$

Example 4: Finding Zeros of a Polynomial Function

Find all of the real zeros.

a. $f(x) = 10x^4 - 11x^3 - 42x^2 + 7x + 12$

$\frac{12}{10}$ $\begin{matrix} 1 & 2 & 3 & 4 & 6 & 12 \\ 1 & 2 & 5 & 10 \end{matrix}$

$\pm 1 \pm \frac{1}{2} \pm \frac{1}{5} \pm \frac{1}{10}$, etc.

Graph to choose!!

$-\frac{1}{2} \,|\; 10 \;\; -11 \;\; -42 \;\; 7 \;\; 12$

$\quad\; -5 \;\; 8 \;\; 17 \;\; -12$

$\frac{3}{5} \,|\; 10 \;\; -16 \;\; -34 \;\; 24 \;\; |\, 0$

$\quad\; 6 \;\; -6 \;\; -24$

$10 \;\; -10 \;\; -40 \;\; |\, 0$

$10x^2 - 10x - 40$

$10(x^2 - x - 4)$

$\frac{1 \pm \sqrt{1 - 4(1)(-4)}}{2}$

$\frac{1 \pm \sqrt{17}}{2}, -\frac{1}{2}, \frac{3}{5}$

b. $f(x) = 6x^4 - 11x^3 - 16x^2 + 2x + 4$

$\frac{1}{2} \,|\; 6 \;\; -11 \;\; -16 \;\; 2 \;\; 4$

$\quad\; 3 \;\; -4 \;\; -10 \;\; -4$

$-\frac{2}{3} \,|\; 6 \;\; -8 \;\; -20 \;\; -8 \;\; |\, 0$

$\quad\; -4 \;\; 8 \;\; 8$

$6 \;\; -12 \;\; -12 \;\; |\, 0$

$6x^2 - 12x - 12$

$1 \pm \sqrt{3}, \frac{1}{2}, -\frac{2}{3}$

The Irrational Conjugates Theorem

In **Example 4**, notice that the irrational zeros are *conjugates* of the form $a + \sqrt{b}$ and $a - \sqrt{b}$. This illustrates the theorem below.

KEY IDEA

The Irrational Conjugates Theorem

Let f be a polynomial function with rational coefficients, and let a and b be rational numbers such that $\sqrt{b}$ is irrational. If $a + \sqrt{b}$ is a zero of f, then $a - \sqrt{b}$ is also a zero of f.

Example 5: Using Zeros to Write a Polynomial Function

a. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 3 and $2 + \sqrt{5}$.

$(x-3)(x-(2+\sqrt{5}))(x-(2-\sqrt{5}))$

$(x-3)(x^2 - x(2-\sqrt{5}) - x(2+\sqrt{5}) + (2+\sqrt{5})(2-\sqrt{5}))$

$(x-3)(x^2 - 2x + x\sqrt{5} - 2x + x\sqrt{5} + 4 - 5)$

$(x-3)(x^2 - 4x - 1)$

$x^3 - 4x^2 - x - 3x^2 + 12x + 3$

$x^3 - 7x^2 + 11x + 3$

b. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and $3+\sqrt{7}$.

$$(x-2)(x-(3+\sqrt{7}))(x-(3-\sqrt{7}))$$

$$(x-2)(x^2 - x(3-\sqrt{7}) - x(3+\sqrt{7}) + (3+\sqrt{7})(3-\sqrt{7}))$$

$$(x-2)(x^2 - 3x + x\sqrt{7} - 3x - x\sqrt{7} + 9 - 7)$$

$$(x-2)(x^2 - 6x + 2)$$

$$x^3 - 6x^2 + 2x - 2x^2 + 12x - 4$$

$$x^3 - 8x^2 + 14x - 4$$

Pg 188,1-43 odds

4.6 The Fundamental Theorem of Algebra

Targets:

1. I can identify the degree of a polynomial.
2. I can explain the Fundamental Theorem of Algebra.
3. I can find all of the zeros of a polynomial function.

Explore It! Finding Zeros of Functions

Work with a partner.

a. Use Desmos to explore each function for several values of *a,b,c* and *d*.

$$f(x) = (x + a)(x + b)$$

$$g(x) = (x + a)(x + b)(x + c)$$

$$h(x) = (x + a)(x + b)(x + c)(x + d)$$

How does the graph change when you change the values of *a,b,c* and *d*? Does the number of real zeros change?

The values of a, b, c, and d change the x-intercepts to -a, -b, -c, and -d. No

b. Use Desmos to explore each function for several values of *a,b,c* and *d*.

$$m(x) = ax^2 + bx + c$$

$$n(x) = ax^3 + bx^2 + cx + d$$

$$p(x) = ax^4 + bx^3 + cx^2 + dx + e$$

How does the graph change when you change the values of *a,b,c* and *d*? Does the number of real zeros change?

The graph changes position and shape. Yes

c. Make a conjecture about the number of real zeros of $y = f(x)$ when the degree of $f(x)$ is a positive number n.

The number of real zeros of $y=f(x)$ is at most n.

The Fundamental Theorem of Algebra

Equation	Degree	Solution(s)	Number of solutions
$2x - 1 = 0$	1	$\frac{1}{2}$	1
$x^2 - 2 = 0$	2	$\pm\sqrt{2}$	2
$x^3 - 8 = 0$	3	$2, -1 \pm i\sqrt{3}$	3
$x^3 + x^2 - x - 1 = 0$	3	$-1, -1, 1$	3

KEY IDEA

The Fundamental Theorem of Algebra

Theorem If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

Corollary If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has exactly n solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

This also means that an nth-degree polynomial function f has exactly n zeros.

Example 1: Finding Solutions of a Polynomial Equation

Identify the number of solutions of the polynomial equation. Then find all the solutions.

a. $x^4 + x^3 + 8x + 8 = 0$ 4

$x^3(x+1) + 8(x+1)$

$(x^3+8)(x+1)$

$(x+2)(x^2-2x+4)(x+1)$

$\frac{2 \pm \sqrt{4-4(1)(4)}}{2}$

$\frac{2 \pm \sqrt{-12}}{2}$

$1 \pm i\sqrt{3}, -2, -1$

b. $x^4 + x - 5 = 5x^3$

$x^4 - 5x^3 + x - 5 = 0$

$x^3(x-5) + 1(x-5) = 0$

$(x^3+1)(x-5) = 0$

$(x+1)(x^2-x+1)(x-5)$

$\frac{1 \pm \sqrt{1-4(1)(1)}}{2}$

$\frac{1 \pm \sqrt{-3}}{2}$

$\frac{1 \pm i\sqrt{3}}{2}, -1, 5$

Example 2: Finding the Zeros of a Polynomial Function

Find all the zeros of the polynomial functions.

a. $f(x) = x^5 + x^3 - 2x^2 - 12x \overset{-}{\cancel{+}} 8$

$\frac{8}{1}$ $\pm 1, 2, 4, 8$

$$\begin{array}{r|rrrrrr} -1 & 1 & 0 & 1 & -2 & -12 & -8 \\ & & -1 & 1 & -2 & 4 & 8 \\ \hline 2 & 1 & -1 & 2 & -4 & -8 & 0 \\ & & 2 & 2 & 8 & 8 & \\ \hline & 1 & 1 & 4 & 4 & 0 & \end{array}$$

$x^3 + x^2 + 4x + 4$

$x^2(x+1) + 4(x+1)$

$x^2 + 4$

$\frac{0 \pm \sqrt{0 - 4(1)(4)}}{2}$

$\pm 2i, -1, -1, 2$

Complex Conjugates

b. $f(x) = x^5 + 3x^4 + 9x^3 + 23x^2 - 36$

$\frac{36}{1}$ $\pm 1 \pm 2 \pm 3 \pm 4 \pm 6 \pm 9 \pm 12 \pm 18 \pm 36$

$$\begin{array}{r|rrrrrr} -2 & 1 & 3 & 9 & 23 & 0 & -36 \\ & & -2 & -2 & -14 & -18 & 36 \\ \hline 1 & 1 & 1 & 7 & 9 & -18 & 0 \\ & & 1 & 2 & 9 & 18 & \\ \hline & 1 & 2 & 9 & 18 & 0 & \end{array}$$

$x^3 + 2x^2 + 9x + 18$

$x^2(x+2) + 9(x+2)$

$x^2 + 9$

$\frac{0 \pm \sqrt{0 - 4(9)}}{2}$ $\quad \frac{0 \pm 6i}{2}$

$\pm 3i, -2, -2, 1$

Paris of complex numbers of the forms $a + bi$ and $a - bi$, where $b \neq 0$, are called complex conjugates. In Example 2, notice that the zeros $2i$ and $-2i$ are complex conjugates. This illustrates the theorem below.

KEY IDEA

The Complex Conjugates Theorem

If f is a polynomial function with real coefficients, and $a + bi$ is an imaginary zero of f, then $a - bi$ is also a zero of f.

Example 3:

a. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and $3 + i$.

$(x+2)(x-(3+i))(x-(3-i))$

$(x+2)(x^2-x(3-i)-x(3+i)+(3+i)(3-i))$

$(x+2)(x^2-3x+xi-3x-xi+9-i^2)$

$(x+2)(x^2-6x+10)$

$x^3-8x^2+22x-20$

b. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 5 and $1 - i$.

$(x-5)(x-(1-i))(x-(1+i))$

$(x-5)(x^2-x(1+i)-x(1-i)+(1-i)(1+i))$

$(x-5)(x^2-x-xi-x+xi+1-i^2)$

$(x-5)(x^2-2x+2)$

$x^3-7x^2+12x-10$

Modeling Real Life

Example 5: A tachometer measures the speed (in revolutions per minute, or RPMs) at which an engine shaft rotates. For a certain boat, the speed x (in hundreds of RPMs) of the engine shaft and the speed s (in miles per hour) of the boat are modeled by $s(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 11$. What is the tachometer reading when the boat travels 15 miles per hour?

$$0.00547(15)^3 - 0.225(15)^2 + 3.62(15) - 11$$

$$x = 19.9$$

1990 RPM's

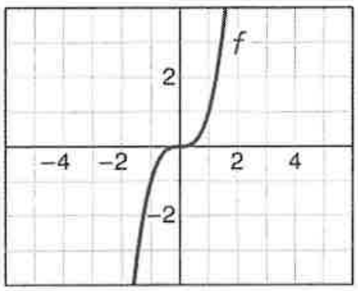
Pg 196,1-33 odds

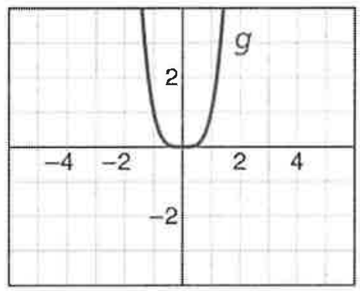
4.7 Transformations of Polynomial Functions

Targets:

- I can describe transformations of polynomial functions
- I can graph transformations of polynomial functions
- I can write functions that represent transformations of polynomial functions

Work with a partner. The graphs of the parent cubic function $f(x) = x^3$ and the parent quartic function $g(x) = x^4$ are shown.





In parts (a)–(h), use technology to explore each function for several values of k, h, and a. How does the graph change when you change the values of k, h, and a?

a. $y = f(x) + k$ Up or Down

b. $y = f(x - h)$ Right or Left

c. $y = a \cdot f(x)$ Stretch / Shrink

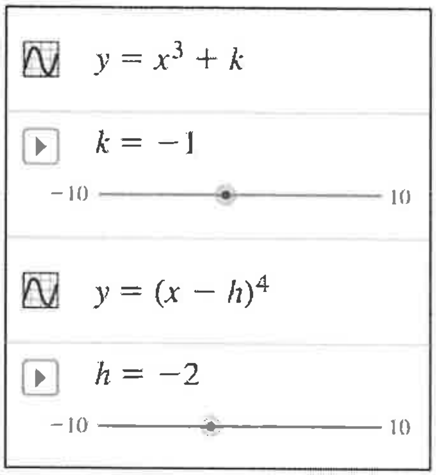
d. $y = f(ax)$ Stretch / Shrink

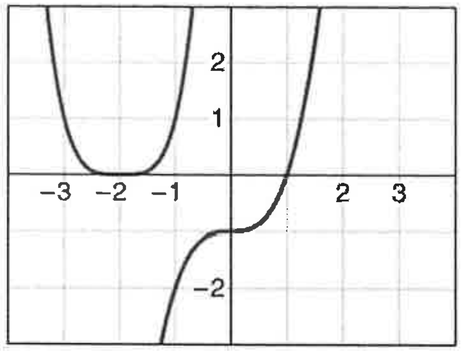
e. $y = g(x) + k$ Up / Down

f. $y = g(x - h)$ Right / Left

g. $y = a \cdot g(x)$ Stretch / Shrink

h. $y = g(ax)$ Stretch / Shrink





Describing Transformations of Polynomial Functions

You can use the same transformations that we used for linear, absolute value, and quadratic when describing transformations of polynomial functions.

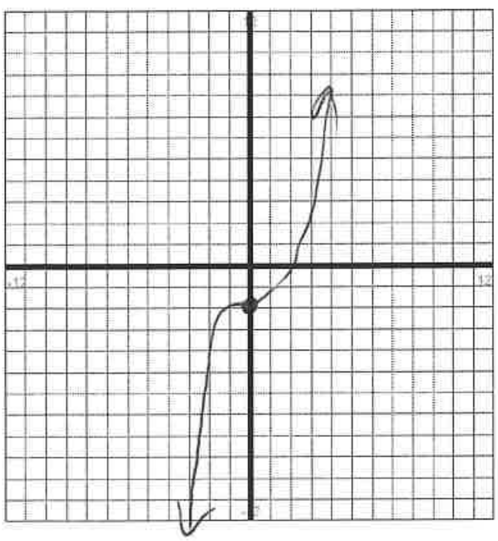
KEY IDEAS

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = (x - 5)^4$ $g(x) = (x + 2)^4$	5 units right 2 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = x^4 + 1$ $g(x) = x^4 - 4$	1 unit up 4 units down
Reflection Graph flips over a line.	$f(-x)$ $-f(x)$	$g(x) = (-x)^4 = x^4$ $g(x) = -x^4$	in the y-axis in the x-axis
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis by a factor of a.	$a \cdot f(x)$	$g(x) = 8x^4$ $g(x) = \frac{1}{4}x^4$	stretch by a factor of 8 shrink by a factor of $\frac{1}{4}$

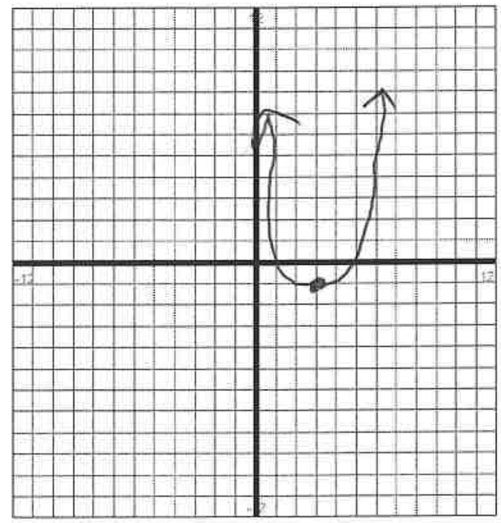
Example 1: Describe the transformation of $f(x) = x^3$ represented by $g(x) = (x + 5)^3 + 2$. Graph $g(x)$.

Left 5 and up 2

a. $g(x) = x^3 - 2$ Down 2

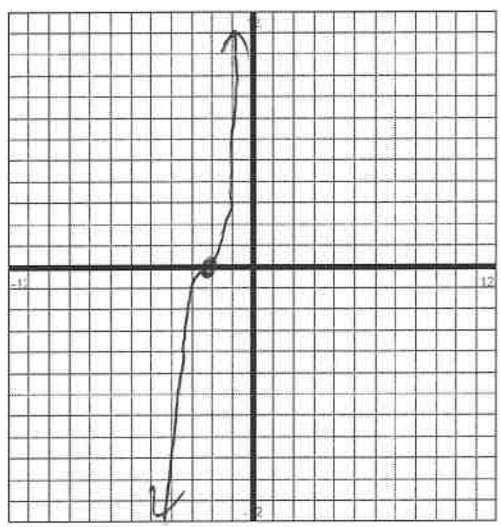


b. $g(x) = (x-3)^4 - 1$ Right 3, Down 1



c. $g(x) = 4(x+2)^3$

Left 2
Stretch 4



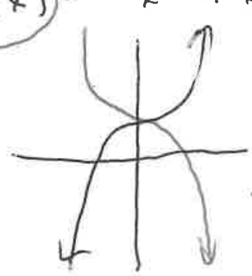
Writing Transformations of Polynomial Functions

Example 3: Let $f(x) = x^3 + x^2 + 1$. Write a rule for g and describe the transformation from f to g. Then, graph the two functions to check your answers.

a. $g(x) = f(-x)$

$g(x) = f(-x) = (-x)^3 + (-x)^2 + 1$

$g(x) = -x^3 + x^2 + 1$



Reflection in the y-axis

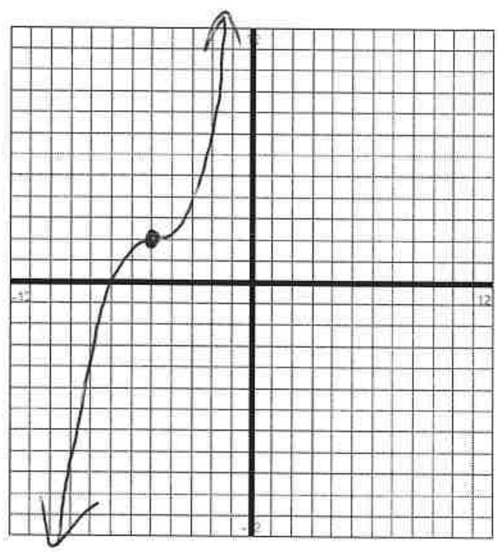
b. $g(x) = 3f(x)$

$g(x) = 3f(x) = 3(x^3 + x^2 + 1)$

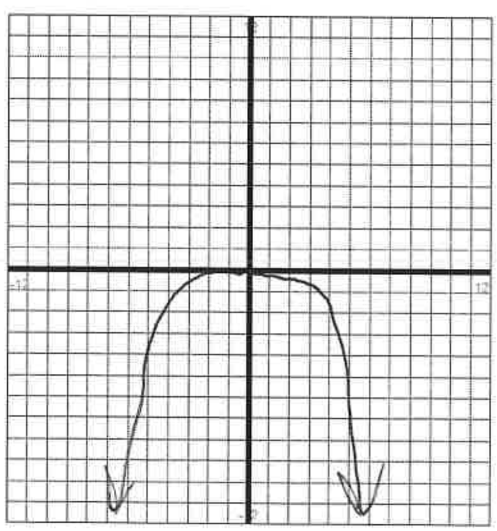
$g(x) = 3x^3 + 3x^2 + 3$

Vert Stretch 3

Try It!



Example 2: Describe the transformation of $f(x) = x^4$ represented by $g(x) = -\frac{1}{4}x^4$. Then, graph $g(x)$.



Vert. Shrink

Reflectin

Try It!

Describe the transformation of the parent function in words and graph $g(x)$.

Let g be a vertical stretch by a factor of 2, followed by a translation 3 units up, of the graph of $f(x) = x^4 - 2x^2$.

$$f(x) = x^4 - 2x^2$$
$$g(x) = 2\,f(x)$$
$$= 2(x^4 - 2x^2)$$
$$= 2x^4 - 4x^2$$
$$= 2x^4 - 4x^2 + 3$$

Try It!

Let $f(x) = x^5 - 4x + 6$ *and* $g(x) = -f(x)$. Write a rule for g, describe the transformation, and graph the two functions on your calculator to check your answers.

$$f(x) = x^5 - 4x + 6$$
$$g(x) = -f(x)$$
$$= -(x^5 - 4x + 6)$$
$$= -x^5 + 4x - 6$$

Reflectin in X-axis

Let the graph of g be a horizontal stretch by a factor of 2, followed by a translation 3 units right of the graph of

$f(x) = 8x^3 + 3$. Write a rule for g.

$$g(x) = 2(8x^3 + 3)$$
$$= 16x^3 + 6$$
$$= 16(x-3)^3 + 6$$

Pg 203,1-25 odds