7.5 Solving Rational Equations

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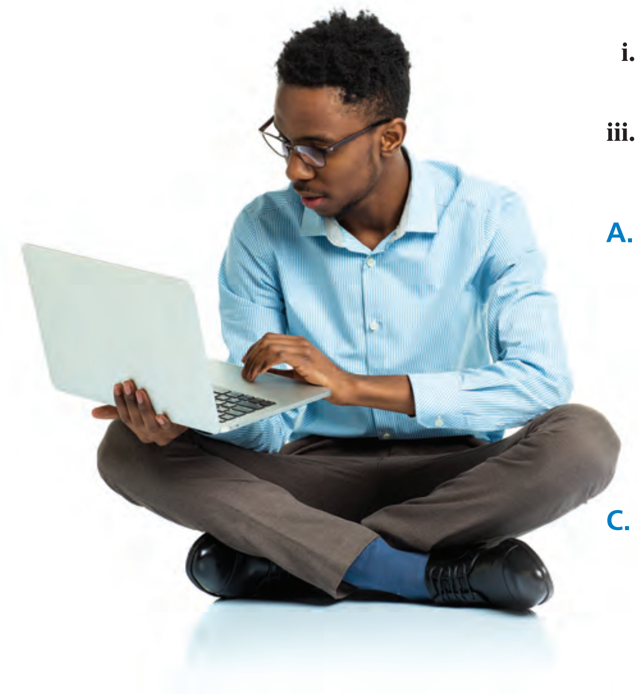
Learning Target Solve rational equations.

Success Criteria
- I can solve rational equations by cross multiplying and by using least common denominators.
- I can identify extraneous solutions of rational equations.
- I can solve real-life problems using inverses of rational functions.

EXPLORE IT! Solving Rational Equations



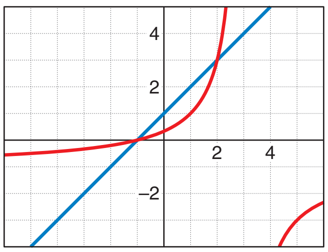
Work with a partner.

a. Match each equation with its related graph. Explain your reasoning. Then use the graph to approximate the solution(s) of the equation.

i. $\dfrac{2}{x-1} = 1$ **ii.** $\dfrac{-x-1}{x-3} = x + 1$

iii. $\dfrac{1}{x} = x^2$ **iv.** $\dfrac{1}{x} = \dfrac{-1}{x-2}$

A.



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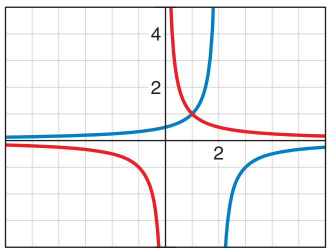

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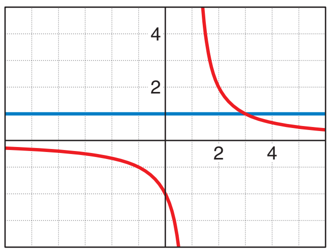

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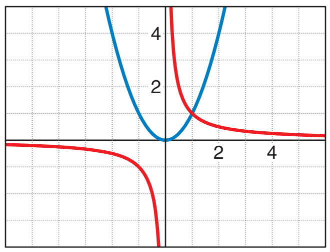

b. Solve the proportion $\dfrac{3}{4} = \dfrac{8}{x}$ algebraically. Explain your method. Then solve the rational equations in part (a) using the same algebraic method. Compare your answers with your approximations in part (a).

c. A student solves the equation $\dfrac{5x}{x^2-4} = \dfrac{3}{x+2}$ as shown. Explain the student's method. Is the answer correct? Explain.

$$\frac{5x}{x^2-4} = \frac{3}{x+2}$$
$$5x^2 + 10x = 3x^2 - 12$$
$$2x^2 + 10x + 12 = 0$$
$$2(x+3)(x+2) = 0$$
So, $x = -3$ and $x = -2$.

Math Practice

Evaluate Results
Why is it always important to check your solutions in the original equation?

Solving by Cross Multiplying

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You can *cross multiply* to solve a rational equation when each side of the equation is a single rational expression.

EXAMPLE 1 Solving a Rational Equation by Cross Multiplying

WATCH

Solve $\frac{3}{x+1} = \frac{9}{4x+5}$.

SOLUTION

$$\frac{3}{x+1} = \frac{9}{4x+5}$$ Write original equation.

$$3(4x+5) = 9(x+1)$$ Cross multiply.

$$12x + 15 = 9x + 9$$ Distributive Property

$$3x + 15 = 9$$ Subtract $9x$ from each side.

$$3x = -6$$ Subtract 15 from each side.

$$x = -2$$ Divide each side by 3.

▶ The solution is $x = -2$. Check this in the original equation.

Check

$$\frac{3}{-2+1} \stackrel{?}{=} \frac{9}{4(-2)+5}$$

$$\frac{3}{-1} \stackrel{?}{=} \frac{9}{-3}$$

$$-3 = -3 \checkmark$$

EXAMPLE 2 Modeling Real Life

WATCH INFO

An *alloy* is formed by mixing two or more metals. Sterling silver is an alloy composed of 92.5% silver and 7.5% copper by weight. You have 15 ounces of 800 grade silver, which is 80% silver and 20% copper by weight. How much pure silver should you mix with the 800 grade silver to make sterling silver?

SOLUTION

$$\text{percent of copper in mixture} = \frac{\text{weight of copper in mixture}}{\text{total weight of mixture}}$$

$$\frac{7.5}{100} = \frac{(0.2)(15)}{15+x}$$ x is the amount of silver added.

$$7.5(15+x) = 100(0.2)(15)$$ Cross multiply.

$$112.5 + 7.5x = 300$$ Simplify.

$$7.5x = 187.5$$ Subtract 112.5 from each side.

$$x = 25$$ Divide each side by 7.5.

▶ You should mix 25 ounces of pure silver with the 15 ounces of 800 grade silver.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the equation by cross multiplying. Check your solution(s).

1. $\frac{3}{5x} = \frac{2}{x-7}$

2. $\frac{-4}{x+3} = \frac{5}{x-3}$

3. $\frac{1}{2x+5} = \frac{x}{11x+8}$

4. **WHAT IF?** You have 12 ounces of an alloy that is 90% silver and 10% copper by weight. How much pure silver should you mix with the alloy to make sterling silver?

Solving by Using the Least Common Denominator

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When a rational equation is not expressed as a proportion, you can solve it by multiplying each side of the equation by the least common denominator of the rational expressions.

EXAMPLE 3 **Solving Rational Equations by Using the LCD**

WATCH

Solve each equation.

a. $\frac{5}{x} + \frac{7}{4} = -\frac{9}{x}$ **b.** $1 - \frac{8}{x-5} = \frac{3}{x}$

SOLUTION

a.

$\frac{5}{x} + \frac{7}{4} = -\frac{9}{x}$ Write original equation.

$4x\left(\frac{5}{x} + \frac{7}{4}\right) = 4x\left(-\frac{9}{x}\right)$ Multiply each side by the LCD, $4x$.

$20 + 7x = -36$ Simplify.

$7x = -56$ Subtract 20 from each side.

$x = -8$ Divide each side by 7.

The solution is $x = -8$. Check this in the original equation.

Check

$\frac{5}{-8} + \frac{7}{4} \stackrel{?}{=} -\frac{9}{-8}$

$-\frac{5}{8} + \frac{14}{8} \stackrel{?}{=} \frac{9}{8}$

$\frac{9}{8} = \frac{9}{8}$ ✓

b.

$1 - \frac{8}{x-5} = \frac{3}{x}$ Write original equation.

$x(x-5)\left(1 - \frac{8}{x-5}\right) = x(x-5) \cdot \frac{3}{x}$ Multiply each side by the LCD, $x(x - 5)$.

$x(x-5) - 8x = 3(x-5)$ Simplify.

$x^2 - 5x - 8x = 3x - 15$ Distributive Property

$x^2 - 16x + 15 = 0$ Write in standard form.

$(x-1)(x-15) = 0$ Factor.

$x = 1$ or $x = 15$ Zero-Product Property

The solutions are $x = 1$ and $x = 15$. Check these in the original equation.

Check

$1 - \frac{8}{1-5} \stackrel{?}{=} \frac{3}{1}$ Substitute for x. $1 - \frac{8}{15-5} \stackrel{?}{=} \frac{3}{15}$

$1 + 2 \stackrel{?}{=} 3$ Simplify. $1 - \frac{4}{5} \stackrel{?}{=} \frac{1}{5}$

$3 = 3$ ✓ $\frac{1}{5} = \frac{1}{5}$ ✓

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the equation by using the LCD. Check your solution(s).

5. $\frac{15}{x} + \frac{4}{5} = \frac{7}{x}$

6. $\frac{3x}{x+1} - \frac{5}{2x} = \frac{3}{2x}$

7. $\frac{4x+1}{x+1} = \frac{12}{x^2-1} + 3$

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When solving a rational equation, you may obtain solutions that are extraneous. Be sure to check for extraneous solutions by checking your solutions in the *original* equation.

EXAMPLE 4 Solving an Equation with an Extraneous Solution

WATCH

Solve $\frac{6}{x-3} = \frac{8x^2}{x^2-9} - \frac{4x}{x+3}$.

SOLUTION

Write each denominator in factored form. The LCD is $(x + 3)(x - 3)$.

$$\frac{6}{x-3} = \frac{8x^2}{(x+3)(x-3)} - \frac{4x}{x+3}$$

$$(x+3)(x-3) \bullet \frac{6}{x-3} = (x+3)(x-3) \bullet \frac{8x^2}{(x+3)(x-3)} - (x+3)(x-3) \bullet \frac{4x}{x+3}$$

$$6(x+3) = 8x^2 - 4x(x-3)$$

$$6x + 18 = 8x^2 - 4x^2 + 12x$$

$$0 = 4x^2 + 6x - 18$$

$$0 = 2x^2 + 3x - 9$$

$$0 = (2x - 3)(x + 3)$$

$$2x - 3 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = \frac{3}{2} \quad \text{or} \quad x = -3$$

Check

Check $x = \frac{3}{2}$:

$$\frac{6}{\frac{3}{2} - 3} \stackrel{?}{=} \frac{8\left(\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2 - 9} - \frac{4\left(\frac{3}{2}\right)}{\frac{3}{2} + 3}$$

$$\frac{6}{-\frac{3}{2}} \stackrel{?}{=} \frac{18}{-\frac{27}{4}} - \frac{6}{\frac{9}{2}}$$

$$-4 \stackrel{?}{=} -\frac{8}{3} - \frac{4}{3}$$

$$-4 = -4 \checkmark$$

Check $x = -3$:

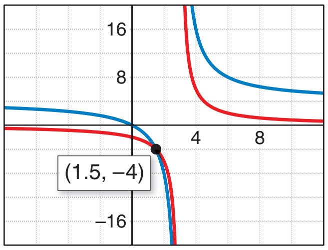
$$\frac{6}{-3 - 3} \stackrel{?}{=} \frac{8(-3)^2}{(-3)^2 - 9} - \frac{4(-3)}{-3 + 3}$$

$$\frac{6}{-6} \stackrel{?}{=} \frac{72}{0} - \frac{-12}{0} \quad ✗$$

Division by zero is undefined.

ANOTHER WAY

You can also graph each side of the equation and find the x-value where the graphs intersect.



▶ The apparent solution $x = -3$ is extraneous. So, the only solution is $x = \frac{3}{2}$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the equation. Check your solution(s).

8. $\frac{3}{x-1} - 1 = \frac{6}{x^2-1}$

9. $\frac{9}{x-2} + \frac{6x}{x+2} = \frac{9x^2}{x^2-4}$

Using Inverses of Functions

EXAMPLE 5 Finding the Inverse of a Rational Function

WATCH

Consider the function $f(x) = \dfrac{2}{x+3}$. Determine whether the inverse of f is a function. Then find the inverse.

SOLUTION

Graph the function f. Notice that no horizontal line intersects the graph more than once. So, the inverse of f is a function. Find the inverse.

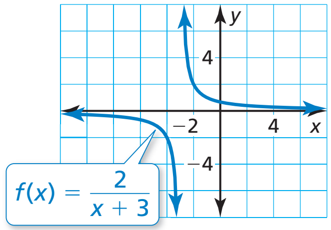
$y = \dfrac{2}{x+3}$ Set y equal to $f(x)$.

$x = \dfrac{2}{y+3}$ Switch x and y.

$x(y+3) = 2$ Cross multiply.

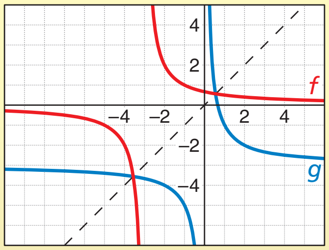
$y + 3 = \dfrac{2}{x}$ Divide each side by x.

$y = \dfrac{2}{x} - 3$ Subtract 3 from each side.



▶ So, the inverse of f is $f^{-1}(x) = \dfrac{2}{x} - 3$.

Check



EXAMPLE 6 Modeling Real Life

WATCH

In Section 7.2 Example 5, you wrote the function $c(x) = \dfrac{50x + 1000}{x}$ to represent the average cost (in dollars) of making x prosthetic hands using a 3-D printer. How many hands must be printed for the average cost per hand to fall to $90?

SOLUTION

Method 1 Substitute 90 for $c(x)$ and solve.

$90 = \dfrac{50x + 1000}{x}$

$90x = 50x + 1000$

$40x = 1000$

$x = 25$

Method 2 Find $c^{-1}(x)$. Then evaluate $c^{-1}(90)$.

$y = \dfrac{50x + 1000}{x}$ Set y equal to $c(x)$.

$x = \dfrac{50y + 1000}{y}$ Switch x and y.

$xy = 50y + 1000$ Cross multiply.

$y = \dfrac{1000}{x - 50}$ Solve for y.

The inverse of c is $c^{-1}(x) = \dfrac{1000}{x-50}$.

$c^{-1}(90) = \dfrac{1000}{90 - 50} = 25$

▶ So, the average cost falls to $90 per hand after 25 hands are printed.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

10. Consider the function $f(x) = \dfrac{1}{x} - 2$. Determine whether the inverse of f is a function. Then find the inverse.

11. WHAT IF? How does the answer in Example 6 change when $c(x) = \dfrac{50x + 800}{x}$?

7.5 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–8, solve the equation by cross multiplying. Check your solution(s). *Example 1*

1. $\frac{4}{2x} = \frac{5}{x + 6}$

2. $\frac{9}{3x} = \frac{4}{x + 2}$

3. $\frac{6}{x - 1} = \frac{9}{x + 1}$

4. $\frac{8}{3x - 2} = \frac{2}{x - 1}$

5. $\frac{x}{2x + 7} = \frac{x - 5}{x - 1}$

6. $\frac{-2}{x - 1} = \frac{x - 8}{x + 1}$

7. $\frac{x^2 - 3}{x + 2} = \frac{x - 3}{2}$

8. $\frac{-1}{x - 3} = \frac{x - 4}{x^2 - 27}$

9. **MP PROBLEM SOLVING** A game show contestant has answered 37 out of 44 trivia questions correctly so far. Solve the equation $\frac{90}{100} = \frac{37 + x}{44 + x}$ to find the number of consecutive questions the contestant needs to answer correctly to raise the correct answer percentage to 90%.

10. **MP PROBLEM SOLVING** Your friend has 12 hits out of 60 times at-bat so far this baseball season. Solve the equation $0.360 = \frac{12 + x}{60 + x}$ to find the number of consecutive hits your friend needs to raise his batting average to 0.360.

11. **MODELING REAL LIFE** Brass is an alloy composed of 55% copper and 45% zinc by weight. You have 25 ounces of copper. How many ounces of zinc do you need to make brass? *Example 2*

12. **MODELING REAL LIFE** You have 0.2 liter of an acid solution whose acid concentration is 16 moles per liter. You want to dilute the solution with water so that its acid concentration is only 12 moles per liter. Use the given model to determine how many liters of water you should add to the solution.

$$\text{Concentration of new solution} = \frac{\text{Concentration of original solution} \cdot \text{Volume of original solution}}{\text{Volume of original solution} + \text{Volume of water added}}$$

MP STRUCTURE **In Exercises 13–16, identify the LCD of the rational expressions in the equation.**

13. $\frac{x}{x + 3} + \frac{1}{x} = \frac{3}{x}$

14. $\frac{5x}{x - 1} - \frac{7}{x} = \frac{9}{x}$

15. $\frac{2}{x + 1} + \frac{x}{x + 4} = \frac{1}{2}$

16. $\frac{4}{x + 9} + \frac{3x}{2x - 1} = \frac{10}{3}$

In Exercises 17–28, solve the equation by using the LCD. Check your solution(s). *Examples 3 and 4*

17. $\frac{3}{2} + \frac{1}{x} = 2$

18. $\frac{2}{3x} + \frac{1}{6} = \frac{4}{3x}$

19. $\frac{8}{x - 4} + \frac{4}{x} = \frac{2x}{x - 4}$

20. $\frac{2}{x - 3} + \frac{1}{x} = \frac{x - 1}{x - 3}$

21. $\frac{6x}{x + 4} + 4 = \frac{2x + 2}{x - 1}$

22. $\frac{10}{x} + 3 = \frac{x + 9}{x - 4}$

23. $\frac{18}{x^2 - 3x} - \frac{6}{x - 3} = \frac{5}{x}$

24. $\frac{10}{x^2 - 2x} + \frac{4}{x} = \frac{5}{x - 2}$

25. $\frac{x + 1}{x + 6} + \frac{1}{x} = \frac{2x + 1}{x + 6}$

26. $\frac{x + 3}{x - 3} + \frac{x}{x - 5} = \frac{x + 5}{x - 5}$

27. $\frac{5}{x} - 2 = \frac{2}{x + 3}$

28. $\frac{5}{x^2 + x - 6} = 2 + \frac{x - 3}{x - 2}$

ERROR ANALYSIS **In Exercises 29 and 30, describe and correct the error in the first step of solving the equation.**

29. ✗

$$\frac{5}{3x} + \frac{2}{x^2} = 1$$

$$x^2 \cdot \frac{5}{3x} + 3x \cdot \frac{2}{x^2} = 1 \cdot 3x^3$$

30. ✗

$$\frac{7x + 1}{2x + 5} + 4 = \frac{10x - 3}{3x}$$

$$(2x + 5)3x \cdot \frac{7x + 1}{2x + 5} + 4$$

$$= \frac{10x - 3}{3x} \cdot (2x + 5)3x$$

31. **COLLEGE PREP** Which of the following equations have an extraneous solution? Select all that apply.

Ⓐ $\frac{x}{2x + 3} = \frac{x - 2}{x - 6}$

Ⓑ $\frac{5}{x - 2} - \frac{2}{x} = \frac{x + 3}{x - 2}$

Ⓒ $\frac{x + 8}{x - 4} - 1 = \frac{6}{x}$

Ⓓ $\frac{15}{x^2 - 3x} - \frac{5}{x - 3} = \frac{7}{x}$

32. **OPEN-ENDED** Give an example of a rational equation that you would solve using cross multiplication and one that you would solve using the LCD. Explain your reasoning.

33. **MP PROBLEM SOLVING** You can clean a park in 2 hours. Working together, you and your friend can clean the park in 1.2 hours.

   a. Let t be the time (in hours) your friend takes to clean the park when working alone. Complete the table. (*Hint*: (Work done) = (Work rate) × (Time))

	Work rate	Time	Work done
You	$\frac{1 \text{ park}}{2 \text{ hours}}$	1.2 hours	
Friend		1.2 hours	

   b. Explain what the sum of the expressions represents in the last column. Write and solve an equation to find how long your friend takes to clean the park when working alone.

34. **DIG DEEPER** A kayaker paddles upstream for 2 miles and downstream for 2 miles. The speed of the current is 1 mile per hour. The entire trip takes 2 hours and 40 minutes. Write and solve an equation to find the average speed at which the kayaker paddles.

In Exercises 35–42, determine whether the inverse of f is a function. Then find the inverse. *Example 5*

35. $f(x) = \dfrac{2}{x - 4}$

36. $f(x) = \dfrac{7}{x + 6}$

37. $f(x) = \dfrac{3}{x} - 2$

38. $f(x) = \dfrac{5}{x} - 6$

39. $f(x) = \dfrac{4}{11 - 2x}$

40. $f(x) = \dfrac{8}{9 + 5x}$

41. $f(x) = \dfrac{1}{x^2} + 4$

42. $f(x) = \dfrac{1}{x^4} - 7$

43. **MODELING REAL LIFE** The recommended percent (in decimal form) of nitrogen (by volume) in the air that a diver breathes is given by $p(d) = \dfrac{105.07}{d + 33}$, where d is the depth (in feet) of the diver. Find the depth when the air contains 47% recommended nitrogen by (i) solving an equation, and (ii) using the inverse of the function. *Example 6*

44. **MODELING REAL LIFE** The model shown gives the cost of fueling a car for 1 year. Last year your friend drove 9000 miles, paid an average of \$2.89 per gallon of gasoline, and spent a total of \$1239 on gasoline.

$$\text{Fuel cost for 1 year} = \frac{\text{Miles driven} \cdot \text{Price per gallon of fuel}}{\text{Fuel-efficiency rate}}$$

   a. Use the model to write a function c that represents the fuel cost for 1 year in terms of the fuel-efficiency rate r.

   b. Find the fuel-efficiency rate of the car by (i) solving an equation and (ii) using the inverse of the function you wrote in part (a).

MP USING TOOLS In Exercises 45–48, use technology to solve the equation $f(x) = g(x)$.

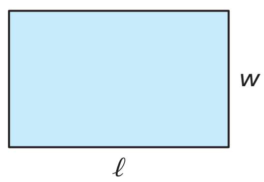
45. $f(x) = \dfrac{2}{3x}$, $g(x) = x$

46. $f(x) = -\dfrac{3}{5x}$, $g(x) = -x$

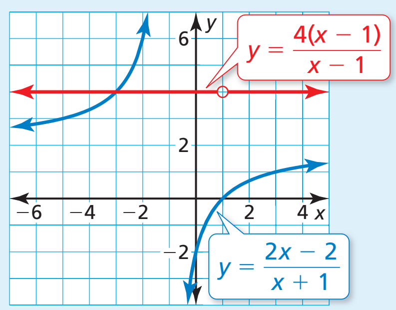
47. $f(x) = \dfrac{1}{x} + 1$, $g(x) = x^2$

48. $f(x) = \dfrac{2}{x} + 1$, $g(x) = x^2 + 1$

49. **CONNECTING CONCEPTS** *Golden rectangles* are rectangles for which the ratio of the width w to the length ℓ is equal to the ratio of ℓ to $\ell + w$. The ratio of the length to the width for these rectangles is called the golden ratio. Find the value of the golden ratio using a rectangle with a width of 1 unit.



50. **HOW DO YOU SEE IT?** Use the graph to identify the solution(s) of $\dfrac{4(x - 1)}{x - 1} = \dfrac{2x - 2}{x + 1}$. Explain your reasoning.



MP STRUCTURE In Exercises 51 and 52, find the inverse of the function. (*Hint:* Try rewriting the function by using either inspection or long division.)

51. $f(x) = \dfrac{3x + 1}{x - 4}$ **52.** $f(x) = \dfrac{4x - 7}{2x + 3}$

53. ABSTRACT REASONING Find the inverse of rational functions of the form $f(x) = \dfrac{ax + b}{cx + d}$. Verify your answer is correct by using it to find $f^{-1}(x)$ in Exercises 51 and 52.

54. THOUGHT PROVOKING
Is it possible to write a rational equation that has the given number of solutions? Justify your answers.

a. no solution **b.** exactly one solution
c. exactly two solutions
d. infinitely many solutions

55. CRITICAL THINKING Let a be a nonzero real number. Tell whether each statement is *always true*, *sometimes true*, or *never true*. Explain your reasoning.

a. For the equation $\dfrac{1}{x - a} = \dfrac{x}{x - a}$, $x = a$ is an extraneous solution.

b. The equation $\dfrac{3}{x - a} = \dfrac{x}{x - a}$ has exactly one solution.

c. The equation $\dfrac{1}{x - a} = \dfrac{2}{x + a} + \dfrac{2a}{x^2 - a^2}$ has no solution.

56. MAKING AN ARGUMENT Is it possible for a rational equation of the form $\dfrac{x - a}{b} = \dfrac{x - c}{d}$, where $a, b, c,$ and d are constants, $b \neq 0$, and $d \neq 0$, to have extraneous solutions? Explain your reasoning.

REVIEW & REFRESH

In Exercises 57 and 58, evaluate the function for the given value of x.

57. $f(x) = x^3 - 2x + 7;\ x = -2$

58. $g(x) = -2x^4 + 7x^3 + x - 2;\ x = 3$

59. MODELING REAL LIFE The linear function $t = 2p$ represents the total cost t (in dollars) of p pounds of broccoli.

a. Find the domain of the function. Is the domain discrete or continuous? Explain.

b. Graph the function using its domain.

In Exercises 60–63, perform the operation.

60. $\dfrac{x^2}{x + 2} - \dfrac{4}{x + 2}$

61. $\dfrac{x - 1}{-2x + 6} + \dfrac{2x + 3}{x^2 + 3x - 18}$

62. $\dfrac{x^2 + 5x - 14}{4x^3 + 28x^2} \div \dfrac{x - 2}{2x^3}$

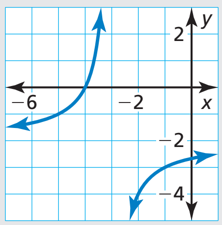
63. $\dfrac{2x^2 + 4x}{x^2 - 3x - 10} \cdot \dfrac{x^2 - 8x + 15}{2x}$

64. Let $f(x) = 3x^2 + 1$ and $g(x) = \sqrt{x - 4}$. Find $f(g(8))$ and $g(f(-2))$.

MP STRUCTURE In Exercises 65–68, match the function with its graph. Explain your reasoning.

65. $g(x) = \dfrac{-2}{x}$ **66.** $f(x) = \dfrac{3}{x} + 2$

67. $y = \dfrac{-2}{x + 3} - 2$ **68.** $h(x) = \dfrac{2x + 2}{3x + 1}$

A.


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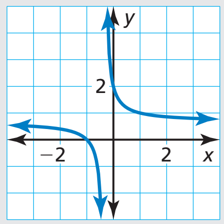

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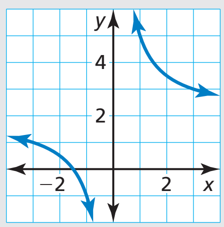
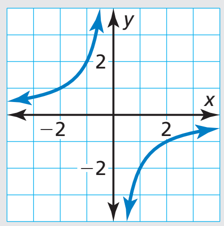


D.


In Exercises 69 and 70, solve the equation. Check your solution(s).

69. $\dfrac{-3}{x - 1} = \dfrac{6}{x + 4}$

70. $\dfrac{x}{x^2 - 2x} + \dfrac{4}{x} = \dfrac{x - 4}{x - 2}$