45. $\dfrac{2x^2 + 3x - 1}{x^3 - 1}$

46. $\dfrac{x^3 - 2x^2 + x - 2}{x^4 - x^3 + x - 1}$

Enrichment Exercises

There is a shortcut for finding some partial fraction decompositions of quadratic polynomials that do not factor over the real numbers. Exercises 47 and 48 give one method and some examples.

47. Show that for real numbers a and b with $a \neq b$,

$$\frac{1}{(b-a)[p(x)+a]} + \frac{1}{(a-b)[p(x)+b]} = \frac{1}{[p(x)+a][p(x)+b]}$$

48. Use the result of Exercise 47 to find the partial fraction decomposition of the following.

a. $\dfrac{1}{(x^2+4)(x^2+1)}$

b. $\dfrac{1}{(x^2+1)(x^2+9)}$

c. $\dfrac{1}{(x^2+x+1)(x^2+x+2)}$

d. $\dfrac{1}{(x^2+2x+4)(x^2+2x+9)}$

SECTION 6.5 Inequalities in Two Variables and Systems of Inequalities

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A40.

PS1. Graph: $y = -2x + 3$ [2.3]

PS2. Graph: $y = -x^2 + 3x + 4$ [2.4]

PS3. Graph: $y = |x| + 1$ [2.2]

PS4. Graph: $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ [5.3]

PS5. Graph: $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$ [5.2]

PS6. Graph: $(y+2)^2 = 4x$ [5.1]

Graphing an Inequality

Two examples of inequalities in two variables are

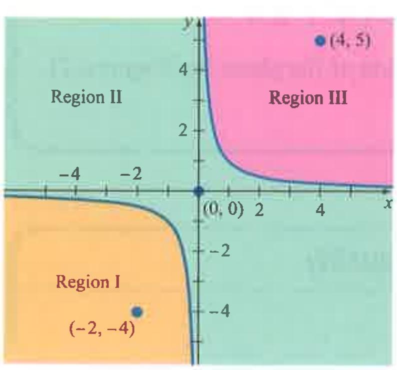
$$2x + 3y > 6 \qquad \text{and} \qquad xy \leq 1$$

A solution of an inequality in two variables is an ordered pair (x, y), with real coordinates, that satisfies the inequality. For example, $(-2, 4)$ is a solution of the first inequality because $2(-2) + 3(4) > 6$. The ordered pair $(2, 1)$ is not a solution of the second inequality because $(2)(1) \nleq 1$.

The **solution set of an inequality** in two variables is the set of all ordered pairs, with real coordinates, that satisfy the inequality. The **graph** of an inequality is the graph of the solution set.

To sketch the graph of an inequality, first replace the inequality symbol with an equality sign and sketch the graph of the equation. Use a dashed graph for $<$ or $>$ to indicate that the curve is not part of the solution set. Use a solid graph for $\leq$ or $\geq$ to show that the curve *is* part of the solution set.

It is important to test an ordered pair in each region of the plane defined by the graph. If the ordered pair satisfies the inequality, shade that entire region. Do this for each region into which the graph divides the plane. For example, consider the inequality $xy \geq 1$. Figure 6.19 shows the three regions of the plane defined by this inequality. Because the inequality is $\geq$, a solid graph is used.



$xy \geq 1$

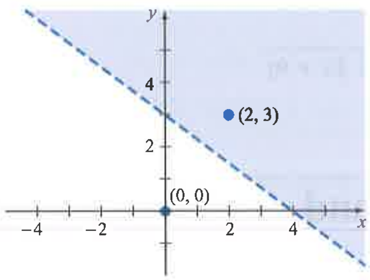
Figure 6.19

Choose an ordered pair in each of the three regions and determine whether that ordered pair satisfies the inequality. In Region I, choose a point, say $(-2, -4)$. Because $(-2)(-4) \geq 1$, Region I is part of the solution set. In Region II, choose a point, say $(0,0)$. Because $0 \cdot 0 \ngeq 1$, Region II is not part of the solution set. In Region III, choose $(4,5)$. Because $4 \cdot 5 \geq 1$, Region III is part of the solution set.

You may choose the coordinates of any point not on the graph of the equation as a test ordered pair; $(0,0)$ is often a good choice.

Question • Is $(0,0)$ a solution of $y \geq x^2 + 2x - 3$?

A half-plane is the set of points on one side of a line in a plane. The graph of a linear inequality in two variables will always include one of the half-planes that are produced by graphing the given linear inequality with the inequality symbol replaced with an equal sign.



$3x + 4y > 12$

Figure 6.20

EXAMPLE 1 Graph a Linear Inequality

Graph: $3x + 4y > 12$

Solution

Graph the line $3x + 4y = 12$ using a dashed line.

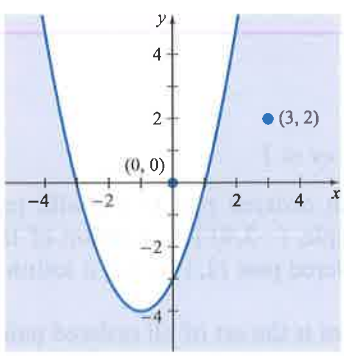
$$\text{Test the ordered pair } (0,0)\text{:}\quad 3(0) + 4(0) = 0 \not> 12$$

Because $(0,0)$ does not satisfy the inequality, do not shade this region.

$$\text{Test the ordered pair } (2,3)\text{:}\quad 3(2) + 4(3) = 18 > 12$$

Because $(2,3)$ satisfies the inequality, the half-plane that includes $(2,3)$, shown in blue in Figure 6.20, is the solution set.

▶ **Try Exercise 12, page 521**



$y \leq x^2 + 2x - 3$

Figure 6.21

EXAMPLE 2 Graph a Nonlinear Inequality

Graph: $y \leq x^2 + 2x - 3$

Solution

Graph the parabola $y = x^2 + 2x - 3$ using a solid curve.

$$\text{Test the ordered pair } (0,0)\text{:}\quad 0 \nleq 0^2 + 2(0) - 3$$

Because $(0,0)$ does not satisfy the inequality, do not shade the portion of the plane that includes (0, 0).

$$\text{Test the ordered pair } (3,2)\text{:}\quad 2 \leq (3)^2 + 2(3) - 3$$

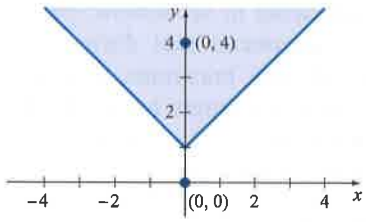
Because $(3,2)$ satisfies the inequality, shade this region of the plane. See Figure 6.21.

▶ **Try Exercise 18, page 521**

EXAMPLE 3 Graph an Absolute Value Inequality

Graph: $y \geq |x| + 1$

Answer • Yes.



$y \geq |x| + 1$

Figure 6.22

Solution

Graph the equation $y = |x| + 1$ using a solid graph.

$$\text{Test the ordered pair } (0,0)\text{:} \quad 0 \ngeq |0| + 1$$

Because $0 \ngeq 1$, (0, 0) does not belong to the solution set. Do not shade the portion of the plane that contains (0, 0).

$$\text{Test the ordered pair } (0,4)\text{:} \quad 4 \geq |0| + 1$$

Because (0, 4) satisfies the inequality, shade this region. See Figure 6.22.

▶ **Try Exercise 26, page 521**

Systems of Inequalities in Two Variables

The **solution set of a system of inequalities** is the intersection of the solution sets of the individual inequalities. To graph the solution set of a system of inequalities, first graph the solution set of each inequality. The solution set of the system of inequalities is the region of the plane represented by the intersection of the shaded regions.

EXAMPLE 4 Graph a System of Linear Inequalities

Graph the solution set of the system of inequalities.

$$\begin{cases} 3x - 2y > 6 \\ 2x - 5y \leq 10 \end{cases}$$

Solution

Graph the line $3x - 2y = 6$ using a dashed line. Test the ordered pair (0, 0). Because $3(0) - 2(0) \not> 6$, (0, 0) does not belong to the solution set. Do not shade the region that contains (0, 0). Instead, shade the region below and to the right of the graph of $3x - 2y = 6$, because any ordered pair from this region satisfies $3x - 2y > 6$. See Figure 6.23.

Graph the line $2x - 5y = 10$ using a solid line. Test the ordered pair (0, 0). Because $2(0) - 5(0) \leq 10$, shade the region that contains (0, 0). See Figure 6.23.

The solution set is the region of the plane represented by the intersection of the solution sets of the individual inequalities. See Figure 6.24.

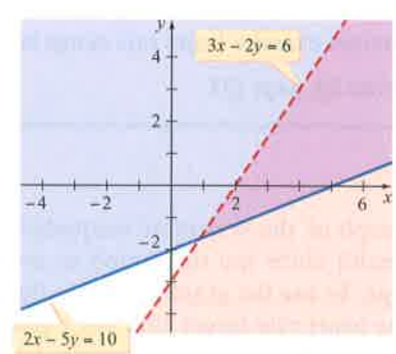


Figure 6.23

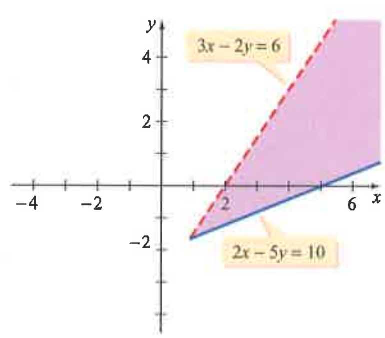


Figure 6.24

▶ **Try Exercise 32, page 521**

Many fitness experts recommend that you participate in an aerobic exercise program at least three times a week. They also recommend that during these workouts your heart rate stays within 60% to 80% of your maximum heart rate. Several popular methods are used to establish a person's maximum heart rate. The following maximum heart rate formula is based on research by the Department of Kinesiology and Applied Physiology at the University of Colorado.[1]

$$\text{Maximum heart rate} = 208 - 0.7x$$

where x is a person's age in years.

EXAMPLE 5 Find a Targeted Exercise Heart Rate Range

A health club uses the following system of inequalities to determine targeted exercise heart rate ranges based on a person's age. In this system, y is the person's exercise heart rate in beats per minute and x is the person's age in years.

$$\begin{cases} y \geq 0.60(208 - 0.7x) & (1) \\ y \leq 0.80(208 - 0.7x) & (2) \\ 20 \leq x \leq 70 & (3) \end{cases}$$

Inequality (1) is used to determine the minimum of a person's targeted exercise heart rate range, and Inequality (2) is used to determine the maximum of a person's targeted exercise heart rate range.

Determine the targeted exercise heart rate range for Emily, who just turned 30. Round minimum and maximum values to the nearest beat per minute.

Solution

Substitute Emily's age, 30, for x in Inequality (1) to determine the minimum of her targeted exercise heart rate range. Substitute 30 for x in Inequality (2) to determine the maximum of her targeted exercise heart rate range.

$$\begin{aligned} y &\geq 0.60(208 - 0.7x) \\ y &\geq 0.60(208 - 0.7(30)) \\ y &\geq 0.60(208 - 21) \\ y &\geq 0.60(187) \\ y &\geq 112.2 \end{aligned} \qquad \begin{aligned} y &\leq 0.80(208 - 0.7x) \\ y &\leq 0.80(208 - 0.7(30)) \\ y &\leq 0.80(208 - 21) \\ y &\leq 0.80(187) \\ y &\leq 149.6 \end{aligned}$$

Emily's target exercise heart rate range is 112 to 150 beats per minute.

▶ **Try Exercise 52, page 521**

A graph of the system of inequalities in Example 5 is shown in Figure 6.25. Some health clubs use this graph to estimate a person's targeted exercise heart rate range. To use the graph, estimate the *height* of the lower and upper boundaries of the heart rate target zone for a given age x. The red dashed lines show that Emily, age 30, has a targeted exercise heart rate range of about 112 to 150 beats per minute.

[1] *Exercise and Heart Rate* by Stan Reents, May 6, 2007, http://www.athleteinme.com/ArticleView.aspx?id=275.

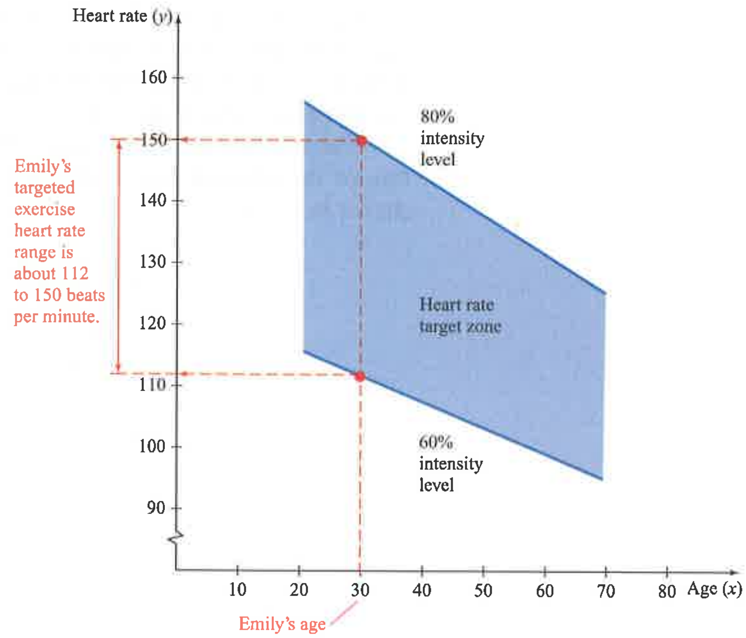


Figure 6.25

Nonlinear Systems of Inequalities

EXAMPLE 6 Graph a Nonlinear System of Inequalities

Graph the solution set of the system of inequalities.

$$\begin{cases} x^2 - y^2 \leq 9 \\ x + 3y > 3 \end{cases}$$

Solution

Graph the hyperbola $x^2 - y^2 = 9$ by using a solid graph. Test the ordered pair $(0, 0)$. Because $0^2 - 0^2 \leq 9$, shade the region containing the origin, which is the region between the branches of the hyperbola. By choosing points in the other two regions, you should determine that those regions are not part of the solution set. See Figure 6.26.

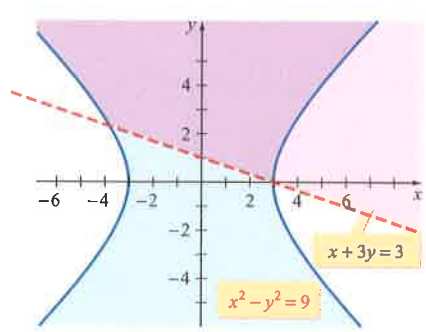


Figure 6.26

(continued)

Graph the line $x + 3y = 3$ by using a dashed graph. Test the ordered pair $(0, 0)$. Because $0 + 3(0) \ngtr 3$, do not shade the half-plane below the dashed line. Testing the ordered pair $(4, 4)$ will show that we need to shade the half-plane above the line $x + 3y = 3$. See Figure 6.26.

The solution set is the region of the plane represented by the intersection of the solution sets of the individual inequalities. This intersection is shown in Figure 6.27.

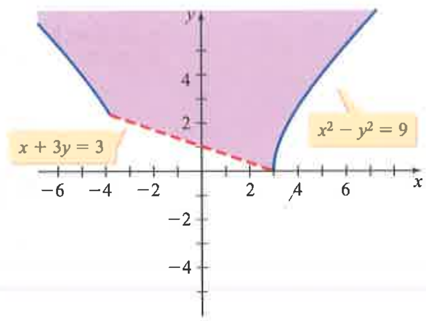


Figure 6.27

▶ Try Exercise 42, page 521

EXAMPLE 7 Identify a System of Inequalities with No Solution

Graph the solution set of the system of inequalities

$$\begin{cases} x^2 + y^2 \leq 16 \\ x^2 - y^2 \geq 36 \end{cases}$$

Solution

Graph the circle $x^2 + y^2 = 16$ by using a solid graph. Test the ordered pair $(0, 0)$. Because $0^2 + 0^2 \leq 16$, shade the inside of the circle. See Figure 6.28.

Graph the hyperbola $x^2 - y^2 = 36$ by using a solid graph. Use ordered pairs from each of the regions defined by the hyperbola to determine that the solution of $x^2 - y^2 > 36$ consists of the region to the right of the right branch of the hyperbola and the region to the left of the left branch. See Figure 6.28.

Because the solution sets of the inequalities do not intersect, the system has no solution. The solution set is the empty set.

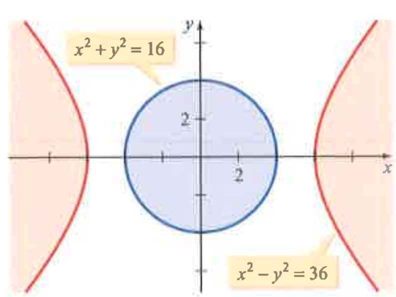


Figure 6.28

▶ Try Exercise 44, page 521

EXERCISE SET 6.5

Concept Check

1. What is a half-plane?

2. Is it possible for a system of inequalities to have no solution? If so, give an example. If not, explain why not.

3. Is (3, 4) a solution of $-2x + 5y > 11$?

4. Is (−2, 3) a solution of $6x + 4y \le 0$?

5. Is (4, 5) a solution of the following system of inequalities?

$$\begin{cases} 3x - 2y \le 6 \\ x + 3y < 14 \\ x \ge 2, y > 3 \end{cases}$$

6. Is (2, 1) a solution of the following system of inequalities?

$$\left\{\begin{array}{l} 2x + 7y \ge 3 \\ -x + 6y \le 14 \\ x \ge 0, y \ge 2 \end{array}\right\}$$

In Exercises 7 to 28, sketch the graph of each inequality.

7. $y \le -2$

8. $x + y > -2$

9. $y \ge 2x + 3$

10. $y < -2x + 1$

11. $2x - 3y < 6$

12. $3x + 4y \le 4$

13. $4x + 3y \le 12$

14. $5x - 2y < 8$

15. $y < x^2$

16. $x > y^2$

17. $y \ge x^2 - 2x - 3$

18. $y < 2x^2 - x - 3$

19. $(x - 2)^2 + (y - 1)^2 < 16$

20. $(x + 2)^2 + (y - 3)^2 > 25$

21. $\dfrac{(x-3)^2}{9} - \dfrac{(y+1)^2}{16} > 1$

22. $\dfrac{(x+1)^2}{25} - \dfrac{(y-3)^2}{16} \le 1$

23. $4x^2 + 9y^2 - 8x + 18y \ge 23$

24. $25x^2 - 16y^2 - 100x - 64y < 64$

25. $y \ge |2x - 4|$

26. $y < |x|$

27. $y < 2^{x-1}$

28. $y > \log_3 x$

In Exercises 29 to 50, sketch the graph of the solution set of each system of inequalities.

29. $\begin{cases} 1 \le x < 3 \\ -2 < y \le 4 \end{cases}$

30. $\begin{cases} -2 < x < 4 \\ y \ge -1 \end{cases}$

31. $\begin{cases} x + y \le 2 \\ x - y < 2 \end{cases}$

32. $\begin{cases} 2x - 5y < -6 \\ 3x + y < 8 \end{cases}$

33. $\begin{cases} 2x - y \ge -4 \\ 4x - 2y \le -17 \end{cases}$

34. $\begin{cases} 4x + 2y > 5 \\ 6x + 3y > 10 \end{cases}$

35. $\begin{cases} 2x + 3y < 6 \\ 3x - 2y \ge -6 \end{cases}$

36. $\begin{cases} 3x + 5y \ge -8 \\ 2x - 3y \ge 1 \end{cases}$

37. $\begin{cases} y < 2x + 3 \\ y > 2x - 2 \end{cases}$

38. $\begin{cases} y > 3x + 1 \\ y < 3x - 2 \end{cases}$

39. $\begin{cases} y > x - 1 \\ y \le -x^2 + 4 \end{cases}$

40. $\begin{cases} y \le 2x + 7 \\ y > x^2 + 3x + 1 \end{cases}$

41. $\begin{cases} x^2 + y^2 \le 49 \\ 9x^2 + 4y^2 \ge 36 \end{cases}$

42. $\begin{cases} y < 2x - 1 \\ y > x^2 - 2x + 2 \end{cases}$

43. $\begin{cases} (x-1)^2 + (y+1)^2 \le 16 \\ (x-1)^2 + (y+1)^2 \ge 4 \end{cases}$

44. $\begin{cases} (x+2)^2 + (y-3)^2 > 25 \\ (x+2)^2 + (y-3)^2 < 16 \end{cases}$

45. $\begin{cases} \dfrac{x^2}{4} - \dfrac{y^2}{16} > 1 \\ \dfrac{x^2}{16} + \dfrac{y^2}{4} < 1 \end{cases}$

46. $\begin{cases} \dfrac{(x+1)^2}{36} + \dfrac{(y-2)^2}{25} < 1 \\ \dfrac{(x+1)^2}{25} + \dfrac{(y-2)^2}{36} < 1 \end{cases}$

47. $\begin{cases} 6x + y \ge 30 \\ x + 4y \ge 40 \\ 2x + 3y \ge 60 \\ x \ge 0, y \ge 0 \end{cases}$

48. $\begin{cases} 5x + y \le 9 \\ 2x + 3y \le 14 \\ x \ge -2, y \ge 2 \end{cases}$

49. $\begin{cases} x + 4y \le 80 \\ x + y \le 35 \\ 2x + y \le 60 \\ x \ge 0, y \ge 0 \end{cases}$

50. $\begin{cases} 4x + y \ge 13 \\ 3x + 2y \ge 16 \\ x \le 15, y \le 12 \end{cases}$

51. **Physical Fitness** The instructor of an aerobics exercise class for beginners uses the following system of inequalities to find the targeted exercise heart rate ranges for the members of the class.

$$\begin{cases} y \ge 0.55(208 - 0.7x) \\ y \le 0.75(208 - 0.7x) \\ 20 \le x \le 50 \end{cases}$$

In this system, y is the person's exercise heart rate in beats per minute and x is the person's age in years. Use the system of inequalities to determine the targeted exercise heart rate range for Ashley, who is 35. Round the minimum and maximum targeted heart rates to the nearest beat per minute.

52. **Physical Fitness** The sprinters on a track team use the following system of inequalities to determine their targeted exercise heart rate ranges for their workouts.

$$\begin{cases} y \ge 0.80(208 - 0.7x) \\ y \le 0.85(208 - 0.7x) \\ 20 \le x \le 28 \end{cases}$$

Indicates Try It Exercises

In this system, y is the person's exercise heart rate in beats per minute and x is the person's age in years. Use the system of inequalities to determine the targeted exercise heart rate range for a sprinter who is 26 years old. Round the minimum and maximum targeted heart rates to the nearest beat per minute.

In Exercises 53 to 58, sketch the graph of the inequality.

53. $|y| \geq |x|$

54. $|y| \leq |x - 1|$

55. $|x + y| \leq 1$

56. $|x - y| > 1$

57. $|x| + |y| \leq 1$

58. $|x| - |y| > 1$

Enrichment Exercises

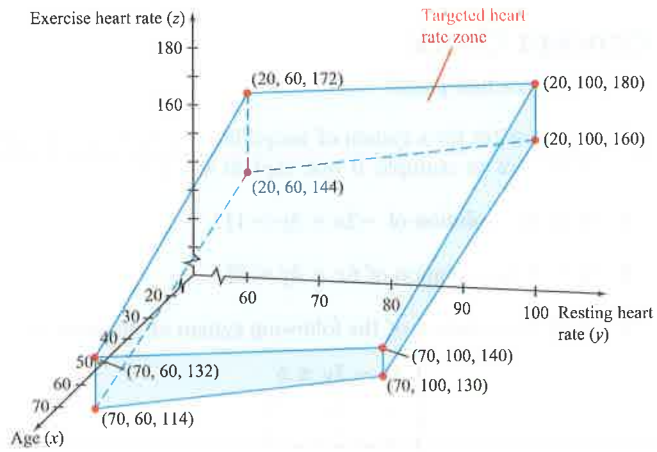
59. Sketch the graphs of $xy > 1$ and $y > \frac{1}{x}$. Note that the two graphs are not the same, yet the second inequality can be derived from the first by dividing each side by x. Explain.

60. Sketch the graph of $\frac{x}{y} < 1$ and the graph of $x < y$. Note that the two graphs are not the same, yet the second inequality can be derived from the first by multiplying each side by y. Explain.

61. Physical Fitness The *Karvonen method* is often used to find a person's targeted exercise heart rate range. This method is generally considered more reliable than the method used in Example 5 because it uses both a person's age x, in years, and the person's resting heart rate y, in beats per minute, to establish a targeted exercise heart rate range, which is displayed on a vertical z-axis. The Karvonen method is defined by the following system of inequalities.

$$\begin{cases} z \geq 0.60(220 - x) + 0.40y & (1) \\ z \leq 0.80(220 - x) + 0.20y & (2) \\ 20 \leq x \leq 70 & (3) \\ 60 \leq y \leq 100 & (4) \end{cases}$$

The graph of the Karvonen system of inequalities is the three-dimensional solid shown below.



The coordinates of a point on the solid are given in the order (x, y, z). For instance, the two points (20, 60, 144) and (20, 60, 172) indicate that a 20-year-old person with a resting heart rate of 60 beats per minute has a targeted exercise heart rate range from 144 to 172 beats per minute. For any given age x, $20 \leq x \leq 70$, the graph of the Karvonen system of inequalities is a trapezoidal cross section of the above solid.

a. Use the Karvonen inequalities to graph the targeted exercise heart rate zone for people who are 25 years of age. (*Hint:* Use a sheet of graph paper. Substitute 25 for x to produce a system of inequalities that involves only two variables. Label the horizontal axis as the y-axis and the vertical axis as the z-axis. The domain will be $\{y \mid 60 \leq y \leq 100\}$.)

b. Tyler is 25 years old and has a resting heart rate of 80 beats per minute. Use your graph from **a** to estimate Tyler's targeted heart rate range.

c. Use the Karvonen inequalities to determine your targeted exercise heart rate range. (*Note:* Your resting heart rate is your heart rate after you have rested for 7 to 8 hours.)

SECTION 6.6 Linear Programming

Introduction to Linear Programming
Solving Optimization Problems

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A41.

PS1. Graph: $2x + 3y \leq 12$ [6.5]

PS2. Evaluate $C = 3x + 4y$ at (0, 5), (2, 3), (6, 1), and (9, 0). [P.3]

PS3. Evaluate $C = 6x + 4y + 15$ at (0, 20), (4, 18), (10, 10), and (15, 0). [P.3]