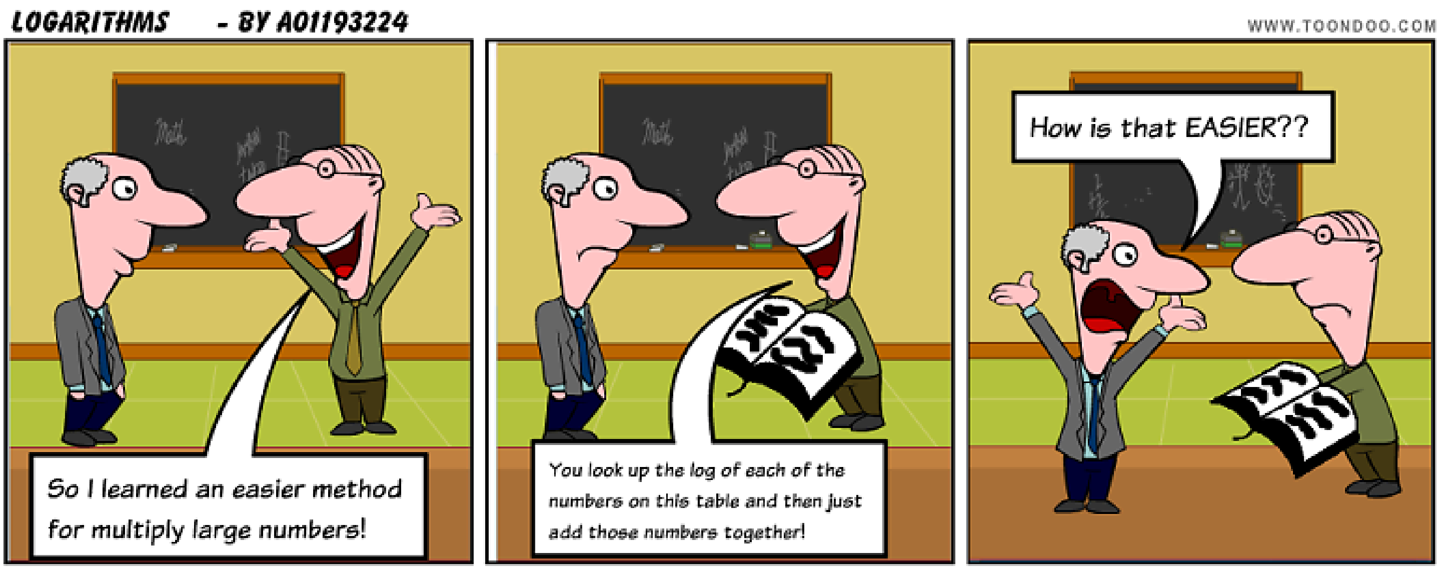
Chapter 4 Notes



Exponential and Logarithmic Functions

Section 4.1 Inverse Functions

Targets: I can find the inverse to a function and state the domain and range of the inverse function.

Definition of an Inverse Function
If the ordered pairs of a function g are the ordered pairs of a function f with the order of the coordinates reversed, then g is the **inverse function** of f. ***Example:*** Observe that the ordered pairs of g are the ordered pairs of f with the order of the coordinates reversed. The following two examples illustrate this concept, $f(x) = 2x$ $g(x) = \frac{1}{2}x$ $f(5) = 2(5) = 10$ $g(10) = \frac{1}{2}(10) = 5$ **Ordered Pair:** $(5,10)$ **Ordered Pair:** $(10,5)$

Definition of a One-to-One Function
A function f is a **one-to-one function** if and only if $f(a) = f(b)$ implies $a = b$.

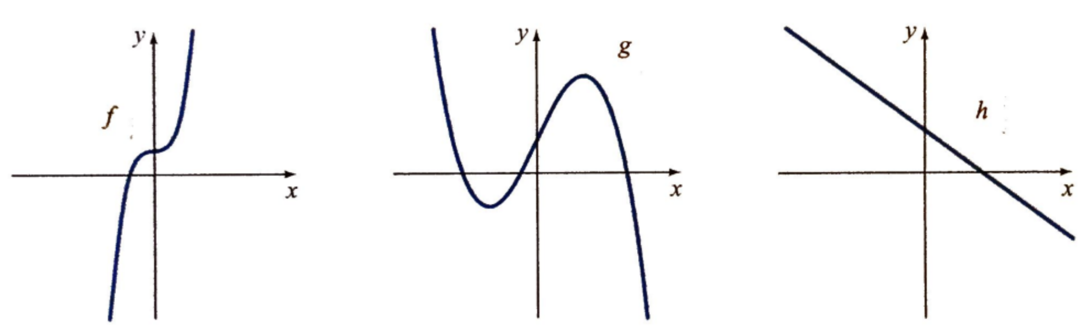
Horizontal Line Test for a One-to-One Function
If every horizontal line intersects the graph of a function at most once, then the graph is the graph of a one-to-one function.

Condition for an Inverse Function
A function f has an inverse function if and only if f is a one-to-one function.

A Property of Increasing Functions and Decreasing Functions

If f is an increasing function or a decreasing function, then f has an inverse function.

Which of the functions graphed below has an inverse function?



Composition of Inverse Functions Property
If f is a one-to-one function, then f^{-1} is the inverse function of f if and only if $(f \circ f^{-1})(x) = f\left[f^{-1}(x)\right] = x$ for all x in the domain of f^{-1} and $(f^{-1} \circ f)(x) = f^{-1}\left[f(x)\right] = x$ for all x in the domain of f.

Use composition of function to determine whether f and g are inverses of one another.

1. $f(x) = 3x - 6;\ g(x) = \frac{1}{3}x + 2$

2. $f(x) = \frac{2}{3}x + 4;\ g(x) = \frac{3}{2}x - 6$

Domain and Range of an Inverse Function
D_x of $f(x) = R_y$ of $f^{-1}(x)$ R_y of $f(x) = D_x$ of $f^{-1}(x)$

Steps for Finding the Inverse of a Function
To find the equation of the inverse f^{-1} of the one-to-one function f, follow these steps. 1. Substitute y for $f(x)$. 2. Solve for x. 3. Interchange x and y. 4. Substitute $f^{-1}(x)$ for y.

Find $f^{-1}(x)$. State the domain and range of $f^{-1}(x)$.

3. $f(x) = 3x + 8$

4. $f(x) = \dfrac{2x+1}{x},\ x \neq 0$

5. $f(x) = \dfrac{6x}{x+5},\ x \neq -5$

6. $f(x) = \sqrt{8-x},\ x \leq 8$

Inverse Functions Application Problems

Targets: I can solve application problems using inverse function.

1. A merchant uses the function $S(x) = \frac{4}{3}x + 100$ to determine the retail selling price *S*, in dollars, of a gold bracelet for which she has paid a wholesale price of *x* dollars.

   a. The merchant paid a wholesale price of $672 for a gold bracelet. Use *S* to determine the retail selling price of this bracelet.

   b. Find S^{-1} and use it to determine the merchant's wholesale price for a gold bracelet that retails at $1596.

Section 4.2 Exponential Functions and Their Applications

Targets: I can evaluate and graph exponential functions.

I can use the properties of translations, stretches and reflections to describe what happens to the graph.

I can solve application problems use exponential functions.

Definition of Exponential Function
The **exponential function with base *b*** is defined by $f(x)=b^x$, where $b>0,\ b\neq 1,$ and x is a real number.

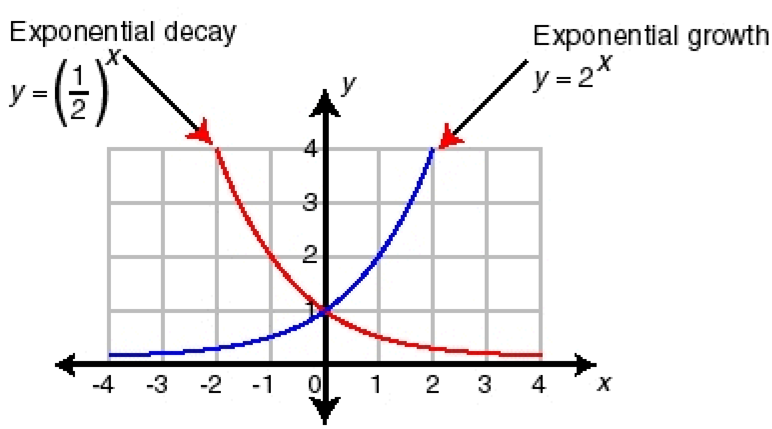
Evaluate the exponential function.

1. $f(x)=3^x$ a. $x=2$ b. $x=-4$ c. $x=\pi$

2. $f(x)=\left(\frac{3}{4}\right)^x$ a. $x=3$ b. $x=-2$ c. $x=\sqrt{2}$

Graphs of Exponential Functions

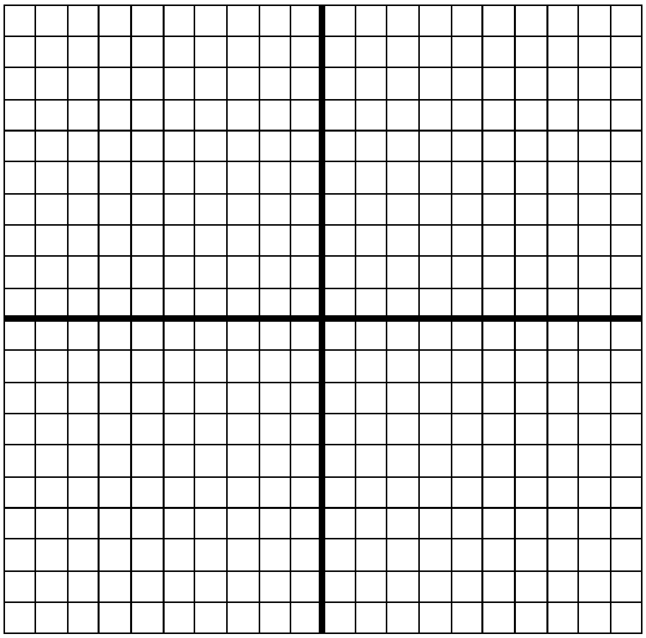
Properties of $f(x)=b^x$
For all positive real numbers $b,\ b\neq 1,$ the exponential function defined by $f(x)=b^x$ has the following properties: • The function *f* is a one-to-one function. It has the set of real numbers as its domain and the set of positive real numbers as its range. • The graph of *f* is a smooth, continuous curve with a *y*-intercept of (0,1), and the graph passes through (1,b). • If $b>1$, *f* is an ***increasing function*** and the graph of *f* is asymptotic to the negative *x*-axis. $\left[\text{As } x\to\infty, f(x)\to\infty, \text{ and as } x\to-\infty, f(x)\to 0\right]$ • If $0<b<1$, *f* is ***decreasing function*** and the graph of *f* is asymptotic to the positive *x*-axis. $\left[\text{As } x\to-\infty, f(x)\to\infty, \text{ and as } x\to\infty, f(x)\to 0\right]$

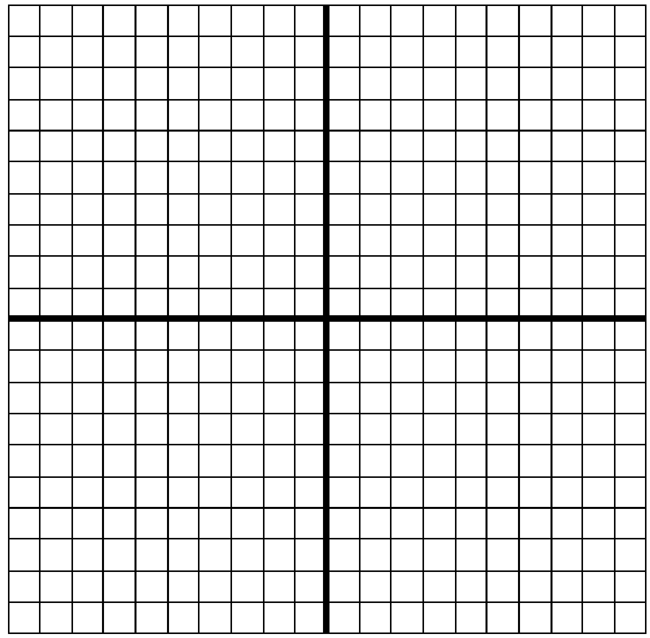


Graph the exponential function.

3. $f(x) = 2^x$

4. $f(x) = \left(\frac{3}{4}\right)^x$





Translations, Stretching or Reflecting of Exponential Graphs

5. Explain how to use the graph of $f(x) = 2^x$ to produce the graph of $F(x) = 2^x - 3$.

6. Explain how to use the graph of $f(x) = 2^x$ to produce the graph of $G(x) = 2^{x-3}$.

7. Explain how to use the graph of $f(x) = 2^x$ to produce the graph of $M(x) = 2(2^x)$.

8. Explain how to use the graph of $f(x) = 2^x$ to produce the graph of $N(x) = 2^{-x}$.

Natural Exponential Functions

Definition of *e*
The letter ***e*** represents the number that $\left(1+\frac{1}{n}\right)^n$ approaches as n increases without bound.

Definition of the Natural Exponential Function
For all real numbers x, the function defined by $f(x)=e^x$ is called the **natural exponential function**.

The e^x is located above the LN key on you calculator.

9. A cup of coffee is heated to $160°F$ and placed in a room that maintains a temperature of $70°F$. The temperature T of the coffee, in degrees Fahrenheit, after t minutes is given by $T=70+90e^{-0.0485t}$.

   a. Find the temperature of the coffee, to the nearest degree, 20 minutes after it is placed in the room.

   b. Using your calculator determine when the temperature of the coffee will reach $90°F$.

10. The weekly revenue R, in dollars, form the sale of a product varies with time according to the function $R(x)=\dfrac{1760}{8+14e^{-0.03x}}$, where x is the number of weeks that have passed since the product was put on the market. What will the weekly revenue approach as time goes by?

Section 4.3: Logarithmic Functions and Their Applications

Targets: I can transition between logarithmic functions to exponential functions and vice versa.

I can accurately graph logarithmic function.

I can accurately find the domain of logarithmic functions.

I can use common and natural logarithmic functions to solve application problems accurately.

Definition of a Logarithm and a Logarithmic Function
If $x > 0$ and b is a positive constant $(b \neq 1)$, then $y = \log_b x$ if and only if $b^y = x$ The notation $\log_b x$ is read "the **logarithm** (or log) base *b* of *x*." The function defined by $f(x) = \log_b x$ is a **logarithm function** with the base *b*. This function is the inverse of the exponential function $g(x) = b^x$. **When using a calculator** $\log x$ **is base 10.**

Change of Base Formula
Use the change of base formula to put a logarithmic function into the calculator that is not base 10. Example: $\log_2 x = \dfrac{\log x}{\log 2}$

Composition of Logarithmic and Exponential Functions
Let $g(x) = b^x$ and $f(x) = \log_b x$ $(x > 0,\ b > 0,\ b \neq 1)$. Then $g(f(x)) = b^{\log_b x} = x$ and $f(g(x)) = \log_b b^x = x$

Definition of Exponential Form and Logarithmic Form
The **exponential form** of $y = \log_b x$ is $b^y = x$. The **logarithmic form** of $b^y = x$ is $y = \log_b x$.

Write each equation in its exponential form.

1. $3 = \log_2 8$
2. $2 = \log_{10}(x+5)$
3. $\log_e x = 4$
4. $\log_b b^3 = 3$

Write each equation in its logarithmic form.

5. $3^2 = 9$
6. $5^3 = x$
7. $a^b = c$
8. $b^{\log_b 5} = 5$

Basic Logarithmic Properties			
1. $\log_b b = 1$	2. $\log_b 1 = 0$	3. $\log_b \left(b^x\right) = x$	4. $b^{\log_b x} = x$

Evaluate each of the following logarithmic expressions.

9. $\log_8 1$ 10. $\log_5 5$ 11. $3\log_2(2^4)$ 12. $\log_3 81^2$ 13. $\log_{3/4}\left(\frac{16}{9}\right)$ 14. $\log_2 \sqrt{8}$

Properties of $f(x) = \log_b x$

For all positive real numbers b, $b \neq 1$, the function $f(x) = \log_b x$ has the following properties

- The domain of f consists of the set of positive real numbers, and its range consists of the set of all real numbers.
- The graph of f has an x-intercept of (1,0) and passes through (b, 1).
- If $b > 1$, f is an increasing function and its graph is asymptotic to the negative y-axis. $\left[\text{As } x \to \infty, f(x) \to \infty, \text{and as } x \to 0 \text{ from the right}, f(x) \to -\infty.\right]$
- If $0 < b < 1$, f is a decreasing function and its graph is asymptotic to the positive y-axis. $\left[\text{As } x \to \infty, f(x) \to -\infty, \text{and as } x \to 0 \text{ from the right}, f(x) \to \infty.\right]$

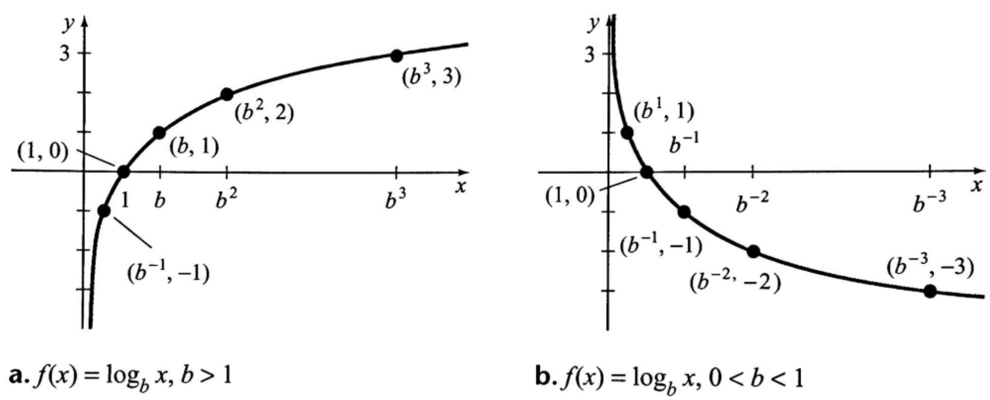


Figure 4.28

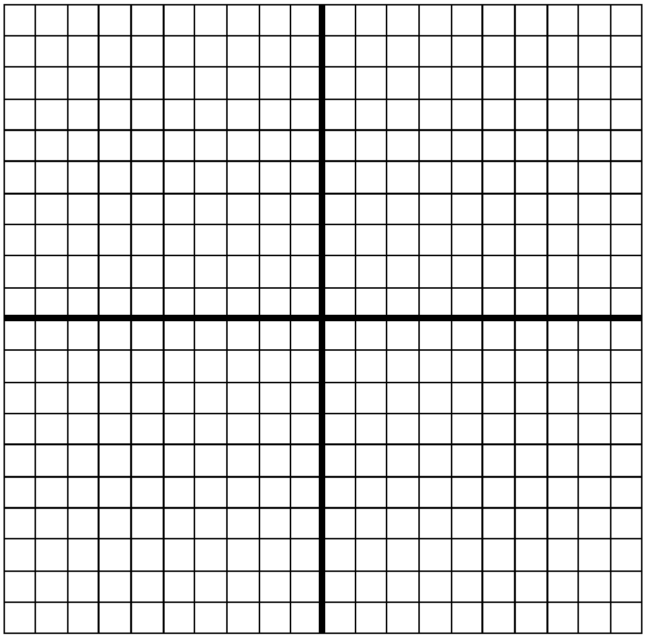
Common and Natural Logarithms

Definition of Common and Natural Logarithms

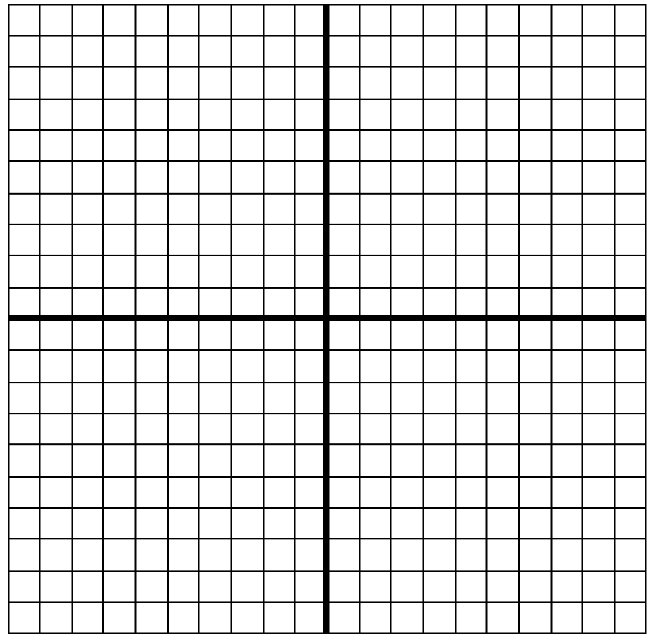
- The function defined by $f(x) = \log_{10} x$ is called the **common logarithmic function**. It is customarily written as $f(x) = \log x$, without stating the base.
- The function defined by $f(x) = \log_e x$ is called the **natural logarithmic function**. It is customarily written as $f(x) = \ln x$.

Graphing Logarithmic Functions

15. Graph $f(x) = \log_3 x$.



16. Graph $y = \log_{2/3} x$.



Find the domain of each of the following logarithmic functions.

17. $f(x) = \log_6(x-3)$

18. $F(x) = \log_2\left(x^2 + 4x + 3\right)$

19. $R(x) = \log_5\left(\dfrac{x}{8-x}\right)$

Translation of Logarithmic Functions

20. Explain how to use the graph of

a. $f(x) = \log_4 x$ to produce the graph of $f(x) = \log_4(x+3)$.

b. $f(x) = \log_4 x$ to produce the graph of $f(x) = \log_4 x + 3$.

21. In the study *The Pace of Life*, M. H. Bornstein and H. G. Bornstein (*Natural*, Vol. 259, pp. 557-558, 1976) reported that as the population of a city increases, the average walking speed of a pedestrian also increases. An approximate relation between the average pedestrian walking speed s, in miles per hour, and the population x, in thousands, of a city is given by the function $s(x) = 0.37 \ln x + 0.05$.

a. Determine the average walking speed, to the nearest tenth of a mile per hour, in San Francisco, which has a population of 805,000, and in Green Bay, Wisconsin, which has a population of 101,000.

b. Estimate the population of a city for which the average pedestrian walking speed is 3.1 miles per hour. Round to the nearest hundred thousand.

4.4 Properties of Logarithms and Logarithmic Scales Notes

Targets: I can use the properties of logarithms to simplify problems.

I can use the change of base formula to evaluate logarithms.

Properties of Logarithms	
In the following properties, b, M, and N are positive real numbers $(b \neq 1)$	
Product Property	$\log_b(MN) = \log_b M + \log_b N$
Quotient Property	$\log_b \frac{M}{N} = \log_b M - \log_b N$
Power Property	$\log_b(M^p) = p \log_b M$
Logarithm-of-each-side Property	$M = N$ implies $\log_b M = \log_b N$
One-to-one Property	$\log_b M = \log_b N$ implies $M = N$

Use the properties of logarithms to expand the following logarithmic expressions. Assume all variable expressions represent positive real numbers. When possible, evaluate logarithmic expressions.

1. $\log_5(xy^2)$

2. $\ln\left(\frac{e\sqrt{y}}{z^3}\right)$

3. $\log\left(\frac{100}{xy^3}\right)$

4. $\log \sqrt[3]{x\sqrt{y}}$

5. $\log_5 \left(\dfrac{\sqrt[3]{xy}}{z} \right)$

6. $\ln \left(\dfrac{x\sqrt{y}}{z^{-2}} \right)$

Use the properties of logarithms to rewrite each expression as a single logarithm with a coefficient of 1. Assume all variable expressions represent positive real numbers.

7. $2\ln x + \dfrac{1}{2}\ln(x+4)$

8. $\log_5 (x^2 - 4) + 3\log_5 y - \log_5 (x-2)^2$

9. $2\log x - \left[\log y + 3\log(z-1)\right]$

Change of Base Formula
If x, a, and b are positive real numbers with $a \neq 1$ and $b \neq 1$, then $$\log_b x = \frac{\log_a x}{\log_a b}$$ Because most calculators use only common logarithms (a = 10) or natural logarithms (a = e), the change-of-base formula is used most often in the following form. If x and b are positive real nubers and $b \neq 1$, then $$\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

Evaluate the logarithm. Round to the nearest ten-thousandth.

10. $\log_3 18$

11. $\log_{12} 400$

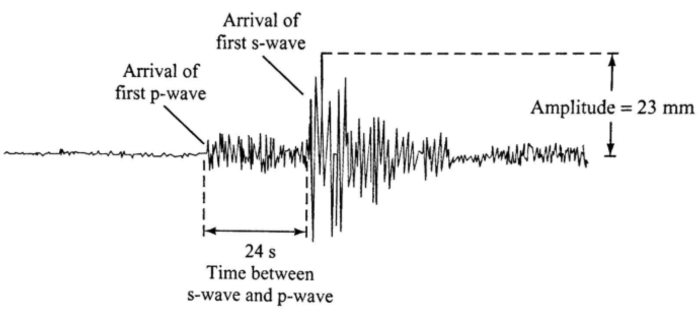
Richter Scale Magnitude of an Earthquake
An earthquake with an intensity of I has a **Richter scale magnitude** of $M = \log\left(\frac{I}{I_0}\right)$ where I_0 is the measure of the intensity of a zero-level earthquake.

12. Find the Richter scale magnitude (to the nearest tenth) of the November 2011 Oklahoma earthquake that had an intensity of $I = 398{,}107 I_0$.

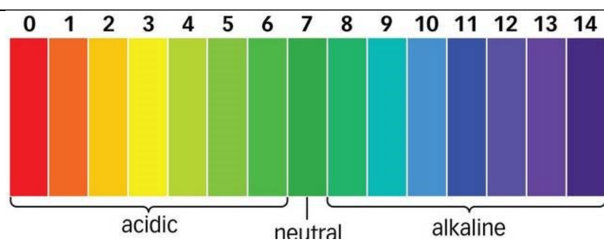
13. The 1960 Chile earthquake had a Richter scale magnitude of 9.5. The 1989 San Francisco earthquake had a Richter scale magnitude of 7.1. Compare the intensities of the earthquakes.

Amplitude-Time-Difference Formula
The Richter scale magnitude M of an earthquake is given by $M = \log A + 3\log 8t - 2.92$ where A is the amplitude, in millimeters of the s-waves on a seismogram and t is the difference in time, in seconds, between the s-waves and the p-waves.

14. Find the Richter scale magnitude of the earthquake that produced the seismogram in the figure below.



Definition of the pH of a Solution
The **pH of a solution** with a hydronium-ion concentration of $\left[H^+\right]$ mole per liter is given by $pH = -\log\left[H^+\right]$.



15. Find the pH of each liquid and determine whether it is an acid or base. Round to the nearest tenth.
   a. Orange juice with $\left[H^+\right] = 2.8\times10^{-4}$ mole per liter
   b. Milk with $\left[H^+\right] = 3.97\times10^{-7}$ mole per liter
   c. Rainwater with $\left[H^+\right] = 6.31\times10^{-5}$ mole per liter

Section 4.5 Exponential & Logarithmic Equations

Targets: I can solve exponential and logarithmic equations accurately.

Equality of Exponents Theorem
If $b^x = b^y$, then $x = y$, provided $b > 0$ and $b \neq 1$.

Solve the exponential equation.

1. $2^{x+1} = 32$
2. $3^{x-2} = 81$
3. $5^x = 40$
4. $4^x = 25$

5. $3^{2x-1} = 5^{x+2}$
6. $5 + e^{x+1} = 20$
7. $\dfrac{2^x + 2^{-x}}{2} = 3$

Solve the logarithmic equation.

8. $\log(3x-5)=2$

9. $\log(9x+1)=3$

10. $\log(2x)-\log(x-3)=1$

11. $\log x+\log(x+15)=2$

12. $\ln(3x+8)=\ln(2x+2)+\ln(x-2)$

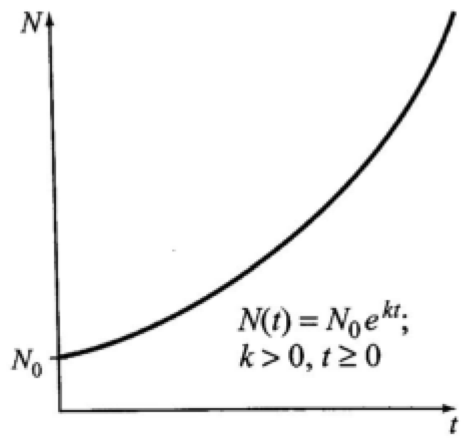
13. During a free fall portion of a jump, the time *t*, in seconds, required for a sky diver to reach a velocity *v*, in feet per second is given by $t=-\frac{175}{32}\ln\left(1-\frac{v}{175}\right),\ 0\le v<175$.

   a. Determine the velocity of the diver after 5 seconds.
   b. The graph of *t* has a vertical asymptote at $v=175$. Explain the meaning of the vertical asymptote in the context of this example.

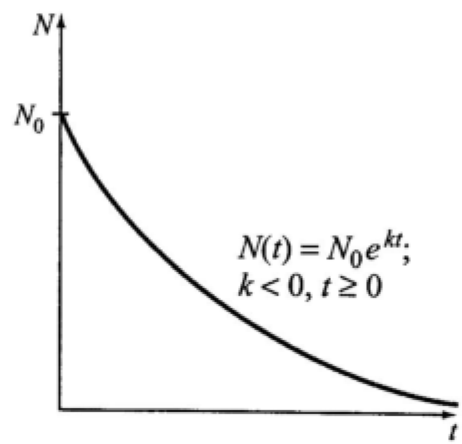
4.6 Exponential Growth and Decay

Targets: I can solve application problems using the exponential growth/decay function.
I can solve carbon dating application problems using the carbon dating function.
I can solve application problems using the compound interest formula.

Exponential Growth and Decay
If a quantity N increases or decreases at a rate proportional to the amount present at time t, then the quantity can be modeled by $N(t)=N_0e^{kt}$ where N_0 is the value of N at time $t=0$ and k is a constant called the **growth rate constant**. • If k is positive, N increases as t increases and $N(t)=N_0e^{kt}$ is called the **exponential growth function**. • If k is negative, N decreases as t increases and $N(t)=N_0e^{kt}$ is called the **exponential decay function**.



Exponential growth function



Exponential decay function

1. The population of a city is growing exponentially. The population of the city was 16,400 in 2003 and 20,200 in 2013.
   a. Find the exponential growth function that models the population growth of the city.
   b. Use the function from **a** to predict, to the nearest 100, the population of the city in 2018.

Table 4.11

Isotope	Half-Life
Carbon (^{14}C)	5730 years
Radium (^{226}R)	1660 years
Polonium (^{210}Po)	138 days
Phosphorus (^{32}P)	14 days
Polonium (^{214}Po)	$1/10,000$ of a second

2. Find the exponential decay function for the amount of phosphorus (^{32}P) that remains in a sample after t days.

Carbon Dating

The percent of carbon-14 present at time t, in years is $P(t) = 0.5^{t/5730}$. This formula can be used to find the percentage of any radioactive isotope that remains after a certain amount of time has passed by replacing 5730 with the half-life for the isotope.

3. Estimate the age of a bone if it now has 85% of the carbon-14 it had at time $t = 0$.

Compound Interest	
Compound Interest Formula	**Continuous Compound Interest Formula**
A principal P invested at an annual interest rate r, expressed as a decimal and compounded n times per year for t years, produces the balance $$A = P\left(1+\frac{r}{n}\right)^{nt}$$	If an account with principal P and annual interest rate r is compounded continuously for t years, then the balance is $$A = Pe^{rt}$$

4. Find the balance if $1,000 is invested at an annual interest rate of 10% for 2 years compounded on the following basis.
   a. Monthly
   b. Daily

5. Find the balance after 4 years on $800 invested at an annual rate of 6% compounded continuously.

6. Find the time required for money invested at annual rate of 6% to double in value is the investment is compounded on the following basis.
   a. Semiannually
   b. Continuously