7.3 Multiplying and Dividing Rational Expressions

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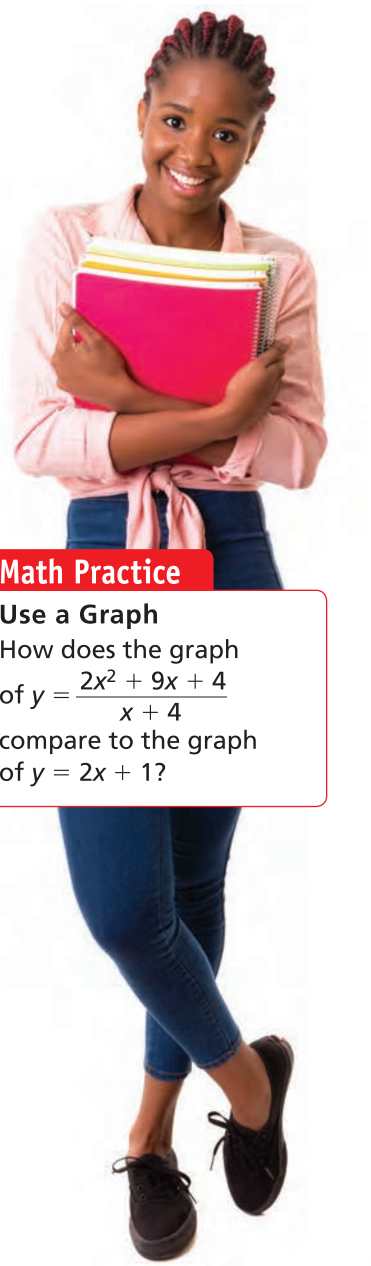
Learning Target Multiply and divide rational expressions.

Success Criteria
- I can simplify rational expressions and identify any excluded values.
- I can multiply rational expressions.
- I can divide rational expressions.

EXPLORE IT! Analyzing Rational Expressions



Work with a partner.

a. A student divides $2x^2 + 9x + 4$ by $x + 4$ below as shown. Justify each solution step and describe the method.

$$\begin{array}{r} 2x + 1 \\ x+4 \overline{)\,2x^2 + 9x + 4} \\ \underline{2x^2 + 8x} \\ x + 4 \\ \underline{x + 4} \\ 0 \end{array}$$

So, $(2x^2 + 9x + 4) \div (x + 4) = 2x + 1$.

What other methods can you use to divide these polynomials?

Math Practice

Use a Graph

How does the graph of $y = \dfrac{2x^2 + 9x + 4}{x + 4}$ compare to the graph of $y = 2x + 1$?

b. **MP CHOOSE TOOLS** Create a table of values for $\dfrac{2x^2 + 9x + 4}{x + 4}$ and $2x + 1$.

What happens when $x = -4$? Why does this happen? Explain how this affects your answer in part (a).

c. You can multiply and divide rational expressions in the same way that you multiply and divide fractions. Find each product or quotient. Determine whether you need to restrict the domains for any of the products or quotients.

Expression	Product or Quotient
i. $\dfrac{x}{x+1} \cdot \dfrac{1}{x}$	
ii. $\dfrac{1}{x-2} \cdot \dfrac{x-2}{x+1}$	
iii. $\dfrac{1}{x} \div \dfrac{x}{x+1}$	
iv. $\dfrac{x}{x+2} \div \dfrac{x}{x-1}$	

d. Is the set of rational expressions closed under multiplication? under division? Justify your answer.

Vocabulary

rational expression, *p. 370*
simplified form of a rational expression, *p. 370*

Simplifying Rational Expressions

A **rational expression** is a fraction whose numerator and denominator are nonzero polynomials. The *domain* of a rational expression excludes values that make the denominator zero. A rational expression is in **simplified form** when its numerator and denominator have no common factors (other than ± 1).

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KEY IDEA

Simplifying Rational Expressions

Let a, b, and c be expressions with $b \neq 0$ and $c \neq 0$.

Property $\dfrac{a\cancel{c}}{b\cancel{c}} = \dfrac{a}{b}$ Divide out common factor c.

Examples $\dfrac{15}{65} = \dfrac{3 \cdot \cancel{5}}{13 \cdot \cancel{5}} = \dfrac{3}{13}$ Divide out common factor 5.

$\dfrac{4\cancel{(x+3)}}{(x+3)\cancel{(x+3)}} = \dfrac{4}{x+3}$ Divide out common factor $x + 3$.

Simplifying a rational expression usually requires two steps. First, factor the numerator and denominator. Then divide out any factors that are common to both the numerator and denominator.

EXAMPLE 1 Simplifying a Rational Expression

WATCH

Simplify $\dfrac{x^2 - 4x - 12}{x^2 - 4}$.

SOLUTION

$$\frac{x^2 - 4x - 12}{x^2 - 4} = \frac{(x+2)(x-6)}{(x+2)(x-2)}$$ Factor numerator and denominator.

$$= \frac{\cancel{(x+2)}(x-6)}{\cancel{(x+2)}(x-2)}$$ Divide out common factor.

$$= \frac{x-6}{x-2}, \quad x \neq -2$$ Simplified form

The original expression is undefined when $x = -2$. To make the original and simplified expressions equivalent, restrict the domain of the simplified expression by excluding $x = -2$. Both expressions are undefined when $x = 2$, so it is not necessary to list it.

COMMON ERROR

Do not divide out variable terms that are not factors.

$$\frac{x-6}{x-2} \neq \frac{-6}{-2}$$

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Simplify the rational expression, if possible.

1. $\dfrac{2(x+1)}{(x+1)(x+3)}$

2. $\dfrac{x+4}{x^2 - 16}$

3. $\dfrac{4}{x(x+2)}$

4. $\dfrac{x^2 - 2x - 3}{x^2 - x - 6}$

5. **WRITING** When simplifying rational expressions, explain why you must restrict the domain of the simplified expression in some cases.

Multiplying Rational Expressions

The rule for multiplying rational expressions is the same as the rule for multiplying numerical fractions: multiply numerators, multiply denominators, and write the new fraction in simplified form. Similar to rational numbers, rational expressions are closed under multiplication.

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KEY IDEA

Multiplying Rational Expressions

Let a, b, c, and d be expressions with $b \neq 0$ and $d \neq 0$.

Property $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$ Simplify $\dfrac{ac}{bd}$ if possible.

Example $\dfrac{5x^2}{2xy^2} \cdot \dfrac{6xy^3}{10y} = \dfrac{30x^3y^3}{20xy^3} = \dfrac{\cancel{10} \cdot 3 \cdot \cancel{x} \cdot x^2 \cdot \cancel{y^3}}{\cancel{10} \cdot 2 \cdot \cancel{x} \cdot \cancel{y^3}} = \dfrac{3x^2}{2}, \quad x \neq 0, y \neq 0$

ANOTHER WAY

In Example 2, you can simplify each rational expression before multiplying, and then simplify the result.

$$\frac{8x^3y}{2xy^2} \cdot \frac{7x^4y^3}{4y}$$
$$= \frac{4x^2}{y} \cdot \frac{7x^4y^2}{4}$$
$$= \frac{\cancel{4} \cdot 7 \cdot x^6 \cdot \cancel{y} \cdot y}{\cancel{4} \cdot \cancel{y}}$$
$$= 7x^6y, \quad x \neq 0, y \neq 0$$

EXAMPLE 2 Multiplying Rational Expressions

WATCH

Find the product $\dfrac{8x^3y}{2xy^2} \cdot \dfrac{7x^4y^3}{4y}$.

SOLUTION

$$\frac{8x^3y}{2xy^2} \cdot \frac{7x^4y^3}{4y} = \frac{56x^7y^4}{8xy^3}$$ Multiply numerators and denominators.

$$= \frac{\cancel{8} \cdot 7 \cdot \cancel{x} \cdot x^6 \cdot \cancel{y^3} \cdot y}{\cancel{8} \cdot \cancel{x} \cdot \cancel{y^3}}$$ Factor and divide out common factors.

$$= 7x^6y, \quad x \neq 0, y \neq 0$$ Simplified form

EXAMPLE 3 Multiplying Rational Expressions

WATCH

Find the product $\dfrac{3x - 3x^2}{x^2 + 4x - 5} \cdot \dfrac{x^2 + x - 20}{3x}$.

SOLUTION

$$\frac{3x - 3x^2}{x^2 + 4x - 5} \cdot \frac{x^2 + x - 20}{3x} = \frac{3x(1 - x)}{(x - 1)(x + 5)} \cdot \frac{(x + 5)(x - 4)}{3x}$$ Factor numerators and denominators.

$$= \frac{3x(1 - x)(x + 5)(x - 4)}{(x - 1)(x + 5)(3x)}$$ Multiply numerators and denominators.

$$= \frac{3x(-1)(x - 1)(x + 5)(x - 4)}{(x - 1)(x + 5)(3x)}$$ Rewrite $1 - x$ as $(-1)(x - 1)$.

$$= \frac{\cancel{3x}(-1)\cancel{(x - 1)}\cancel{(x + 5)}(x - 4)}{\cancel{(x - 1)}\cancel{(x + 5)}\cancel{(3x)}}$$ Divide out common factors.

$$= -x + 4, \quad x \neq -5, x \neq 0, x \neq 1$$ Simplified form

Check Use technology to check your answer. The values are the same, except when $x = -5$, $x = 0$, and $x = 1$.

x	$\dfrac{3x - 3x^2}{x^2 + 4x - 5} \cdot \dfrac{x^2 + x - 20}{3x}$	$-x + 4$
-5	undefined	9
-4	8	8
-3	7	7
-2	6	6
-1	5	5
0	undefined	4
1	undefined	3
2	2	2

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EXAMPLE 4 Multiplying a Rational Expression by a Polynomial

WATCH

Find the product $\frac{x+2}{x^3-27} \cdot (x^2+3x+9)$.

SOLUTION

$$\frac{x+2}{x^3-27} \cdot (x^2+3x+9) = \frac{x+2}{x^3-27} \cdot \frac{x^2+3x+9}{1}$$ Write polynomial as a rational expression.

$$= \frac{(x+2)(x^2+3x+9)}{(x-3)(x^2+3x+9)}$$ Multiply. Factor denominator.

$$= \frac{(x+2)\cancel{(x^2+3x+9)}}{(x-3)\cancel{(x^2+3x+9)}}$$ Divide out common factor.

$$= \frac{x+2}{x-3}$$ Simplified form

STUDY TIP

Notice that $x^2 + 3x + 9$ does not equal zero for any real value of x. So, no values must be excluded from the domain to make the simplified form equivalent to the original.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Find the product.

6. $\frac{3x^5y^2}{8xy} \cdot \frac{6xy^2}{9x^3y}$

7. $\frac{2x^2-10x}{x^2-25} \cdot \frac{x+3}{2x^2}$

8. $\frac{x+5}{x^3-1} \cdot (x^2+x+1)$

Dividing Rational Expressions

The rule for dividing rational expressions is the same as the rule for dividing fractions: multiply the first by the reciprocal of the second, and write the result in simplified form. Rational expressions are closed under nonzero division.

KEY IDEA

Dividing Rational Expressions

Let a, b, c, and d be expressions with $b \neq 0$, $c \neq 0$, and $d \neq 0$.

Property $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ Simplify $\frac{ad}{bc}$ if possible.

Example $\frac{7}{x+1} \div \frac{x+2}{2x-3} = \frac{7}{x+1} \cdot \frac{2x-3}{x+2} = \frac{7(2x-3)}{(x+1)(x+2)}, \ x \neq \frac{3}{2}$

EXAMPLE 5 Dividing Rational Expressions

WATCH

Find the quotient $\frac{7x}{2x-10} \div \frac{x^2-6x}{x^2-11x+30}$.

SOLUTION

$$\frac{7x}{2x-10} \div \frac{x^2-6x}{x^2-11x+30} = \frac{7x}{2x-10} \cdot \frac{x^2-11x+30}{x^2-6x}$$ Multiply by reciprocal.

$$= \frac{7x}{2(x-5)} \cdot \frac{(x-5)(x-6)}{x(x-6)}$$ Factor.

$$= \frac{7\cancel{x}\cancel{(x-5)}\cancel{(x-6)}}{2\cancel{(x-5)}(\cancel{x})\cancel{(x-6)}}$$ Multiply. Divide out common factors.

$$= \frac{7}{2}, \quad x \neq 0, x \neq 5, x \neq 6$$ Simplified form

EXAMPLE 6 Dividing a Rational Expression by a Polynomial

Find the quotient $\frac{6x^2 + x - 15}{4x^2} \div (3x^2 + 5x)$.

SOLUTION

$$\begin{aligned} \frac{6x^2 + x - 15}{4x^2} \div (3x^2 + 5x) &= \frac{6x^2 + x - 15}{4x^2} \cdot \frac{1}{3x^2 + 5x} && \text{Multiply by reciprocal.} \\ &= \frac{(3x + 5)(2x - 3)}{4x^2} \cdot \frac{1}{x(3x + 5)} && \text{Factor.} \\ &= \frac{\cancel{(3x + 5)}(2x - 3)}{4x^2(x)\cancel{(3x + 5)}} && \text{Multiply. Divide out common factor.} \\ &= \frac{2x - 3}{4x^3}, \quad x \neq -\frac{5}{3} && \text{Simplified form} \end{aligned}$$

EXAMPLE 7 Modeling Real Life

The total amount E (in millions of dollars) of healthcare expenditures and the residential population P (in millions) of the United States can be modeled by

$$E = \frac{111{,}082t + 1{,}422{,}329}{1 - 0.002t} \quad \text{and} \quad P = 2.515t + 283.37$$

where t represents the number of years since 2000. Estimate the annual healthcare expenditures per resident in 2022.

SOLUTION

Find a model M for the annual healthcare expenditures per resident by dividing the total amount E by the population P.

$$\begin{aligned} M &= \frac{111{,}082t + 1{,}422{,}329}{1 - 0.002t} \div (2.515t + 283.37) && \text{Divide } E \text{ by } P. \\ &= \frac{111{,}082t + 1{,}422{,}329}{1 - 0.002t} \cdot \frac{1}{2.515t + 283.37} && \text{Multiply by reciprocal.} \\ &= \frac{111{,}082t + 1{,}422{,}329}{(1 - 0.002t)(2.515t + 283.37)} && \text{Multiply.} \end{aligned}$$

To estimate the annual healthcare expenditures per resident in 2022, let $t = 22$ in the model.

$$\begin{aligned} M &= \frac{111{,}082 \cdot 22 + 1{,}422{,}329}{(1 - 0.002 \cdot 22)(2.515 \cdot 22 + 283.37)} && \text{Substitute 22 for } t. \\ &\approx 11{,}940 && \text{Use technology.} \end{aligned}$$

▶ In 2022, the annual healthcare expenditures per resident was about $11,940.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

9. Find (a) $\frac{4x}{5x - 20} \div \frac{x^2 - 2x}{x^2 - 6x + 8}$ and (b) $\frac{2x^2 + 3x - 5}{6x} \div (2x^2 + 5x)$.

10. WHAT IF? In Example 7, estimate the annual healthcare expenditures per resident in 2022 when $P = -0.028t^2 + 3.03t + 281.8$.

7.3 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–8, simplify the expression, if possible.
Example 1

1. $\dfrac{2x^2}{3x^2 - 4x}$

2. $\dfrac{-x^2}{2x^2 + 5x}$

3. $\dfrac{x^2 - 3x - 18}{x^2 - 7x + 6}$

4. $\dfrac{x^2 + 13x + 36}{x^2 - 7x + 10}$

5. $\dfrac{x^2 + 11x + 18}{x^3 + 8}$

6. $\dfrac{x^2 - 7x + 12}{x^3 - 27}$

7. $\dfrac{32x^4 - 50}{4x^3 - 12x^2 - 5x + 15}$

8. $\dfrac{3x^3 - 3x^2 + 7x - 7}{27x^4 - 147}$

In Exercises 9–18, find the product.
Examples 2, 3 and 4

9. $\dfrac{4xy^3}{x^2y} \cdot \dfrac{y}{8x}$

10. $\dfrac{3x^3y}{xy^2} \cdot \dfrac{x}{9y}$

11. $\dfrac{x^2(x - 4)}{x - 3} \cdot \dfrac{(x - 3)(x + 6)}{x^3}$

12. $\dfrac{x^3(x + 5)}{x - 9} \cdot \dfrac{(x - 9)(x + 8)}{3x^3}$

13. $\dfrac{x^2 - 3x}{x - 2} \cdot \dfrac{x^2 + x - 6}{x}$

14. $\dfrac{x^2 - 4x}{x - 1} \cdot \dfrac{x^2 + 3x - 4}{2x}$

15. $\dfrac{x^2 + 3x - 4}{x^2 + 4x + 4} \cdot \dfrac{2x^2 + 4x}{x^2 - 4x + 3}$

16. $\dfrac{x^2 - x - 6}{4x^3} \cdot \dfrac{2x^2 + 2x}{x^2 + 5x + 6}$

17. $\dfrac{x^2 + 5x - 36}{x^2 - 49} \cdot (x^2 - 11x + 28)$

18. $\dfrac{x^2 - x - 12}{x^2 - 16} \cdot (x^2 + 2x - 8)$

19. ERROR ANALYSIS Describe and correct the error in simplifying the rational expression.

$$\frac{x^2 + \overset{2}{\cancel{16}}x + \overset{3}{\cancel{48}}}{x^2 + \underset{1}{\cancel{8}}x + \underset{1}{\cancel{16}}} = \frac{x^2 + 2x + 3}{x^2 + x + 1}$$

20. ERROR ANALYSIS Describe and correct the error in finding the product.

$$\frac{x^2 - 25}{3 - x} \cdot \frac{x - 3}{x + 5}$$
$$= \frac{(x + 5)(x - 5)}{3 - x} \cdot \frac{x - 3}{x + 5}$$
$$= \frac{\cancel{(x + 5)}(x - 5)\cancel{(x - 3)}}{\cancel{(3 - x)}\cancel{(x + 5)}}$$
$$= x - 5, x \neq 3, x \neq -5$$

21. COMPARING METHODS Find the product below by multiplying the numerators and denominators, then simplifying. Then find the product by simplifying each expression, then multiplying. Which method do you prefer? Explain.

$$\frac{4x^2y}{2x^3} \cdot \frac{12y^4}{24x^2}$$

22. COLLEGE PREP For what value of k is $2(y + 1)$, $x \neq 0$, $y \neq -3$ the simplified form of $\dfrac{y^2 + 4y + 3}{x} \cdot \dfrac{2x}{y + k}$?

Ⓐ $k = -3$ Ⓑ $k = -1$

Ⓒ $k = 1$ Ⓓ $k = 3$

In Exercises 23–30, find the quotient.
Examples 5 and 6

23. $\dfrac{32x^3y}{y^8} \div \dfrac{y^7}{8x^4}$

24. $\dfrac{2xyz}{x^3z^3} \div \dfrac{6y^4}{2x^2z^2}$

25. $\dfrac{x^2 - x - 6}{2x^4 - 6x^3} \div \dfrac{x + 2}{4x^3}$

26. $\dfrac{2x^2 - 12x}{x^2 - 7x + 6} \div \dfrac{2x}{3x - 3}$

27. $\dfrac{x^2 - x - 6}{x + 4} \div (x^2 - 6x + 9)$

28. $\dfrac{x^2 - 5x - 36}{x + 2} \div (x^2 - 18x + 81)$

29. $\dfrac{x^2 + 9x + 18}{x^2 + 6x + 8} \div \dfrac{x^2 - 3x - 18}{x^2 + 2x - 8}$

30. $\dfrac{x^2 - 3x - 40}{x^2 + 8x - 20} \div \dfrac{x^2 + 13x + 40}{x^2 + 12x + 20}$

In Exercises 31 and 32, use the following information.

Manufacturers often package products in a way that uses the least amount of material. One measure of the efficiency of a package is the ratio of its surface area S to its volume V. The smaller the ratio, the more efficient the packaging.

31. You are examining three cylindrical containers.

a. Write an expression for the efficiency ratio of a cylinder with height h and radius r.

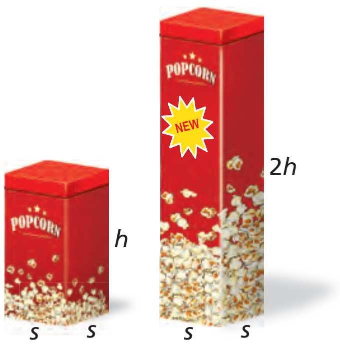
b. Find the efficiency ratio for each cylindrical can listed in the table. Rank the three cans according to efficiency.

	Soup	Coffee	Paint
Height, *h*	10.2 cm	15.9 cm	19.4 cm
Radius, *r*	3.4 cm	7.8 cm	8.4 cm

32. A popcorn company is designing a new tin with the same square base and twice the height of the old tin.

a. Write an expression for the efficiency ratio of each tin.

b. Did the company make a good decision by creating the new tin? Explain.



33. **MODELING REAL LIFE** The total amount E (in thousands of dollars) of educational technology expenditures and the total number of students P (in thousands) in a school system is modeled by

$$E = \frac{-6.984t + 550}{1 - 0.027t}$$

and

$$P = 0.006t + 2.09$$

where t represents the number of years since 2000. Estimate the annual educational technology expenditures per student in 2022. *Example 7*

34. **MODELING REAL LIFE** The total amount S (in millions of dollars) of revenue and the number of users U (in millions) of a social media platform is modeled by

$$S = \frac{3.520t - 3.33}{1 - 0.066t}$$

and

$$U = 0.203t + 0.48$$

where t represents the number of years since 2010. Estimate the revenue per user in 2022.

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35. **MP STRUCTURE** Refer to the population model P in Exercise 33.

a. Interpret the meaning of the coefficient of t.

b. Interpret the meaning of the constant term.

36. **CRITICAL THINKING** Find the expression that makes the following statement true. Assume $x \neq -2$ and $x \neq 5$.

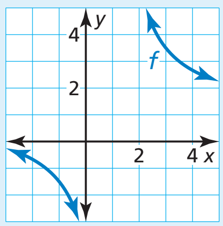
$$\frac{x - 5}{x^2 + 2x - 35} \div \frac{\boxed{}}{x^2 - 3x - 10} = \frac{x + 2}{x + 7}$$

37. **DRAWING CONCLUSIONS** Complete the table for the function $y = \dfrac{x + 4}{x^2 - 16}$. Then use technology to explain the behavior of the function at $x = -4$.

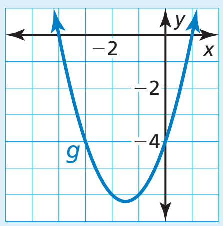
x	*y*
−3.5	
−3.8	
−3.9	
−4.1	
−4.2	

38. **HOW DO YOU SEE IT?**
Use the graphs of f and g to determine the excluded values of each function.

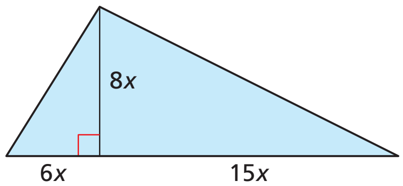
a. $h(x) = (fg)(x)$



b. $k(x) = \left(\dfrac{f}{g}\right)(x)$



39. **CONNECTING CONCEPTS** Find the ratio of the perimeter to the area of the triangle shown.



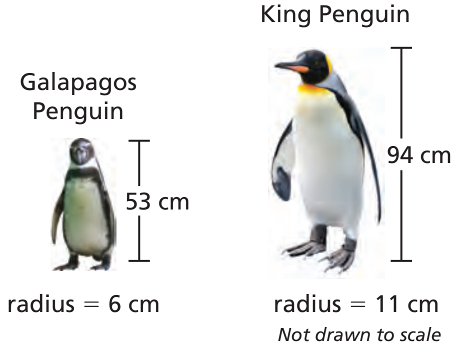
40. **THOUGHT PROVOKING**
Is it possible to write two radical functions whose product represents a parabola and whose quotient represents a hyperbola? Justify your answer.

41. **REASONING** Find two rational functions f and g that have the stated product and quotient.

$$(fg)(x) = x^2;\ \left(\frac{f}{g}\right)(x) = \frac{(x-1)^2}{(x+2)^2}$$

42. **PERFORMANCE TASK** Animals can better conserve body heat as their surface area to volume ratios decrease.

a. Which penguin below is better equipped to live in a colder climate? Explain your reasoning.



b. Research other factors that influence the survival of penguins in cold climates. How might these factors support or change your answer?

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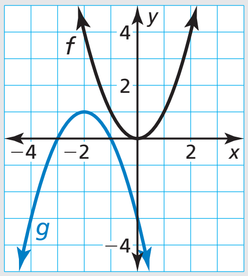
REVIEW & REFRESH

WATCH

43. Describe the transformation of $f(x) = x^2$ represented by g.



In Exercises 44 and 45, graph the function. Find the domain and range.

44. $f(x) = \dfrac{x+1}{x-1}$

45. $y = \dfrac{-2x+5}{-x+10}$

In Exercises 46 and 47, the variables x and y vary inversely. Use the given values to write an equation relating x and y. Then find y when $x = 6$.

46. $x = 1, y = -4$

47. $x = 4, y = 3$

In Exercises 48 and 49 determine the type of function represented by the data. Explain your reasoning.

48.

x	0	2	4	6	8
y	1	2	4	8	16

49. $(-4, 48), (-3, 43), (-2, 38), (-1, 33), (0, 28)$

In Exercises 50 and 51, find the product or quotient.

50. $\dfrac{3xy^2}{xy^4} \cdot \dfrac{y}{6x}$

51. $\dfrac{4xyz}{x^5z^3} \div \dfrac{8y^2}{4x^2z^3}$

52. **MODELING REAL LIFE** A mobile provider maintains a list of active cell phones. The number N (in hundreds) of active cell phones of a particular model since year t can be modeled by

$$N = 0.622t^3 + 0.31t^2 - 1.1t + 20$$

where $0 \le t \le 12$. Since what year are there at least 10,000 active phones of the particular model?

In Exercises 53 and 54, factor the polynomial completely.

53. $8x^3 + 27$

54. $3y^6 - 15y^4 + 18y^2$

55. Write an exponential growth function represented by the graph.