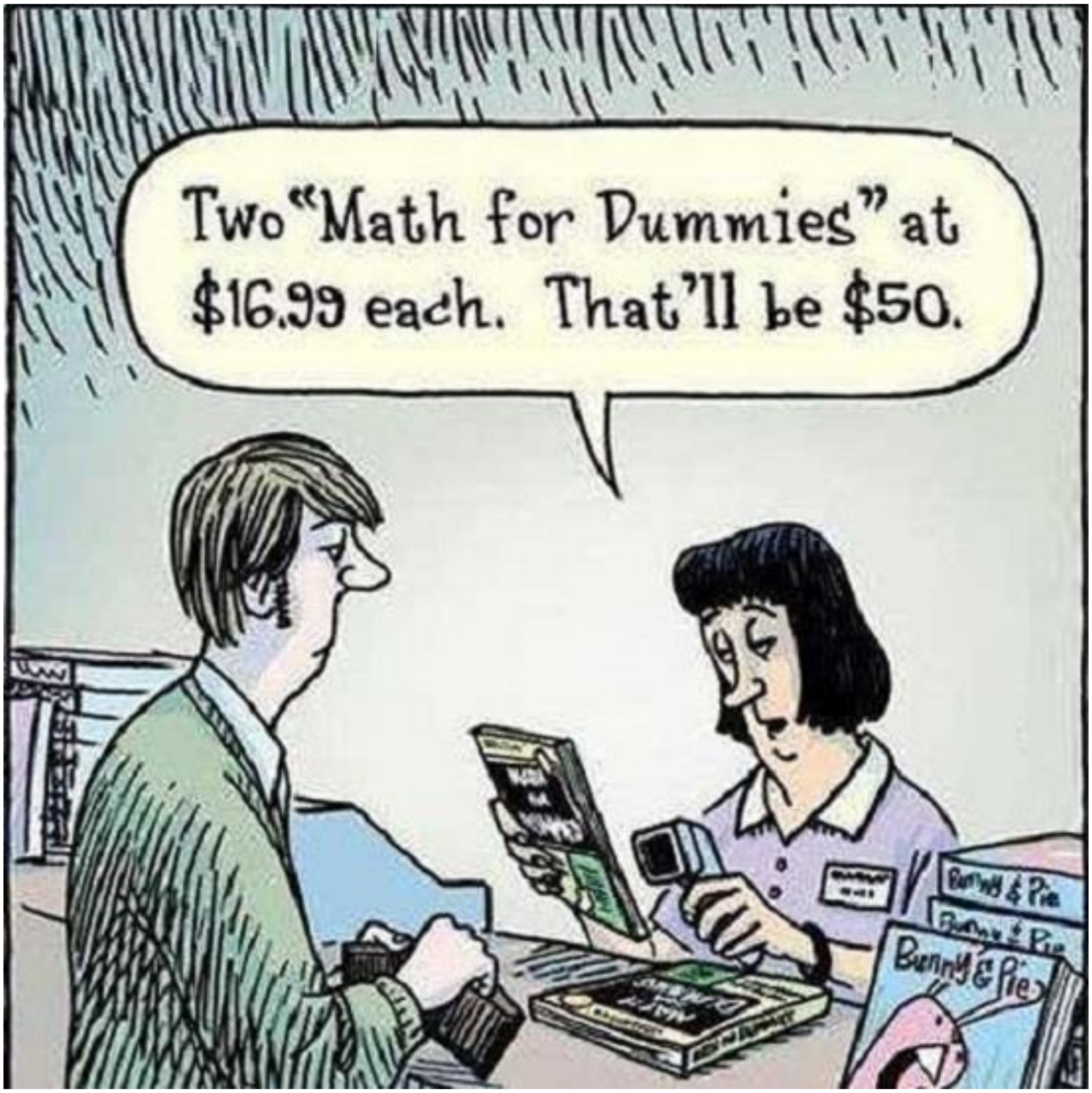
Chapter 3 Notes

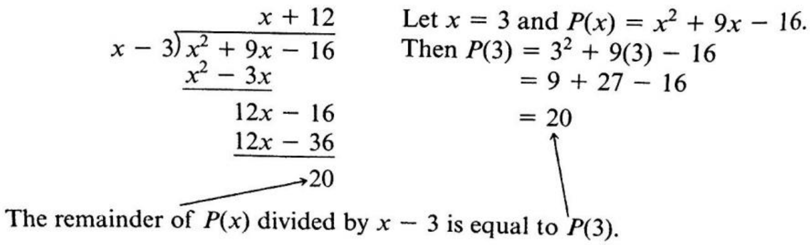


Polynomial and Rational Functions

3.1 Remainder Theorem and Factor Theorem

Targets: I can use long division and synthetic division to determine if the binomial is a factor of the polynomial .

Remainder Theorem
If a polynomial $P(x)$ is divided by $x-c$, then the remainder equals $P(c)$.



Factor Theorem
A polynomial $P(x)$ has a factor $(x-c)$ if and only if $P(c)=0$. That is, $(x-c)$ is a factor of $P(x)$ if and only if c is a ***zero*** of P.
• A polynomial $P(x)$ has a factor of $(x-c)$ if and only if $P(x)$ has a ***remainder*** of ***zero*** when divided by $(x-c)$.

Division of Polynomials

Divide.

1. $\dfrac{16x^3-8x^2+12x}{4x}$

2. $\dfrac{x^4-2x^3-7x-110}{x-4}$

3. $\dfrac{2x^5+5x^2-9x-x^3+6}{2x^2+2x-3}$

The quotient of $P(x)$ is called the ***reduced polynomial***, or a ***depressed polynomial*** because its degree is 1 less than the degree of $P(x)$.

Synthetic Division

Use synthetic Division to divide.

4. $\dfrac{x^4 - 4x^2 + 7x + 15}{x+4}$

5. $\dfrac{2x^3 - 56x + x^4 + 7}{x-3}$

Use synthetic division to determine whether the given binomial is a factor of $P(x)$.

6. $P(x) = x^4 + x^3 - 21x^2 - x + 20, \quad x+5$

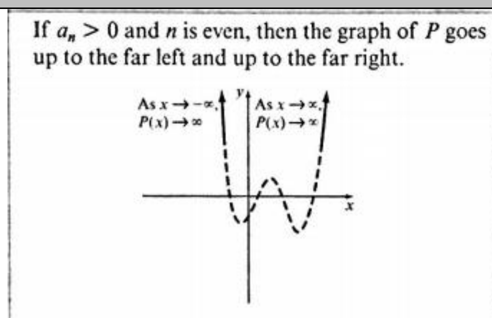
7. $P(x) = x^4 + x^3 - 21x^2 - x + 20, \quad x-2$

3.2 Polynomial Functions of Higher Degree

Targets: I can determine the far-left and far-right behavior of the graph of a polynomial function.
I can accurately find relative minimum(s), relative maximum(s) and absolute minimum or maximum.

Polynomial Function $P(x)$	**Graph**
$P(x) = a$ (degree 0)	Horizontal line through $(0, a)$
$P(x) = ax + b$ (degree 1), $a \neq 0$	Line with *y*-intercept $(0, b)$ and slope *a*
$P(x) = ax^2 + bx + c$ (degree 2), $a \neq 0$	Parabola with vertex $\left(-\frac{b}{2a}, P\left(-\frac{b}{2a}\right)\right)$

End Behavior

<table>
<tr><th colspan="3">The far-left and far-right behavior of a graph of the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + . . . + a_1 x + a_0$ can be determined by examining its leading term $a_n x^n$.</th></tr>
<tr><th></th><th>n Is Even</th><th>n Is Odd</th></tr>
<tr><td>$a_n > 0$</td><td>


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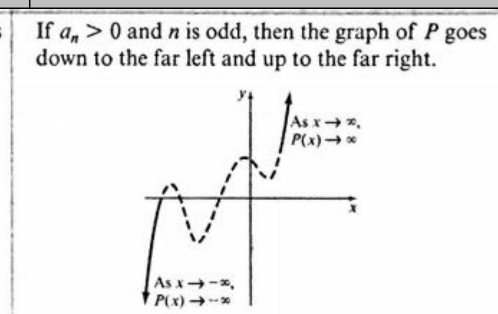

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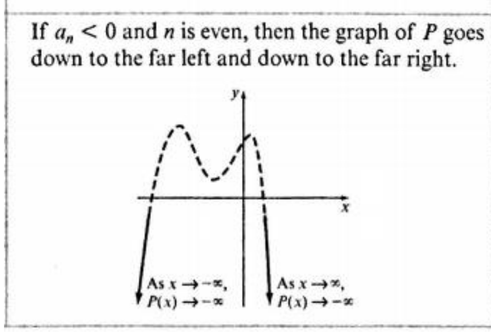
<tr><td>$a_n < 0$</td><td>


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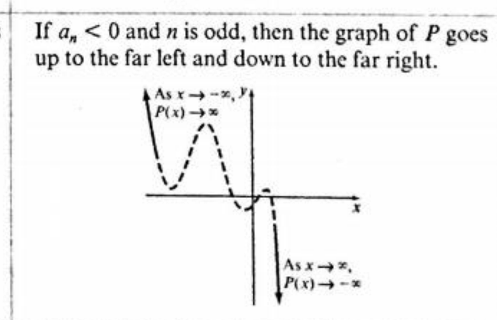

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Examine the leading term to determine the far-left and far-right behavior of the graph of each polynomial function.

1. $P(x) = x^3 - x$
2. $S(x) = \frac{1}{2}x^4 - \frac{5}{2}x^2 + 2$
3. $T(x) = -2x^3 + x^2 + 7x - 6$
4. $U(x) = 9 + 8x^2 - x^4$

Maximum and Minimum Values

Absolute Minimum – the smallest range value (y-value) of the function.

Absolute Maximum – the largest range value (y-value) of the function.

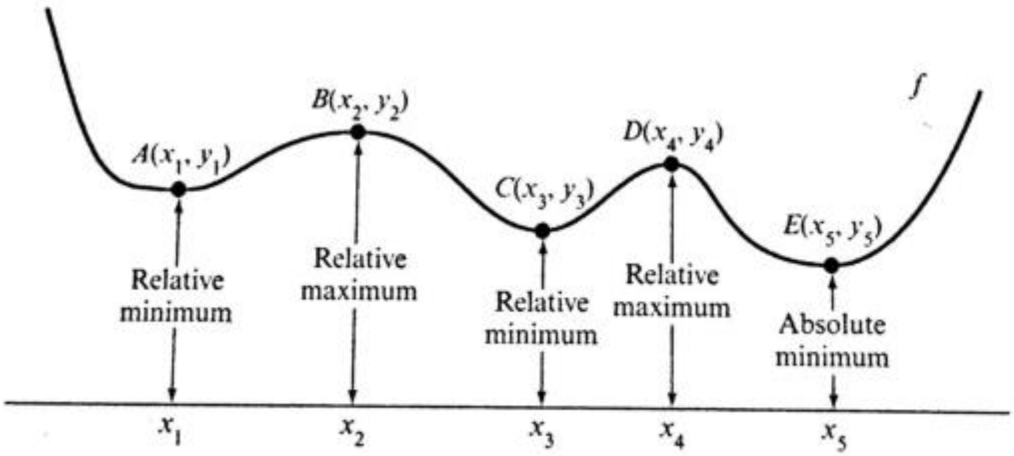
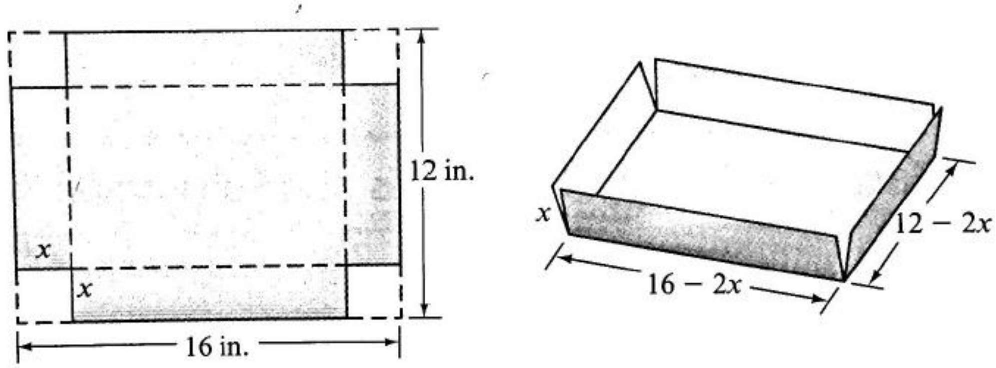


Figure 3.11

Find all relative and absolute extreme values.

5. $P(x) = 0.3x^3 - 2.8x^2 + 6.4x + 2$

6. A rectangular piece cardboard measures 12 inches by 16 inches. An open box is formed by cutting squares that measure x inches from each of the corners of the cardboard and folding up the sides, as shown below

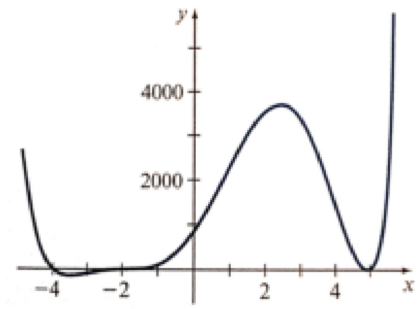


a. Express the volume V of the box as a function of x.

b. Determine (to the nearest tenth of an inch) the x value that maximizes the volume.

3.3 Zeros of Polynomial Functions

Targets: I can use Rational Zero Theorem, Descartes' Rule of Signs, and Synthetic division to determine all possible positive and negative real zeros of a polynomial functions.

Definition of Multiple Zeros of a Polynomial Function
If a polynomial function P has $(x-r)$ has a factor *k* times, then *r* is a **zero of multiplicity *k*** of the polynomial function P.
Example The graph of the polynomial function $P(x)=(x-5)^2(x+2)^3(x+4)$ is shown in Figure 3.18. This polynomial function has • 5 as a zero of multiplicity 2 (aka double zero). • -2 as a zero of multiplicity 3 (aka triple zero). • -4 as a zero of multiplicity 1.  $P(x) = (x-5)^2(x+2)^3(x+4)$ **Figure 3.18**

Number of Zeros of a Polynomial Function
A polynomial function P of the degree *n* has at most *n* zeros, where each zero of multiplicity *k* is counted *k* times.

Rational Zero Theorem
If $P(x)=a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ has *integer* coefficients $(a \neq 0)$ and $\frac{p}{q}$ is a rational zero (in simplest form) of P, then • p is a factor of the constant term a_0. • q is a factor of the leading coefficient a_n.

Use the Rational Zero Theorem to list all possible rational zeros of the polynomial function.

1. $P(x)=4x^4+x^3-40x^2+38x+12$

2. $P(x)=4x^3-8x^2-9x+18$

Finding Upper and Lower Bounds

According to the Upper- and Lower-Bound Theorem, what is the smallest positive integer that is an upper bound and the largest negative integer that is a lower bound of the real zeros of $P(x) = 2x^3 + 7x^2 - 4x - 14$?

Solution

To find the smallest positive-integer upper bound, use synthetic division with 1, 2, . . . , as test values.

1	2	7	−4	−14
		2	9	5
	2	9	5	−9

2	2	7	−4	−14	
		4	22	36	
	2	11	18	22	• No negative numbers

According to the Upper- and Lower-Bound Theorem, 2 is the smallest positive-integer upper bound.

Now find the largest negative-integer lower bound.

−1	2	7	−4	−14
		−2	−5	9
	2	5	−9	−5

−2	2	7	−4	−14
		−4	−6	20
	2	3	−10	6

−3	2	7	−4	−14
		−6	−3	21
	2	1	−7	7

−4	2	7	−4	−14	
		−8	4	0	
	2	−1	0	−14	• Alternating signs

According to the Upper- and Lower-Bound Theorem, −4 is the largest negative-integer lower bound.

3. According to the Upper- and Lower-Bound Theorem, what is the smallest positive integer that is an upper bound and the largest negative integer that is a lower bound of the real zeros of $P(x) = 6x^3 - 7x^2 - 14x + 15$?

Descartes' Rule of Signs
Let P be a polynomial function with real coefficients and with the terms arranged in order of decreasing powers of *x*. • The number of positive real zeros of P is equal to the number of variations in sign of $P(x)$ or that number decreased by an even number. • The number of negative real zeros of P is equal to the number of variations in sign of $P(-x)$ or to that number decreased by an even integer.

Use Descartes' Rule of Signs to determine both the number of possible positive and the number of possible negative real zeros of each polynomial function.

4. $P(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$

5. $P(x) = 2x^5 + 3x^3 + 5x^2 + 8x + 7$

Putting it ALL Together

Guidelines for finding the Zeros of a Polynomial Function with Integer Coefficients
1. ***Gather general information*** Determine the degree *n* of the polynomial function. The number of distinct zeros of the polynomial function is at most *n*. Apply ***Descartes' Rule of Signs*** to find the possible number of positive zeros and the possible number of negative zeros. **2.** ***Check suspects/candidates*** Apply the **Rational Zero Theorem** to list rational numbers that are possible zeros. Use your graphing calculator to determine one number from you list that is a zero, then use ***synthetic division*** to get the depressed polynomial. **3.** ***Work with the reduced polynomials*** Each time a zero is found, you obtain a reduced polynomial. • If a reduced polynomial is of degree 2, find its zeros either by factoring or by applying the quadratic formula. • If the degree of a reduced polynomial is 3 or greater, repeat the preceding steps for this polynomial.

Find the zeros of the following polynomials.

6. $P(x) = 3x^4 + 23x^3 + 56x^2 + 52x + 16$

7. $P(x) = 2x^4 - 5x^3 - 3x^2 + 7x + 3$

3.4 Fundamental Theorem of Algebra

Targets: I can find the zeros of a polynomial function and use the Linear Factor Theorem to write the function as the product of its leading coefficient and linear factors.
I can find the polynomial function given the zeros and the degree of the function.

Fundamental Theorem of Algebra
If P is a polynomial function of degree $n \geq 1$ with complex coefficients, then P has at least one complex zero.

Linear Factor Theorem
If P is a polynomial function of degree $n \geq 1$ with leading coefficient $a_n \neq 0$, $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x^1 + a_0$ then P has exactly n linear factors $P(x) = a_n(x - c_1)(x - c_2)...(x - c_n)$ where $c_1,\ c_2,\ ...\ ,\ c_n$ are complex numbers.

Number of Zeros of a Polynomial Function Theorem
If P is a polynomial function of degree $n \geq 1$, then P has exactly n complex zeros, provided each zero is counted according to its multiplicity.

Conjugate Pair Theorem
If $a + bi$ $(b \neq 0)$ is a complex zero of a polynomial function *with real coefficients*, then the conjugate $a - bi$ is also a complex zero of the polynomial function.

Find all the zeros of each polynomial function and write each function as a product of its leading coefficient and its linear factors.

1. $P(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

2. $P(x) = 2x^4 + x^3 + 39x^2 + 136x - 78$

Use the given zero to find the remaining zeros of each polynomial functions.

3. $P(x) = x^4 - 4x^3 + 14x^2 - 36x + 45;\ 2 + i$

4. $P(x) = x^4 - 8x^3 + 46x^2 - 40x + 205;\ 4 + 5i$

Find each polynomial function that satisfies the given conditions.

5. Zeros: $\frac{1}{2}$, 2, and -3; degree 3

6. Zeros: $2i$ and $3 - 7i$; degree 4

3.5 Graphs of Rational Functions and Their Applications

Targets: I can find the vertical, horizontal and slant asymptotes of a rational function.

Vertical and Horizontal Asymptotes

If $P(x)$ and $Q(x)$ are polynomials, then the function F is given by $F(x) = \frac{P(x)}{Q(x)}$ is called a **rational function**. **Vertical Asymptotes:** Are the real (not imaginary) zeros of the denominator of the rational function.

Example: $F(x) = \frac{x+2}{(x+3)(x-2)}$ Vertical Asymptotes: $x = -3$ and $x = 2$

Horizontal Asymptotes: Let $F(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$ be a rational function with numerator of degree *n* and denominator of degree *m*.

1. If $n < m$, then the *x*-axis, which is the line given by $y = 0$, is the horizontal asymptote of the graph of F.
2. If $n = m$, then the line given by $y = \frac{a_n}{b_m}$ is the horizontal asymptote of the graph of F.
3. If $n > m$, then the graph of F has no horizontal asymptote.

Find the vertical asymptote of each rational function.

1. $f(x) = \frac{x+1}{x^2 + 2x - 3}$

2. $g(x) = \frac{x}{x^2 - x - 6}$

3. $f(x) = \frac{x^3}{x^2 + 1}$

4. $G(x) = \frac{x^3 + 2x^2}{x^2 - 2x - 2}$

Find the horizontal asymptote of each rational function.

5. $g(x)=\dfrac{4x^2+1}{3x^2}$

6. $f(x)=\dfrac{2x+3}{x^2+1}$

7. $H(x)=\dfrac{(2x+3)(x-1)}{(2+5x)(3-x)}$

8. $h(x)=\dfrac{x^3+1}{x-2}$

Slant Asymptotes
The line given by $y=mx+b,\ m\neq 0$, is a **slant asymptote** of the graph of a function F provided $F(x)\to mx+b$ as $x\to\infty$ or $x\to-\infty$.   **Slant Asymptotes Theorem:** The rational function given by $F(x)=P(x)/Q(x)$, where $P(x)$ and $Q(x)$ have no common factors, has a slant asymptote if the degree of $P(x)$ is 1 greater than the degree of $Q(x)$.

Find the slant asymptote of each rational function.

7. $f(x)=\dfrac{2x^3+5x^2+1}{x^2+x+3}$

8. $h(x)=\dfrac{-x^3-4x^2-x+4}{x^2-x-6}$