7.2 Graphing Rational Functions

Learning Target Describe and graph rational functions.

Success Criteria

- I can graph rational functions.
- I can describe transformations of rational functions.
- I can explain how to find the asymptotes of a rational function from an equation.
- I can write rational functions in different forms.

EXPLORE IT! Graphing Rational Functions

Math Practice

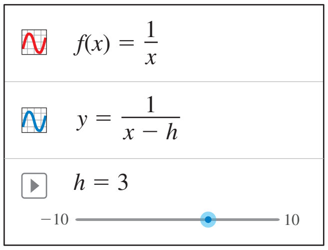
Look for Structure How do the values of h and k affect the asymptotes?

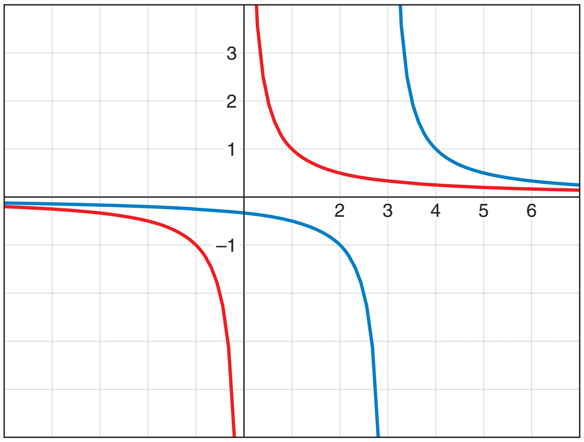
Work with a partner. The function $f(x) = \frac{1}{x}$ is a simple *rational* function.

a. What does the graph of f look like? How is the graph similar to other functions you have studied? How is it different?

b. Describe the end behavior of f. Then describe the behavior of the graph when x is between -1 and 1. Why is there an asymptote at the y-axis?

c. Use technology to explore the graph of $y = f(x - h)$ for several values of h. How does the graph change when you change the value of h?





d. Use technology to explore the graph of $y = f(x) + k$ for several values of k. How does the graph change when you change the value of k?

Graphing Simple Rational Functions

A **rational function** has the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. The inverse variation function $f(x) = \frac{a}{x}$ is a rational function.

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Vocabulary

rational function, *p. 362*

STUDY TIP

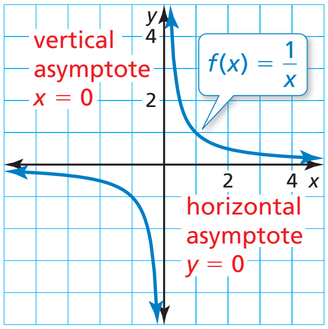
Notice that $\frac{1}{x} \to 0$ as $x \to +\infty$ and as $x \to -\infty$. This explains why $y = 0$ is a horizontal asymptote of the graph of $f(x) = \frac{1}{x}$. You can also analyze y-values as x approaches 0 to see why $x = 0$ is a vertical asymptote.

KEY IDEA

Parent Function for Simple Rational Functions

The graph of the parent function $f(x) = \frac{1}{x}$ is a *hyperbola*, which consists of two symmetrical parts called branches. The domain and range are all nonzero real numbers.

Any function of the form $g(x) = \frac{a}{x}$ $(a \neq 0)$ has the same asymptotes, domain, and range as the function $f(x) = \frac{1}{x}$.



EXAMPLE 1 Graphing a Rational Function of the Form $y = \frac{a}{x}$

Graph $g(x) = \frac{4}{x}$. Compare the graph with the graph of $f(x) = \frac{1}{x}$.

WATCH

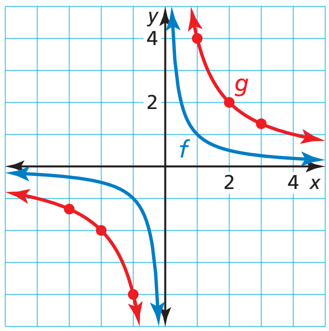
SOLUTION

Step 1 The function is of the form $g(x) = \frac{a}{x}$, so the asymptotes are $x = 0$ and $y = 0$. Draw the asymptotes.

Step 2 Make a table of values and plot the points. Include both positive and negative values of x.

x	-3	-2	-1	1	2	3
y	$-\frac{4}{3}$	-2	-4	4	2	$\frac{4}{3}$

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.



▶ The graph of g lies farther from the axes than the graph of f. Both graphs lie in the first and third quadrants and have the same asymptotes, domain, and range.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Graph the function. Compare the graph with the graph of $f(x) = \frac{1}{x}$.

1. $g(x) = \frac{2}{x}$

2. $g(x) = \frac{-6}{x}$

3. $g(x) = -\frac{3}{x}$

Translating Simple Rational Functions

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KEY IDEA

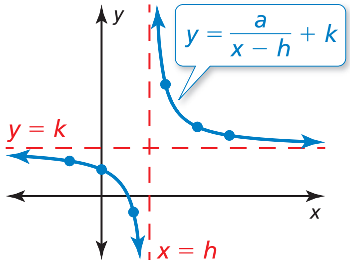
Graphing Translations of Simple Rational Functions

To graph a rational function of the form $y = \dfrac{a}{x - h} + k$, follow these steps.

Step 1 Draw the asymptotes $x = h$ and $y = k$.

Step 2 Plot points to the left and to the right of the vertical asymptote.

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.



EXAMPLE 2 Graphing a Translation of a Rational Function

Graph $g(x) = \dfrac{-4}{x + 2} - 1$. Find the domain and range.

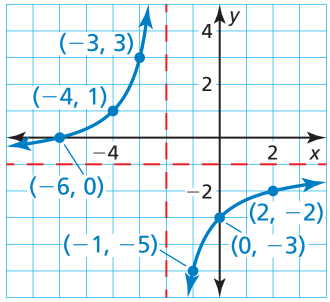
WATCH

SOLUTION

Step 1 Draw the asymptotes $x = -2$ and $y = -1$.

Step 2 Plot points to the left of the vertical asymptote, such as $(-3, 3)$, $(-4, 1)$, and $(-6, 0)$. Plot points to the right of the vertical asymptote, such as $(-1, -5)$, $(0, -3)$, and $(2, -2)$.

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.



▶ The domain is all real numbers except $x = -2$ and the range is all real numbers except $y = -1$.

Math Practice

Look for Structure

Compare the graph of g to the graph of $f(x) = \dfrac{-4}{x}$. Explain your reasoning.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Graph the function. Find the domain and range.

4. $y = \dfrac{3}{x} - 2$

5. $y = \dfrac{-1}{x + 4}$

6. $y = \dfrac{1}{x - 1} + 5$

7. MP **STRUCTURE** Which functions have the same asymptotes as $h(x) = \dfrac{7}{x + 4} + 3$?

$m(x) = \dfrac{1}{x + 3} + 4$

$n(x) = \dfrac{1}{x + 4} + 3$

$p(x) = \dfrac{7}{x - 4} - 3$

$q(x) = \dfrac{-2}{x + 4} + 3$

Graphing Other Rational Functions

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All rational functions of the form $y = \dfrac{ax + b}{cx + d}$ also have graphs that are hyperbolas.

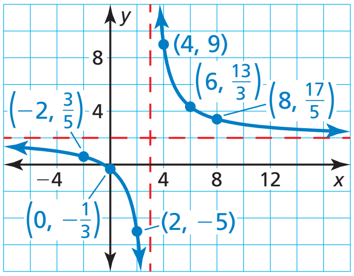
- The vertical asymptote of the graph is the line $x = -\dfrac{d}{c}$ because the function is undefined when the denominator $cx + d$ is zero.
- The horizontal asymptote is the line $y = \dfrac{a}{c}$.

EXAMPLE 3 Graphing a Rational Function of the Form $y = \dfrac{ax + b}{cx + d}$

WATCH

Graph $f(x) = \dfrac{2x + 1}{x - 3}$. Find the domain and range.

SOLUTION



Step 1 Draw the asymptotes. Solve $x - 3 = 0$ for x to find the vertical asymptote $x = 3$. The horizontal asymptote is $y = \dfrac{a}{c} = \dfrac{2}{1} = 2$.

Step 2 Plot points to the left of the vertical asymptote, such as $(2, -5)$, $\left(0, -\dfrac{1}{3}\right)$, and $\left(-2, \dfrac{3}{5}\right)$. Plot points to the right of the vertical asymptote, such as $(4, 9)$, $\left(6, \dfrac{13}{3}\right)$, and $\left(8, \dfrac{17}{5}\right)$.

Step 3 Draw the two branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.

▶ The domain is all real numbers except $x = 3$ and the range is all real numbers except $y = 2$.

Rewriting a rational function may reveal properties of the function and its graph. For example, rewriting a rational function in the form $y = \dfrac{a}{x - h} + k$ reveals that it is a translation of $y = \dfrac{a}{x}$ with vertical asymptote $x = h$ and horizontal asymptote $y = k$.

EXAMPLE 4 Rewriting and Graphing a Rational Function

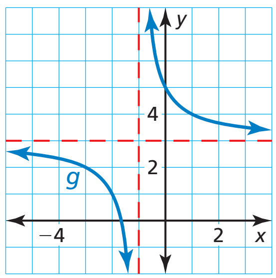
WATCH

Rewrite $g(x) = \dfrac{3x + 5}{x + 1}$ in the form $g(x) = \dfrac{a}{x - h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \dfrac{a}{x}$.

SOLUTION

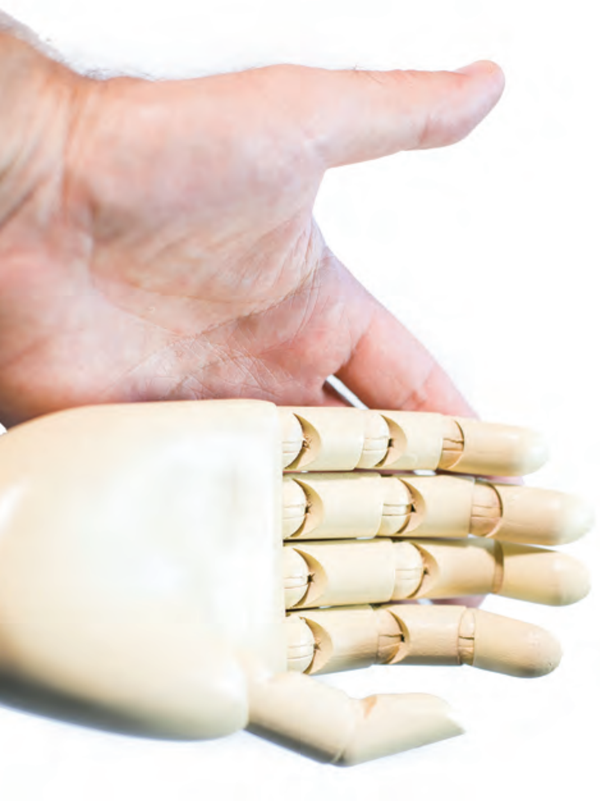
Rewrite the function by using polynomial long division.

$$\begin{array}{r} 3 \\ x + 1 \overline{)\,3x + 5} \\ \underline{3x + 3} \\ 2 \end{array}$$



▶ The rewritten function is $g(x) = \dfrac{2}{x + 1} + 3$.

The graph of g is a translation 1 unit left and 3 units up of the graph of $f(x) = \dfrac{2}{x}$.



EXAMPLE 5 Modeling Real Life

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A 3-D printer builds objects by depositing material one layer at a time. The layers are bonded together, creating a solid object. A medical researcher makes prosthetic hands using a 3-D printer. The printer costs \$1000 and the material for each hand costs \$50.

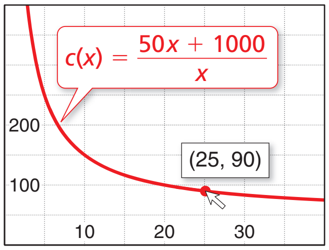
- How many prosthetic hands must be printed for the average cost per hand to fall to \$90?
- What happens to the average cost as more prosthetic hands are printed?

SOLUTION

1. **Understand the Problem** You are given the cost of a printer and the cost to create a prosthetic hand using the printer. You are asked to find the number of hands for which the average cost falls to \$90.
2. **Make a Plan** Write an equation that represents the average cost. Use technology to estimate the number of hands for which the average cost is \$90. Then analyze the graph to determine what happens to the average cost as more hands are printed.
3. **Solve and Check** Let $c(x)$ be the average cost (in dollars) for printing x hands.

$$c(x) = \frac{\text{(Unit cost)(Number printed)} + \text{(Cost of printer)}}{\text{Number printed}} = \frac{50x + 1000}{x}$$

Use technology to graph the function. Because the number of hands and average cost cannot be negative, graph the function in the first quadrant.



▶ The graph shows that the average cost falls to \$90 per hand after 25 hands are printed. Because the horizontal asymptote is $c(x) = 50$, the average cost approaches \$50 as more hands are printed.

Check Use technology to create a table for large values of x. The table shows that the average cost approaches \$50 as more hands are printed.

x	$c(x)$
10	150
100	60
1000	51
10000	50.1
100000	50.01

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Graph the function. Find the domain and range.

8. $f(x) = \frac{x - 1}{x + 3}$

9. $f(x) = \frac{2x + 1}{4x - 2}$

10. $f(x) = \frac{-3x + 2}{-x - 1}$

11. Rewrite $g(x) = \frac{2x + 3}{x + 1}$ in the form $g(x) = \frac{a}{x - h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \frac{a}{x}$.

12. WHAT IF? How do the answers in Example 5 change when the cost of the 3-D printer is \$800?

7.2 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–8, graph the function. Compare the graph with the graph of $f(x) = \frac{1}{x}$. *Example 1*

1. $g(x) = \frac{3}{x}$
2. $g(x) = \frac{10}{x}$
3. $g(x) = \frac{-5}{x}$
4. $g(x) = \frac{-9}{x}$
5. $g(x) = \frac{15}{x}$
6. $g(x) = \frac{-12}{x}$
7. $g(x) = \frac{-0.5}{x}$
8. $g(x) = \frac{0.1}{x}$

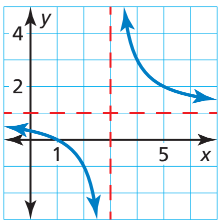
In Exercises 9–16, graph the function. Find the domain and range. *Example 2*

9. $g(x) = \frac{4}{x} + 3$
10. $y = \frac{2}{x} - 3$
11. $h(x) = \frac{6}{x - 1}$
12. $y = \frac{1}{x + 2}$
13. $h(x) = \frac{-3}{x + 2}$
14. $f(x) = \frac{-2}{x - 7}$
15. $g(x) = \frac{-3}{x - 4} - 1$
16. $y = \frac{10}{x + 7} - 5$

ANALYZING RELATIONSHIPS **In Exercises 17–20, match the function with its graph. Explain your reasoning.**

17. $g(x) = \frac{2}{x - 3} + 1$
18. $h(x) = \frac{2}{x + 3} + 1$
19. $f(x) = \frac{2}{x - 3} - 1$
20. $y = \frac{2}{x + 3} - 1$

A.



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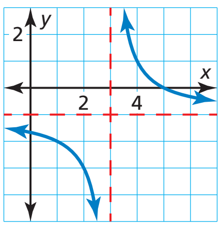

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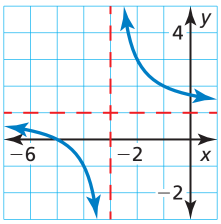

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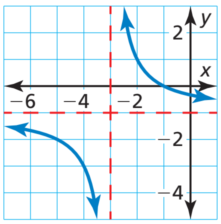


In Exercises 21–28, graph the function. Find the domain and range. *Example 3*

21. $f(x) = \frac{x + 4}{x - 3}$
22. $y = \frac{x - 1}{x + 5}$
23. $y = \frac{x + 6}{4x - 8}$
24. $h(x) = \frac{8x + 3}{2x - 6}$
25. $f(x) = \frac{-5x + 2}{4x + 5}$
26. $g(x) = \frac{6x - 1}{3x - 1}$
27. $h(x) = \frac{-5x}{-2x - 3}$
28. $y = \frac{-2x + 3}{-x + 10}$

ERROR ANALYSIS **In Exercises 29 and 30, describe and correct the error in finding the vertical and horizontal asymptotes of the graph of the function.**

29. $f(x) = \frac{3}{x + 6} - 2$

> The vertical asymptote is $x = 6$.
> The horizontal asymptote is $y = -2$.

30. $g(x) = \frac{10x + 9}{2x + 1}$

> $2x + 1 = 0$
> $2x = -1$
> $x = -\frac{1}{2}$
> The vertical asymptote is $x = -\frac{1}{2}$.
> The horizontal asymptote is $y = \frac{9}{1} = 9$.

In Exercises 31–38, rewrite the function in the form $g(x) = \frac{a}{x - h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \frac{a}{x}$. *Example 4*

31. $g(x) = \frac{5x + 6}{x + 1}$
32. $g(x) = \frac{7x + 4}{x - 3}$
33. $g(x) = \frac{2x - 4}{x - 5}$
34. $g(x) = \frac{4x - 11}{x - 2}$
35. $g(x) = \frac{x + 18}{x - 6}$
36. $g(x) = \frac{x + 2}{x - 8}$
37. $g(x) = \frac{7x - 20}{x + 13}$
38. $g(x) = \frac{9x - 3}{x + 7}$

39. **MODELING REAL LIFE** Your school purchases a math software application. The program has an initial cost of \$500 plus \$20 for each student who subscribes. *Example 5*

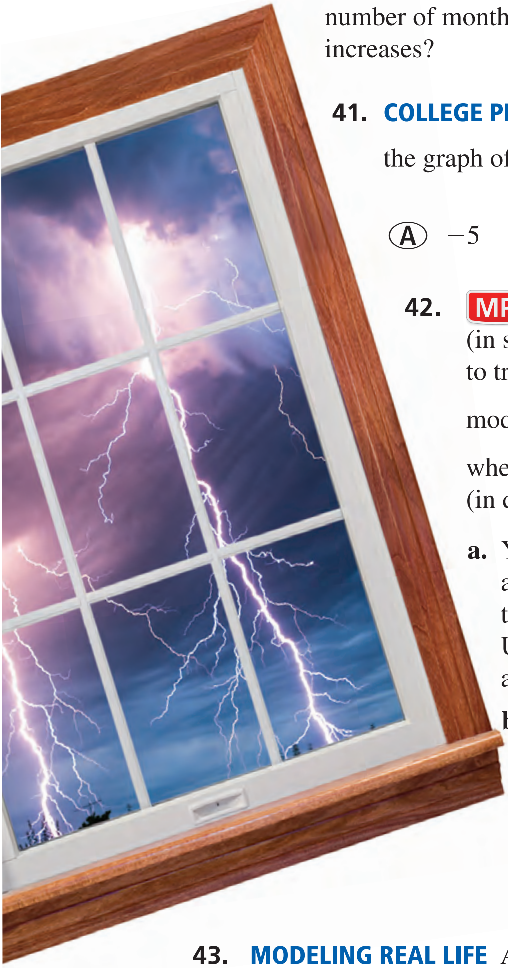
a. How many students must subscribe for the average cost per student to fall to \$30?

b. What happens to the average cost as more students subscribe?

40. **MODELING REAL LIFE** To join a rock climbing gym, you must pay an initial fee of \$100 and a monthly fee of \$59.

a. How many months must you have a membership for the average cost per month to fall to \$69?

b. What happens to the average cost as the number of months that you are a member increases?



41. **COLLEGE PREP** What is the x-intercept of the graph of the function $y = \dfrac{x-5}{x-2}$?

Ⓐ -5 Ⓑ -2 Ⓒ 2 Ⓓ 5

42. **MP USING TOOLS** The time t (in seconds) it takes for sound to travel 1 kilometer can be modeled by $t = \dfrac{1000}{0.6T + 331}$, where T is the air temperature (in degrees Celsius).

a. You are 1 kilometer from a lightning strike. You hear thunder 2.9 seconds later. Use a graph to find the approximate air temperature.

b. Find the average rate of change in the time it takes sound to travel 1 kilometer as the air temperature increases from 0°C to 10°C.

43. **MODELING REAL LIFE** A business is studying the cost to remove a pollutant from the ground at its site. The function $y = \dfrac{15x}{1.1 - x}$ models the estimated cost y (in thousands of dollars) to remove x percent (expressed as a decimal) of the pollutant.

a. Graph the function. Describe a reasonable domain and range.

b. How much does it cost to remove 20% of the pollutant? 40% of the pollutant? 80% of the pollutant? Does doubling the percent of the pollutant removed double the cost? Explain.

44. **MAKING AN ARGUMENT** Is it possible for a rational function to have two vertical asymptotes? Justify your answer.

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MP USING TOOLS In Exercises 45–48, use technology to graph the function. Then determine whether the function is *even*, *odd*, or *neither*.

45. $h(x) = \dfrac{6}{x^2 + 1}$

46. $f(x) = \dfrac{2x^2}{x^2 - 9}$

47. $y = \dfrac{x^3}{3x^2 + x^4}$

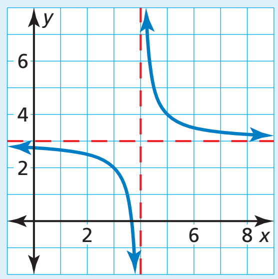
48. $f(x) = \dfrac{4x^2}{2x^3 - x}$

49. **MP PROBLEM SOLVING** Internet service provider A charges a \$50 installation fee and a monthly fee of \$43. The table shows the average monthly costs y of provider B for x months of service. Which provider would you choose if you plan to stay with the provider for 12 months? 18 months? 24 months?

Months, x	Average monthly cost (dollars), y
12	\$46.92
18	\$45.94
24	\$45.46

50. **HOW DO YOU SEE IT?**

The graph of $f(x) = \dfrac{1}{x-h} + k$ is shown. Find the values of h and k. Then describe the end behavior of f.



In Exercises 51 and 52, sketch a graph of the rational function f with the given characteristics.

51. • The domain of f is all real numbers except $x = 1$.
• $f(x) \to -5$ as $x \to -\infty$ and as $x \to +\infty$.

52. • The graph of f does not have a y-intercept.
• The range of f is all real numbers except $y = 6$.

53. **MP REASONING** A rational function f is of the form $f(x) = \dfrac{a}{x - h} + k$. The asymptotes of the graph of f intersect at (3, 2). The point (2, 1) is on the graph of f. Find three other points on the graph of f. Explain your reasoning.

54. **DRAWING CONCLUSIONS** In what line(s) is the graph of $y = \dfrac{1}{x}$ symmetric? What does this symmetry tell you about the inverse of the function $f(x) = \dfrac{1}{x}$?

55. **ABSTRACT REASONING** Describe the intervals where the graph of $y = \dfrac{a}{x}$ is increasing or decreasing when (a) $a > 0$ and (b) $a < 0$. Explain your reasoning.

56. **THOUGHT PROVOKING** There are four basic types of conic sections: parabolas, circles, ellipses, and hyperbolas. Each of these can be represented by the intersection of a double-napped cone and a plane. Three of them are shown. Sketch the intersection for a hyperbola.

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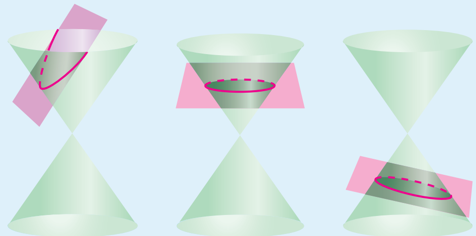

Parabola Circle Ellipse

REVIEW & REFRESH

WATCH

In Exercises 57 and 58, write a function g whose graph represents the indicated transformations of the graph of f.

57. $f(x) = x$; translation 6 units down, followed by a reflection in the x-axis

58. $f(x) = |x|$; translation 4 units left, followed by a horizontal stretch by a factor of 3

In Exercises 59–64, factor the polynomial completely.

59. $4x^2 - 4x - 80$

60. $10x^2 + 31x - 14$

61. $x^2 + 20x + 100$

62. $x^3 - 216$

63. $x^3 + 11x^2 + 28x$

64. $2x^3 - 14x^2 + 5x - 35$

65. **MODELING REAL LIFE** The time t (in minutes) required to empty a tank varies inversely with the pumping rate r (in gallons per minute). The rate of a certain pump is 70 gallons per minute. It takes the pump 20 minutes to empty the tank. Complete the table for the times it takes the pump to empty a tank for the given pumping rates.

Pumping rate (gal/min)	Time (min)
50	
56	

66. Find the discriminant of the equation $4x^2 - 10x + 7 = 0$ and describe the number and type of solutions of the equation.

In Exercises 67 and 68, solve the equation.

67. $4^x = 21$

68. $\log_3(5x + 1) = 4$

In Exercises 69 and 70, simplify the expression.

69. $2^{1/2} \cdot 2^{3/5}$

70. $\dfrac{6^{5/6}}{6^{1/6}}$

71. Rewrite $g(x) = \dfrac{2x + 9}{x + 8}$ in the form $g(x) = \dfrac{a}{x - h} + k$. Graph the function. Describe the graph of g as a transformation of the graph of $f(x) = \dfrac{a}{x}$.

72. Determine the type of function represented by the table. Explain your reasoning.

x	−3	−2	−1	0	1	2
y	128	32	8	2	$\frac{1}{2}$	$\frac{1}{8}$

73. **MP STRUCTURE** Which functions do *not* have a domain of all real numbers?

$f(x) = (x - 8)^2 - 5$ $\qquad g(x) = \sqrt{x - 8} - 5$

$h(x) = \sqrt[3]{x - 8} - 5$ $\qquad p(x) = \dfrac{1}{x - 8} - 5$