6.7 Modeling with Exponential and Logarithmic Functions

GO DIGITAL

Learning Target Write exponential and logarithmic functions to model sets of data.

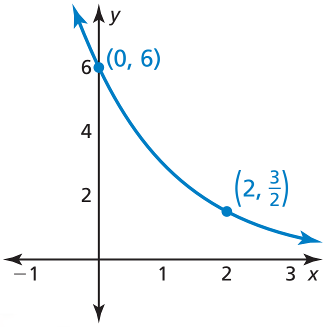
Success Criteria

- I can use a common ratio to determine whether data can be represented by an exponential function.
- I can write an exponential function using two points.
- I can use technology to find exponential models and logarithmic models for sets of data.

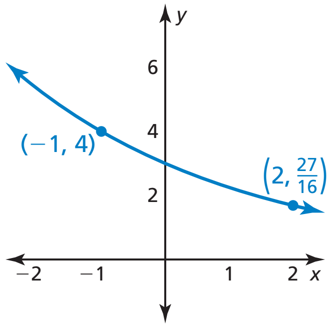
EXPLORE IT! Writing Exponential Functions

Work with a partner. In parts (a) and (b), the graph of a function of the form $f(x) = ab^x$ is shown.

a. Write an exponential function that passes through the points. Explain your reasoning.



b. Write an exponential function that passes through the points. Explain your reasoning. Compare your methods with those of your classmates.



Math Practice

Construct Arguments
Do two points always determine an exponential function of the form $f(x) = ab^x$?

c. A function f is of the form $f(x) = ab^x$, where a is a real number and $b > 0$. Can you find the values of a and b so that f passes through the points (0, 4) and (2, −2)? Explain your reasoning.

d. You perform an experiment where you measure the temperature T (in degrees Fahrenheit) of coffee m minutes after it is poured into a cup. The temperature of the coffee is initially 185°F. After 5 minutes, the temperature of the coffee is about 150°F. Can you use a model of the form $y = ab^x$ to model this situation? Explain. If not, sketch a graph that could model the temperature of the coffee over time. Make several observations about your graph.

GO DIGITAL

Choosing Functions to Model Data

You have analyzed *finite differences* of data with equally-spaced inputs to determine what type of polynomial function can be used to model the data. To determine whether an exponential function can be used to model the data, the outputs must be multiplied by a constant factor. So, consecutive outputs form equivalent ratios.

EXAMPLE 1 Using Differences or Ratios to Identify Functions

WATCH

Determine the type of function represented by each table.

a.

x	−2	−1	0	1	2	3	4
y	0.5	1	2	4	8	16	32

b.

x	−8	−6	−4	−2	0	2	4
y	−1	8	7	2	−1	4	23

SOLUTION

a. The inputs are equally spaced. Look for a pattern in the outputs.

x	−2	−1	0	1	2	3	4
y	0.5	1	2	4	8	16	32

× 2   × 2   × 2   × 2   × 2   × 2

▶ As x increases by 1, y is multiplied by 2. So, the common ratio is 2, and the data represent an exponential function.

b. The inputs are equally spaced. The outputs do not have a common ratio. So, analyze the finite differences.

x	−8	−6	−4	−2	0	2	4
y	−1	8	7	2	−1	4	23

9   −1   −5   −3   5   19   first differences

−10   −4   2   8   14   second differences

6   6   6   6   third differences

REMEMBER
First differences of linear functions are constant, second differences of quadratic functions are constant, and so on.

▶ The third differences are nonzero and constant. So, the data represent a cubic function.

SELF-ASSESSMENT

1 I do not understand.   2 I can do it with help.   3 I can do it on my own.   4 I can teach someone else.

1. **WRITING** Given a table of values, explain how you can determine whether an exponential function can be used to model the data.

Determine the type of function represented by the table. Explain your reasoning.

2.

x	0	10	20	30
y	15	12	9	6

3.

x	0	2	4	6
y	27	9	3	1

Writing Exponential Functions

Math Practice

Consider Similar Problems

Two points can determine an exponential curve. What other type of function can be determined by two points?

EXAMPLE 2 Writing an Exponential Function Using Two Points

Write an exponential function $y = ab^x$ whose graph passes through (1, 6) and (3, 54).

SOLUTION

Step 1 Substitute the coordinates of the two given points into $y = ab^x$.

$6 = ab^1$ Equation 1: Substitute 6 for y and 1 for x.

$54 = ab^3$ Equation 2: Substitute 54 for y and 3 for x.

Step 2 Solve for a in Equation 1 to obtain $a = \frac{6}{b}$ and substitute this expression for a in Equation 2.

$54 = \left(\frac{6}{b}\right)b^3$ Substitute $\frac{6}{b}$ for a in Equation 2.

$54 = 6b^2$ Simplify.

$9 = b^2$ Divide each side by 6.

$3 = b$ Take the positive square root because $b > 0$.

REMEMBER

By the definition of an exponential function, b must be positive.

Step 3 Substitute 3 for b to determine that $a = \frac{6}{b} = \frac{6}{3} = 2$.

▶ So, the exponential function is $y = 2(3)^x$.

Data do not always show an *exact* exponential relationship. When the data in a scatter plot show an *approximately* exponential relationship, you can model the data with an exponential function.

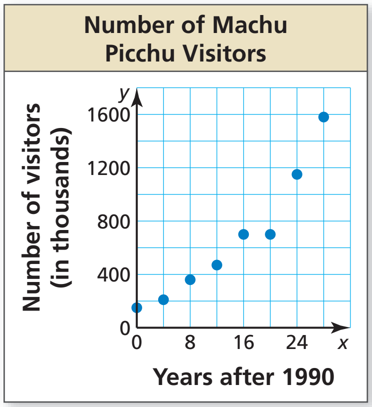
EXAMPLE 3 Finding an Exponential Model

The table shows the numbers y (in thousands) of people who visit Machu Picchu x years after 1990. Write a function that models the data.

Years after 1990, x	0	4	8	12	16	20	24	28
Number of visitors, y	150	210	360	470	700	700	1150	1580

SOLUTION

Step 1 Make a scatter plot of the data. The data appear exponential.



Step 2 Choose any two points to write a function, such as (0, 150) and (20, 700). Substitute the coordinates of these two points into $y = ab^x$.

$150 = ab^0$

$700 = ab^{20}$

The first equation shows that $a = 150$.

Substitute 150 for a in the second equation to obtain $b = \sqrt[20]{\frac{700}{150}} \approx 1.08$.

▶ So, an exponential function that models the data is $y = 150(1.08)^x$.

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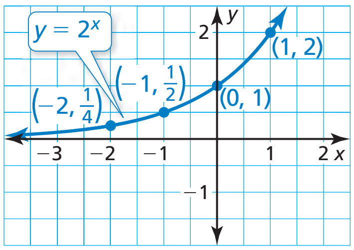
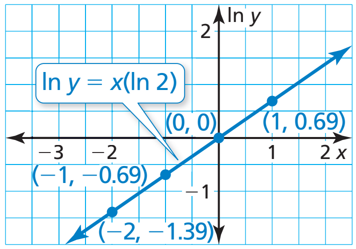
A set of more than two points (x, y) fits an exponential pattern if and only if the set of transformed points $(x, \ln y)$ fits a linear pattern.

Graph of points (x, y)



The graph is an exponential curve.

Graph of points $(x, \ln y)$



The graph is a line.

EXAMPLE 4 Writing a Model Using Transformed Points

WATCH

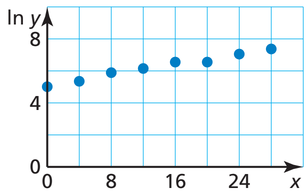
Use the data from Example 3. Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data.

SOLUTION

Step 1 Create a table of data pairs $(x, \ln y)$.

x	0	4	8	12	16	20	24	28
$\ln y$	5.01	5.35	5.89	6.15	6.55	6.55	7.05	7.37

Step 2 Plot the transformed points as shown. The points lie close to a line, so an exponential model should be a good fit for the original data.



Step 3 Find an exponential model $y = ab^x$ by choosing any two points on the line, such as (0, 5.01) and (20, 6.55). Use these points to write an equation of the line. Then solve for y.

$\ln y - 5.01 = 0.08(x - 0)$ — Equation of line

$\ln y = 0.08x + 5.01$ — Simplify.

$y = e^{0.08x + 5.01}$ — Exponentiate each side using base e.

$y = e^{0.08x}(e^{5.01})$ — Use properties of exponents.

$y = 149.90(1.08)^x$ — Simplify.

So, an exponential function that models the data is $y = 149.90(1.08)^x$.

Math Practice

Communicate Precisely

Explain why the line can be represented by the equation $\ln y - 5.01 = 0.08(x - 0)$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Write an exponential function $y = ab^x$ whose graph passes through the given points.

4. (2, 12), (3, 24)

5. (1, 2), (3, 32)

6. (2, 16), (5, 2)

7. A store sells gaming laptops. The table shows the numbers y of gaming laptops sold during the xth month that the store has been open. Repeat Examples 3 and 4 using these data.

Month, x	1	2	3	4	5	6	7
Number of gaming laptops, y	12	16	25	36	50	67	96

Using Technology

You can use technology to find models for exponential and logarithmic data.

EXAMPLE 5 Finding an Exponential Model

Use technology to find an exponential model for the data in Example 3. Then use this model and the models in Examples 3 and 4 to estimate the number of visitors in 2020. Compare the estimates.

SOLUTION

Use technology to enter the data and perform an exponential regression. The model is $y = 163.73(1.08)^x$.

$y = ab^x$
PARAMETERS
$a = 163.729$ $b = 1.08489$
STATISTICS
$r^2 = 0.9799$
$r = 0.9899$

Substitute $x = 30$ into each model to estimate the number of visitors in 2020.

Example 3: $y = 150(1.08)^{30} \approx 1509$

Example 4: $y = 149.90(1.08)^{30} \approx 1508$

Regression model: $y = 163.73(1.08)^{30} \approx 1648$

▶ The estimates for the models in Examples 3 and 4 are close to each other. These estimates are less than the estimate for the regression model.

EXAMPLE 6 Finding a Logarithmic Model

The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is 1 atmosphere (1.033227 kilograms per square centimeter). The table shows the air pressures p (in atmospheres) at several altitudes h (in kilometers). Use technology to find a logarithmic model of the form $h = a + b \ln p$ that represents the data. Estimate the altitude when the air pressure is 0.75 atmosphere.

Air pressure, *p*	1	0.55	0.25	0.12	0.06	0.02
Altitude, *h*	0	5	10	15	20	25

SOLUTION

Use technology to enter the data and perform a logarithmic regression. The model is $h = 0.86 - 6.45 \ln p$.

$y = a + b \ln(x)$
PARAMETERS
$a = 0.862658$ $b = -6.44738$
STATISTICS
$R^2 = 0.9926$

Substitute $p = 0.75$ into the model to obtain

$h = 0.86 - 6.45 \ln 0.75 \approx 2.7.$

▶ So, when the air pressure is 0.75 atmosphere, the altitude is about 2.7 kilometers.

Weather balloons carry instruments that send back information such as wind speed, temperature, and air pressure.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

8. Use technology to find an exponential model for the data in Exercise 7. Then use this model and the models you found in Exercise 7 to predict the number of gaming laptops sold during the eighth month. Compare the predictions.

9. Use technology to find a logarithmic model of the form $p = a + b \ln h$ for the data in Example 6. Explain why the result is an error message.

6.7 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–4, determine the type of function represented by the table. Explain your reasoning. *Example 1*

1.

x	0	3	6	9	12	15
y	0.25	1	4	16	64	256

2.

x	−4	−3	−2	−1	0	1	2
y	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

3.

x	5	10	15	20	25	30
y	4	3	7	16	30	49

4.

x	−3	−1	1	3	5	7
y	61	5	5	13	−19	−139

In Exercises 5–14, write an exponential function $y = ab^x$ whose graph passes through the given points. *Example 2*

5. (1, 3), (2, 12)

6. (2, 24), (3, 144)

7. (−1, 4), (1, 1)

8. (−2, 96), (1, 1.5)

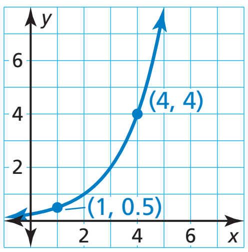
9. (1, 2), (3, 50)

10. (−4, 32), (−2, 2)

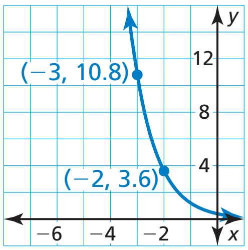
11. (−1, 10), (4, 0.31)

12. (2, 6.4), (5, 409.6)

13.



14.



ERROR ANALYSIS In Exercises 15 and 16, describe and correct the error in determining the type of function represented by the table.

15.

x	0	1	2	3	4
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

×3 ×3 ×3 ×3

The outputs have a common ratio of 3, so the data represent a linear function.

16.

x	−2	−1	1	2	4
y	3	6	12	24	48

×2 ×2 ×2 ×2

The outputs have a common ratio of 2, so the data represent an exponential function.

17. MODELING REAL LIFE A store sells electric scooters. The table shows the numbers y of scooters sold during the xth year that the store has been open. Write a function that models the data. *Example 3*

x	1	2	3	4	5	6	7
y	9	14	19	25	37	53	71

18. MODELING REAL LIFE The table shows the numbers y (in thousands) of visits to a website during the xth month. Write a function that models the data. Then use your model to predict the number of visits after 1 year.

x	1	2	3	4	5	6	7
y	22	39	70	126	227	408	735

In Exercises 19–22, determine whether the data show an exponential relationship. Then write a function that models the data.

19.

x	1	6	11	16	21
y	12	28	76	190	450

20.

x	−3	−1	1	3	5
y	2	7	24	68	194

21.

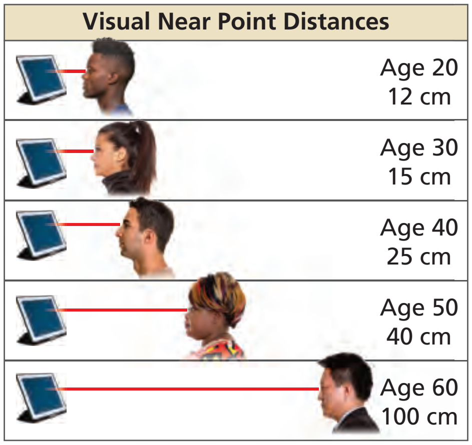
x	0	10	20	30	40	50	60
y	66	58	48	42	31	26	21

22.

x	−20	−13	−6	1	8	15
y	25	19	14	11	8	6

GO DIGITAL

23. **MODELING REAL LIFE** Your visual near point is the closest point at which your eyes can see an object distinctly. The diagram shows the near point y (in centimeters) at age x (in years). Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data. *Example 4*



24. **MODELING REAL LIFE** Use the data from Exercise 17. Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data.

In Exercises 25–28, create a scatter plot of the points $(x, \ln y)$ to determine whether an exponential model fits the data. If so, find an exponential model for the data.

25.

x	1	2	3	4	5
y	18	36	72	144	288

26.

x	1	4	7	10	13
y	3.3	10.1	30.6	92.7	280.9

27.

x	−13	−6	1	8	15
y	9.8	12.2	15.2	19	23.8

28.

x	−8	−5	−2	1	4
y	1.4	1.67	5.32	6.41	7.97

29. **MP USING TOOLS** Use technology to find an exponential model for the data in Exercise 17. Then use the model to predict the number of electric scooters sold during the tenth year. *Example 5*

30. **MP USING TOOLS** A doctor measures an astronaut's heart rate y (in beats per minute) at various times x (in minutes) after the astronaut finishes exercising. The results are shown in the table. Use technology to find an exponential model for the data. Then use the model to predict the astronaut's heart rate after 16 minutes.

x	0	2	4	6	8	10	12
y	172	132	110	92	84	78	75

31. **MP USING TOOLS** A clay pot with a temperature of 160°C is removed from a kiln and placed in a room with a temperature of 20°C. The table shows the temperatures d (in degrees Celsius) of the clay pot at several times t (in hours) after it is removed from the kiln. Use technology to find a logarithmic model of the form $t = a + b \ln d$ that represents the data. Estimate how long it takes for the clay pot to cool to 50°C. *Example 6*

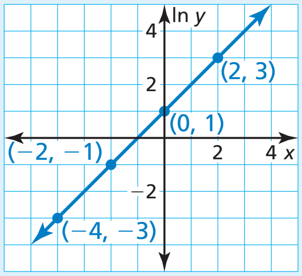
d	160	90	56	38	29	24
t	0	1	2	3	4	5

32. **MP USING TOOLS** The f-stops on a camera control the amount of light that enters the camera. Let s be a measure of the amount of light that strikes the film and let f be the f-stop. The table shows several f-stops on a 35-millimeter camera. Use technology to find a logarithmic model of the form $s = a + b \ln f$ that represents the data. Estimate the amount of light that strikes the film when $f = 5.657$.

f	s
1.414	1
2.000	2
2.828	3
4.000	4
11.314	7

33. **MAKING AN ARGUMENT** Is it possible to find a logarithmic model of the form $d = a + b \ln t$ for the data in Exercise 31? Explain.

34. **HOW DO YOU SEE IT?**
The graph shows a set of data pairs $(x, \ln y)$. Do the data pairs (x, y) fit an exponential pattern? Explain your reasoning.



35. **CRITICAL THINKING** You plant a sunflower seedling in a community garden. The height (in centimeters) of the sunflower after t weeks can be modeled by the *logistic function*

$$h(t) = \frac{256}{1 + 13e^{-0.65t}}.$$

a. Find the time it takes the sunflower to reach a height of 200 centimeters.

b. Use technology to graph the function. Interpret the meaning of the asymptote(s) in this situation.

36. **THOUGHT PROVOKING**
Is it possible to write y as an exponential function of x when p is positive? If so, write the function. If not, explain why not.

x	1	2	3	4	5
y	p	$2p$	$4p$	$8p$	$16p$

REVIEW & REFRESH

In Exercises 37 and 38, tell whether *x* and *y* are in a proportional relationship. Explain your reasoning.

37. $y = \frac{x}{2}$ 38. $y = 3x - 12$

39. **MODELING REAL LIFE** You brew a cup of coffee at a temperature of 200°F. You place the cup on a table until it reaches a drinking temperature of 130°F. When the room temperature is 72°F, the cooling rate of the coffee is $r = 0.04$. Use Newton's Law of Cooling, $T = (T_0 - T_R)e^{-rt} + T_R$, to determine how long you should wait to drink the coffee.

40. Determine whether functions f and g are inverses. Explain your reasoning.

x	−2	−1	0	1	2
f(x)	15	11	7	3	−1

x	15	11	7	3	−1
g(x)	−2	−1	0	1	2

In Exercises 41 and 42, use the change-of-base formula to evaluate the logarithm.

41. $\log_3 20$ 42. $\log_4 \frac{5}{12}$

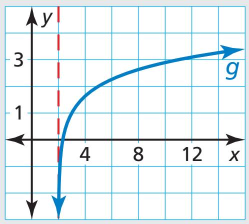
In Exercises 43 and 44, write an exponential function $y = ab^x$ whose graph passes through the given points.

43. (3, 1), (5, 4) 44. (−1, 4), (7, 0.4)

In Exercises 45 and 46, identify the focus, directrix, and axis of symmetry of the parabola. Graph the equation.

45. $x = \frac{1}{8}y^2$ 46. $y^2 = \frac{2}{5}x$

47. The function g is a transformation of $f(x) = \log_3 x$. Write a rule for g.



48. Describe the transformation of $f(x) = x^4$ represented by $g(x) = 2x^4 - 1$. Then graph each function.

In Exercises 49–52, solve the equation.

49. $4^x = 9$ 50. $e^{3x} = e^{5x-6}$

51. $\ln(8x + 5) = \ln 9$ 52. $\log_2(3x - 1) = 5$

53. Show that $x + 5$ is a factor of $f(x) = x^3 - 2x^2 - 23x + 60$. Then factor $f(x)$ completely.

54. Complete the square for $x^2 - 4x$. Then factor the trinomial.