7.4 Adding and Subtracting Rational Expressions

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Learning Target Add and subtract rational expressions.

Success Criteria

- I can add and subtract rational expressions with like denominators.
- I can explain how to find a common denominator for rational expressions.
- I can add and subtract rational expressions with unlike denominators.

EXPLORE IT! Adding and Subtracting Rational Expressions

Work with a partner.

a. Explain how to find each sum or difference.

i. $\frac{3}{8}+\frac{1}{8}$ **ii.** $\frac{9}{10}-\frac{3}{10}$

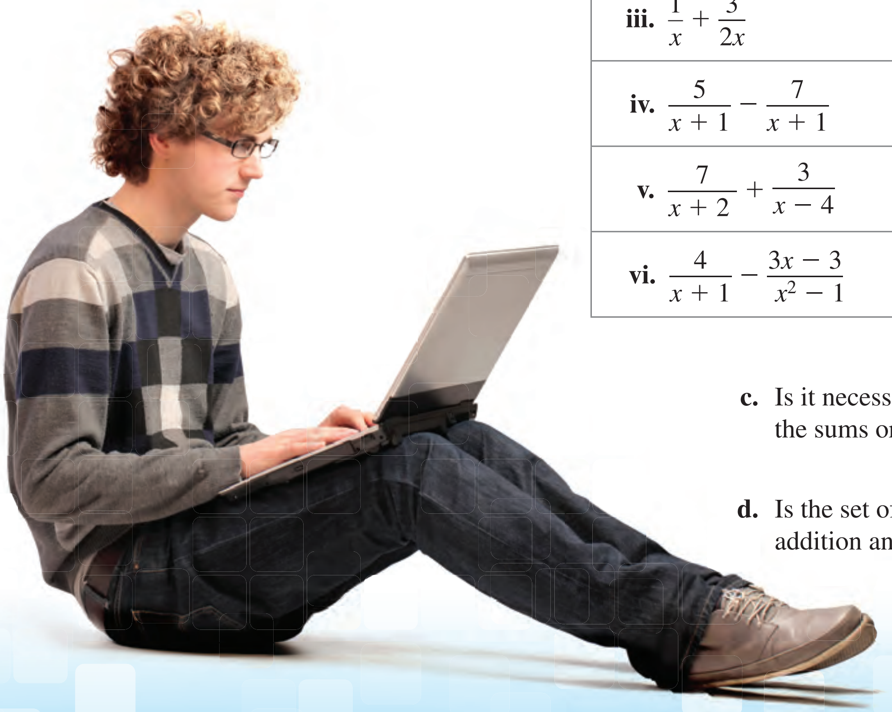
iii. $\frac{1}{2}+\frac{3}{4}$ **iv.** $\frac{5}{8}-\frac{7}{12}$

b. You can add and subtract rational expressions in the same way that you add and subtract fractions. Find each sum or difference. Explain your methods.

Expression	Sum or Difference
i. $\frac{3}{x}+\frac{1}{x}$	
ii. $\frac{9}{x}-\frac{3}{x}$	
iii. $\frac{1}{x}+\frac{3}{2x}$	
iv. $\frac{5}{x+1}-\frac{7}{x+1}$	
v. $\frac{7}{x+2}+\frac{3}{x-4}$	
vi. $\frac{4}{x+1}-\frac{3x-3}{x^2-1}$	

Math Practice

View as Components How is it helpful to view denominators of rational expressions in terms of their factors?



c. Is it necessary to restrict the domain for any of the sums or differences in part (b)? Explain.

d. Is the set of rational expressions closed under addition and subtraction? Justify your answer.

Adding or Subtracting Rational Expressions

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Vocabulary VOCAB

complex fraction, *p. 381*

As with numerical fractions, the procedure used to add or subtract two rational expressions depends upon whether the expressions have like or unlike denominators. To add or subtract rational expressions with like denominators, simply add or subtract their numerators. Then write the result over the common denominator.

KEY IDEA

Adding or Subtracting with Like Denominators

Let a, b, and c be expressions with $c \neq 0$.

Addition

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

Subtraction

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

EXAMPLE 1 Adding or Subtracting with Like Denominators

WATCH

Find each sum or difference.

a. $\dfrac{7}{4x} + \dfrac{3}{4x}$

b. $\dfrac{2x}{x+6} - \dfrac{5}{x+6}$

SOLUTION

a. $\dfrac{7}{4x} + \dfrac{3}{4x} = \dfrac{7+3}{4x} = \dfrac{10}{4x} = \dfrac{5}{2x}$ Add numerators and simplify.

b. $\dfrac{2x}{x+6} - \dfrac{5}{x+6} = \dfrac{2x-5}{x+6}$ Subtract numerators.

To add or subtract rational expressions with *unlike* denominators, find a common denominator. Rewrite each rational expression using the common denominator. Then add or subtract.

KEY IDEA

Adding or Subtracting with Unlike Denominators

Let a, b, c, and d be expressions with $c \neq 0$ and $d \neq 0$.

Addition

$$\frac{a}{c} + \frac{b}{d} = \frac{ad}{cd} + \frac{bc}{cd} = \frac{ad+bc}{cd}$$

Subtraction

$$\frac{a}{c} - \frac{b}{d} = \frac{ad}{cd} - \frac{bc}{cd} = \frac{ad-bc}{cd}$$

You can always find a common denominator of rational expressions by multiplying the denominators, as shown above. However, when you use the least common denominator (LCD), which is the least common multiple (LCM) of the denominators, simplifying your answer may take fewer steps.

To find the LCM of two (or more) expressions, factor the expressions completely. The LCM is the product of the highest power of each factor that appears in either of the expressions.

EXAMPLE 2 Finding a Least Common Multiple (LCM)

WATCH

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Find the least common multiple of $4x^2 - 16$ and $6x^2 - 24x + 24$.

SOLUTION

Step 1 Factor each polynomial. Write numerical factors as products of primes.

$$\begin{aligned} 4x^2 - 16 &= 4(x^2 - 4) \\ &= (2^2)(x + 2)(x - 2) \\ 6x^2 - 24x + 24 &= 6(x^2 - 4x + 4) \\ &= (2)(3)(x - 2)^2 \end{aligned}$$

Step 2 The LCM is the product of the highest power of each factor that appears in either polynomial.

$$\begin{aligned} \text{LCM} &= (2^2)(3)(x + 2)(x - 2)^2 \\ &= 12(x + 2)(x - 2)^2 \end{aligned}$$

EXAMPLE 3 Adding with Unlike Denominators

WATCH

Find the sum $\frac{7}{9x^2} + \frac{x}{3x^2 + 3x}$.

SOLUTION

Method 1 Multiply the denominators to find a common denominator and then add.

$$\begin{aligned} \frac{7}{9x^2} + \frac{x}{3x^2 + 3x} &= \frac{7(3x^2 + 3x) + x(9x^2)}{9x^2(3x^2 + 3x)} && \frac{a}{c} + \frac{b}{d} = \frac{ad + bc}{cd} \\ &= \frac{21x^2 + 21x + 9x^3}{9x^2(3x^2 + 3x)} && \text{Distributive Property} \\ &= \frac{\cancel{3x}(3x^2 + 7x + 7)}{9x^2(x + 1)(\cancel{3x})} && \text{Factor. Divide out common factors.} \\ &= \frac{3x^2 + 7x + 7}{9x^2(x + 1)} && \text{Simplify.} \end{aligned}$$

Method 2 Find the LCD and then add. To find the LCD, find the LCM of the denominators. Note that $9x^2 = 3^2x^2$ and $3x^2 + 3x = 3x(x + 1)$, so the LCD is $(3^2)(x^2)(x + 1) = 9x^2(x + 1)$.

$$\begin{aligned} \frac{7}{9x^2} + \frac{x}{3x^2 + 3x} &= \frac{7}{9x^2} + \frac{x}{3x(x + 1)} && \text{Factor second denominator.} \\ &= \frac{7}{9x^2} \cdot \frac{x + 1}{x + 1} + \frac{x}{3x(x + 1)} \cdot \frac{3x}{3x} && \text{LCD is } 9x^2(x + 1). \\ &= \frac{7x + 7}{9x^2(x + 1)} + \frac{3x^2}{9x^2(x + 1)} && \text{Multiply.} \\ &= \frac{3x^2 + 7x + 7}{9x^2(x + 1)} && \text{Add numerators.} \end{aligned}$$

Note in Examples 1 and 3 that when adding or subtracting rational expressions, the result is a rational expression. Similar to rational numbers, rational expressions are closed under addition and subtraction.

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EXAMPLE 4 Subtracting with Unlike Denominators

WATCH

Find the difference $\dfrac{x+2}{2x-2} - \dfrac{-2x-1}{x^2-4x+3}$.

SOLUTION

$$
\begin{aligned}
\frac{x+2}{2x-2} - \frac{-2x-1}{x^2-4x+3} &= \frac{x+2}{2(x-1)} - \frac{-2x-1}{(x-1)(x-3)} && \text{Factor each denominator.}\\
&= \frac{x+2}{2(x-1)} \cdot \frac{x-3}{x-3} - \frac{-2x-1}{(x-1)(x-3)} \cdot \frac{2}{2} && \text{LCD is } 2(x-1)(x-3).\\
&= \frac{x^2-x-6}{2(x-1)(x-3)} - \frac{-4x-2}{2(x-1)(x-3)} && \text{Multiply.}\\
&= \frac{x^2-x-6-(-4x-2)}{2(x-1)(x-3)} && \text{Subtract numerators.}\\
&= \frac{x^2+3x-4}{2(x-1)(x-3)} && \text{Simplify numerator.}\\
&= \frac{\cancel{(x-1)}(x+4)}{2\cancel{(x-1)}(x-3)} && \text{Factor numerator. Divide out common factor.}\\
&= \frac{x+4}{2(x-3)},\quad x \neq 1 && \text{Simplify.}
\end{aligned}
$$

COMMON ERROR

When subtracting rational expressions, remember to distribute the negative sign to all the terms in the quantity that is being subtracted.

Rewriting Rational Functions

Rewriting a rational function may reveal properties of the function and its graph. In Example 4 of Section 7.2, you used long division to rewrite a rational function. In the next example, you will use inspection.

EXAMPLE 5 Rewriting and Graphing a Rational Function

WATCH

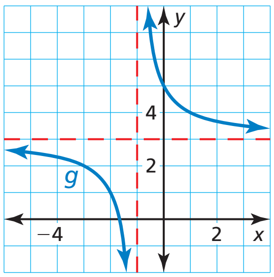
Rewrite $g(x) = \dfrac{3x+5}{x+1}$ in the form $g(x) = \dfrac{a}{x-h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \dfrac{a}{x}$.

SOLUTION

Rewrite by inspection.

$$\frac{3x+5}{x+1} = \frac{3x+3+2}{x+1} = \frac{3(x+1)+2}{x+1} = \frac{3\cancel{(x+1)}}{\cancel{x+1}} + \frac{2}{x+1} = 3 + \frac{2}{x+1}$$

▶ The rewritten function is $g(x) = \dfrac{2}{x+1} + 3$. The graph of g is a translation 1 unit left and 3 units up of the graph of $f(x) = \dfrac{2}{x}$.



SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Find the sum or difference.

1. $\dfrac{2}{3x^2} + \dfrac{1}{3x^2}$

2. $\dfrac{4x}{x-2} - \dfrac{x}{x-2}$

3. $\dfrac{3}{4x} - \dfrac{1}{7}$

4. $\dfrac{1}{3x^2} + \dfrac{x}{9x^2-12}$

5. Rewrite $g(x) = \dfrac{2x-4}{x-3}$ in the form $g(x) = \dfrac{a}{x-h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \dfrac{a}{x}$.

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Complex Fractions

A **complex fraction** is a fraction that contains a fraction in its numerator or denominator. A complex fraction can be simplified using either of the methods below.

KEY IDEA

Simplifying Complex Fractions

Method 1 If necessary, simplify the numerator and denominator by writing each as a single fraction. Then divide by multiplying the numerator by the reciprocal of the denominator.

Method 2 Multiply the numerator and the denominator by the LCD of *every* fraction in the numerator and denominator. Then simplify.

EXAMPLE 6 Simplifying a Complex Fraction

WATCH

Simplify $\dfrac{\frac{5}{x+4}}{\frac{1}{x+4}+\frac{2}{x}}$.

SOLUTION

Method 1

$$\frac{\frac{5}{x+4}}{\frac{1}{x+4}+\frac{2}{x}} = \frac{\frac{5}{x+4}}{\frac{3x+8}{x(x+4)}}$$ Add fractions in denominator.

$$= \frac{5}{x+4} \cdot \frac{x(x+4)}{3x+8}$$ Multiply by reciprocal.

$$= \frac{5x\cancel{(x+4)}}{\cancel{(x+4)}(3x+8)}$$ Multiply. Divide out common factors.

$$= \frac{5x}{3x+8}, \quad x \neq -4, x \neq 0$$ Simplify.

Method 2 The LCD of all the fractions in the numerator and denominator is $x(x+4)$.

$$\frac{\frac{5}{x+4}}{\frac{1}{x+4}+\frac{2}{x}} = \frac{\frac{5}{x+4}}{\frac{1}{x+4}+\frac{2}{x}} \cdot \frac{x(x+4)}{x(x+4)}$$ Multiply numerator and denominator by the LCD.

$$= \frac{\frac{5}{\cancel{x+4}} \cdot x\cancel{(x+4)}}{\frac{1}{\cancel{x+4}} \cdot x\cancel{(x+4)} + \frac{2}{\cancel{x}} \cdot \cancel{x}(x+4)}$$ Multiply. Divide out common factors.

$$= \frac{5x}{x+2(x+4)}$$ Simplify.

$$= \frac{5x}{3x+8}, \quad x \neq -4, x \neq 0$$ Simplify.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Simplify the complex fraction.

6. $\dfrac{\frac{x}{6}-\frac{x}{3}}{\frac{x}{5}-\frac{7}{10}}$

7. $\dfrac{\frac{2}{x}-4}{\frac{2}{x}+3}$

8. $\dfrac{\frac{3}{x+5}}{\frac{2}{x-3}+\frac{1}{x+5}}$

7.4 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–6, find the sum or difference. *Example 1*

1. $\frac{15}{4x} + \frac{5}{4x}$
2. $\frac{9}{16x^2} - \frac{4}{16x^2}$
3. $\frac{9}{x+1} - \frac{2x}{x+1}$
4. $\frac{3x^2}{x-8} + \frac{6x}{x-8}$
5. $\frac{5x}{x+3} + \frac{15}{x+3}$
6. $\frac{4x^2}{2x-1} - \frac{1}{2x-1}$

In Exercises 7–14, find the least common multiple of the expressions. *Example 2*

7. $3x, 3(x-2)$
8. $2x, 2(x+6)$
9. $5x, 5x-10$
10. $4x, 4x-4$
11. $2x^2-18, x^2+x-12$
12. $4x^2-16, x^2+9x+14$
13. $x^2+3x-40, x-8$
14. $x^2-2x-63, x+7$

In Exercises 15–22, find the sum or difference. *Examples 3 and 4*

15. $\frac{12}{5x} - \frac{7}{6x}$
16. $\frac{8}{3x} + \frac{5}{4x}$
17. $\frac{3}{x+4} - \frac{1}{x+6}$
18. $\frac{9}{x-3} + \frac{2x}{x+1}$
19. $\frac{12}{x^2+5x-24} + \frac{3}{x-3}$
20. $\frac{x^2-5}{x^2+5x-14} - \frac{x+3}{x+7}$
21. $\frac{x+2}{x-4} + \frac{2}{x} + \frac{5x}{3x-1}$
22. $\frac{x+3}{x^2-25} - \frac{x-1}{x-5} + \frac{3}{x+3}$

ERROR ANALYSIS In Exercises 23 and 24, describe and correct the error in finding the sum.

23. $\frac{2}{5x} + \frac{4}{x^2} = \frac{2+4}{5x+x^2} = \frac{6}{x(5+x)}$

24. $\frac{x}{x+2} + \frac{4}{x-5} = \frac{x+4}{(x+2)(x-5)}$

25. **MP REASONING** Tell whether the statement is *always, sometimes,* or *never* true. Explain.
    - **a.** The LCD of two rational expressions is the product of the denominators.
    - **b.** The LCD of two rational expressions will have a degree greater than or equal to that of the denominator with the higher degree.

26. **COLLEGE PREP** Which expression is not equivalent to $\frac{x-a}{x^2-a^2}$?

    Ⓐ $\frac{x}{x-a} - \frac{a}{x+a}$
    Ⓑ $\frac{1}{x+a}, x \neq a$
    Ⓒ $\frac{x}{x^2-a^2} - \frac{a}{x^2-a^2}$
    Ⓓ $\frac{x}{x^2-a^2} + \frac{a}{a^2-x^2}$

In Exercises 27–34, rewrite the function in the form $g(x) = \frac{a}{x-h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \frac{a}{x}$. *Example 5*

27. $g(x) = \frac{5x-7}{x-1}$
28. $g(x) = \frac{6x+4}{x+5}$
29. $g(x) = \frac{12x}{x-5}$
30. $g(x) = \frac{8x}{x+13}$
31. $g(x) = \frac{2x+3}{x}$
32. $g(x) = \frac{4x-6}{x}$
33. $g(x) = \frac{3x+11}{x-3}$
34. $g(x) = \frac{7x-9}{x+10}$

In Exercises 35–40, simplify the complex fraction. *Example 6*

35. $\dfrac{\frac{x}{3}-6}{10+\frac{4}{x}}$
36. $\dfrac{15-\frac{2}{x}}{\frac{x}{5}+4}$
37. $\dfrac{\frac{1}{2x-5}-\frac{7}{8x-20}}{\frac{x}{2x-5}}$
38. $\dfrac{\frac{16}{x-2}}{\frac{4}{x+1}+\frac{6}{x}}$
39. $\dfrac{\frac{1}{3x^2-3}}{\frac{5}{x+1}-\frac{x+4}{x^2-3x-4}}$
40. $\dfrac{\frac{3}{x-2}-\frac{6}{x^2-4}}{\frac{3}{x+2}+\frac{1}{x-2}}$

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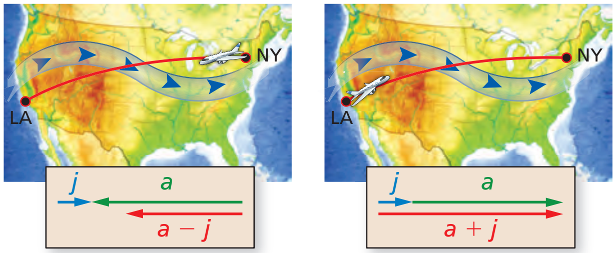
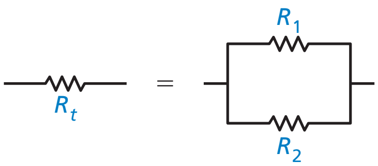
41. **MP PROBLEM SOLVING** The total time T (in hours) needed to fly from New York to Los Angeles and back can be modeled by the equation below, where d is the distance (in miles) each way, a is the average airplane speed (in miles per hour), and j is the average speed (in miles per hour) of the jet stream. Simplify the equation. Then find the total time it takes to fly 2468 miles when $a = 510$ and $j = 115$.

$$T = \frac{d}{a - j} + \frac{d}{a + j}$$



42. **REWRITING A FORMULA** The total resistance R_t of two resistors in a parallel circuit with resistances R_1 and R_2 (in ohms) is given by the equation shown. Simplify the complex fraction. Then find the total resistance when $R_1 = 2000$ and $R_2 = 5600$.

$$R_t = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$



43. **MP PROBLEM SOLVING** You participate in a sprint triathlon that involves swimming, bicycling, and running. The table shows the distances (in miles) and your average speed for each portion of the race.

	Distance (miles)	Speed (miles per hour)
Swimming	0.5	r
Bicycling	12.4	$9r$
Running	3.1	$r + 5$

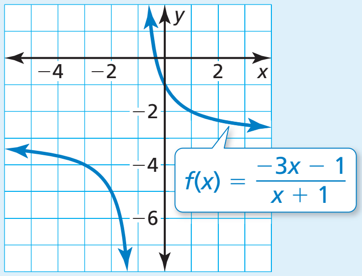
a. Write a model in simplified form for the total time (in hours) it takes to complete the race.

b. How long does it take to complete the race if you can swim at an average speed of 2 miles per hour?

44. **MP PROBLEM SOLVING** A trip involves a 40-mile bus ride and a train ride. The entire trip is 140 miles. The time (in hours) the bus travels is $y_1 = \frac{40}{x}$, where x is the average speed (in miles per hour) of the bus. The time (in hours) the train travels is $y_2 = \frac{100}{x + 30}$. Write and simplify a model that shows the total time y (in hours) of the trip.

45. **MAKING AN ARGUMENT** Is the least common multiple of two expressions always greater than each of the expressions? Explain.

46. **HOW DO YOU SEE IT?**
Use the graph to write a function of the form $f(x) = \frac{a}{x - h} + k$.



47. **MP STRUCTURE** A family borrows P dollars to pay for orthodontics. The family agrees to repay the loan over t years at a monthly interest rate of i (expressed as a decimal). The monthly payment M is given by either formula below.

$$M = \frac{Pi}{1 - \left(\frac{1}{1 + i}\right)^{12t}} \quad \text{or} \quad M = \frac{Pi(1 + i)^{12t}}{(1 + i)^{12t} - 1}$$

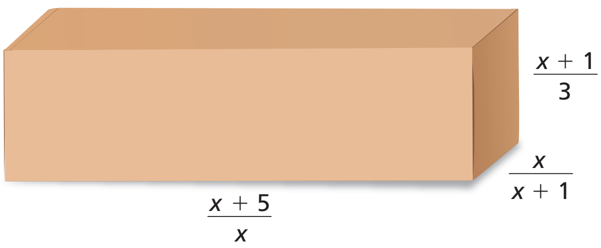
a. Show that the formulas are equivalent by simplifying the first formula.

b. Find the monthly payment when the family borrows \$5000 at a monthly interest rate of 0.5% and repays the loan over 4 years.

48. **THOUGHT PROVOKING**
Is it possible to write two rational functions whose sum is a quadratic function? Justify your answer.

49. **OPEN ENDED** Write a complex fraction that has a value of 3 when $x = 2$ and is undefined when $x = -3$ and $x = 1$.

50. **CONNECTING CONCEPTS** Find an expression for the surface area of the box.



51. **MP PROBLEM SOLVING** One 3-D printer creates 10% of a medical instrument in one minute. Another 3-D printer takes x minutes to create a medical instrument of the same design. Write an expression for the number of instruments the two printers create in one hour when both printers operate for the full hour.

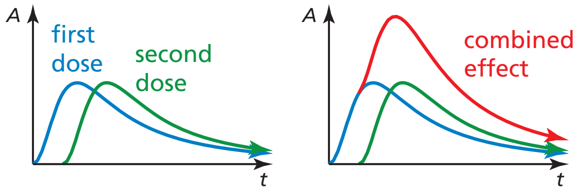
52. **MP STRUCTURE** Find the value of b that completes the equation.

$$\frac{3x-1}{(x+1)(x-3)} = \frac{1}{x+1} + \frac{b}{x-3}$$

53. **DIG DEEPER** The amount A (in milligrams) of aspirin in a person's bloodstream is modeled by

$$A = \frac{391t^2 + 0.112}{0.218t^4 + 0.991t^2 + 1}$$

where t is the time (in hours) after one dose is taken. A second dose is taken 1 hour after the first dose. Write an equation for the total amount of aspirin in the bloodstream after the second dose is taken.



54. **MP PATTERNS** Find the next two expressions in the pattern shown. Then simplify all five expressions. What value do the expressions approach?

$$1 + \cfrac{1}{2 + \cfrac{1}{2}},\ 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}},\ 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}},\ \ldots$$

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REVIEW & REFRESH

WATCH

In Exercises 55 and 56, graph the function. Find the domain and range.

55. $g(x) = \dfrac{4}{x} + 1$

56. $g(x) = \dfrac{-2}{x-6}$

57. Tell whether x and y show *direct variation, inverse variation,* or *neither*.

x	6	8	10	12	14
y	15	20	25	30	35

In Exercises 58 and 59, find the product or quotient.

58. $\dfrac{2x}{4x} \cdot \dfrac{1}{3y}$

59. $\dfrac{xy^3}{x^2} \div \dfrac{y^4}{x^3}$

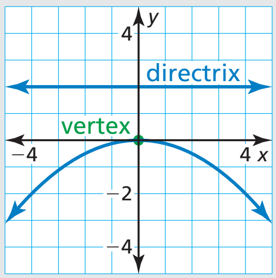
60. **MODELING REAL LIFE** An app store sells 60 apps each day and charges \$8.00 per download. For each \$0.50 decrease in price, the store sells 10 more apps. How much should the store charge to maximize daily revenue?

In Exercises 61 and 62, solve the equation.

61. $|m + 2| = 0$

62. $\left|\dfrac{x}{3}\right| = 3$

63. Write an equation of the parabola.



64. Graph $y \geq 2x^2 - 5x - 3$.

In Exercises 65 and 66, describe the transformation of f represented by g. Then graph each function.

65. $f(x) = \sqrt{x},\ g(x) = \sqrt{x} + 1$

66. $f(x) = \sqrt{x},\ g(x) = 2\sqrt{x-3}$

In Exercises 67 and 68, find the sum or difference.

67. $\dfrac{11}{4x} - \dfrac{1}{2x}$

68. $\dfrac{1}{x^3} + \dfrac{5}{4x}$