6.4 Transformations of Exponential and Logarithmic Functions

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Learning Target Describe and graph transformations of exponential and logarithmic functions.

Success Criteria
- I can describe transformations of exponential and logarithmic functions.
- I can graph transformations of exponential and logarithmic functions.
- I can write functions that represent transformations of exponential and logarithmic functions.

EXPLORE IT! Identifying Transformations

Work with a partner. You can transform graphs of exponential and logarithmic functions in the same way you transformed graphs of functions previously. Each graph shown is a transformation of the parent function

$$f(x) = e^x \qquad \text{or} \qquad f(x) = \ln x.$$

In parts (a)–(f), match each graph with one of the functions in the list at the left. Explain your reasoning. Then describe the transformation of f represented by g.

A. $g(x) = e^{x+2} - 3$

B. $g(x) = -e^{x+2} + 1$

C. $g(x) = -e^{x-2} + 3$

D. $g(x) = e^{x-2} - 1$

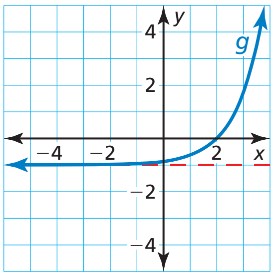
E. $g(x) = \ln(x + 2)$

F. $g(x) = \ln(x - 2)$

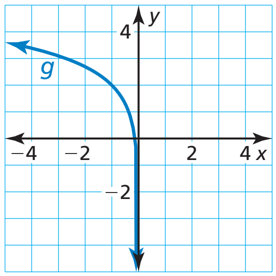
G. $g(x) = 2 + \ln x$

H. $g(x) = 2 + \ln(-x)$

a.



b.



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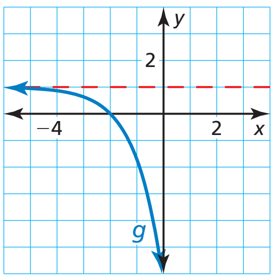

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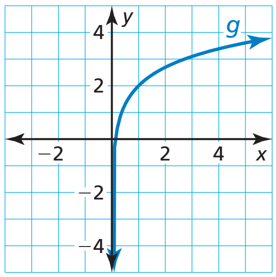
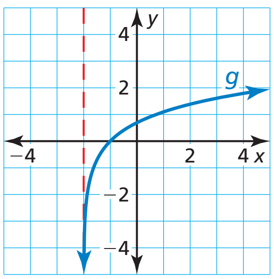
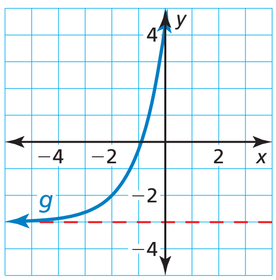


e.



f.



g. The graph of h is a translation 4 units right and 1 unit up of the graph of $f(x) = e^x$. Write a rule for h. Then graph each function.

Math Practice

Look for Structure

How can you use the asymptotes to match the functions and graphs?

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Transforming Graphs of Exponential Functions

You can transform graphs of exponential and logarithmic functions in the same way you transformed graphs of functions in previous chapters. Examples of transformations of the graph of $f(x) = 4^x$ are shown below.

KEY IDEAS

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = 4^{x-3}$ $g(x) = 4^{x+2}$	3 units right 2 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = 4^x + 5$ $g(x) = 4^x - 1$	5 units up 1 unit down
Reflection Graph flips over a line.	$f(-x)$ $-f(x)$	$g(x) = 4^{-x}$ $g(x) = -4^x$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis by a factor of $\frac{1}{a}$.	$f(ax)$	$g(x) = 4^{2x}$ $g(x) = 4^{x/2}$	shrink by a factor of $\frac{1}{2}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis by a factor of a.	$a \cdot f(x)$	$g(x) = 3(4^x)$ $g(x) = \frac{1}{4}(4^x)$	stretch by a factor of 3 shrink by a factor of $\frac{1}{4}$

EXAMPLE 1 Translating an Exponential Function

WATCH

Describe the transformation of $f(x) = \left(\frac{1}{2}\right)^x$ represented by $g(x) = \left(\frac{1}{2}\right)^x - 4$. Then graph each function.

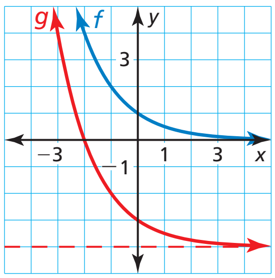
SOLUTION

Notice that the function is of the form $g(x) = \left(\frac{1}{2}\right)^x + k$.

Rewrite the function to identify k.

$$g(x) = \left(\frac{1}{2}\right)^x + \underset{\substack{\uparrow \\ k}}{(-4)}$$

▶ Because $k = -4$, the graph of g is a translation 4 units down of the graph of f.



STUDY TIP

Notice in the graph that the vertical translation also shifts the asymptote 4 units down, so the range of g is $y > -4$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Describe the transformation of *f* represented by *g*. Then graph each function.

1. $f(x) = 3^x$, $g(x) = 3^x + 2$

2. $f(x) = \left(\frac{1}{4}\right)^x$, $g(x) = \left(\frac{1}{4}\right)^{x-2}$

3. $f(x) = 0.5^x$, $g(x) = 0.5^{x+1} - 6$

4. $f(x) = 2^x$, $g(x) = 2^{x-3} + 1$

EXAMPLE 2 Translating a Natural Base Exponential Function

WATCH

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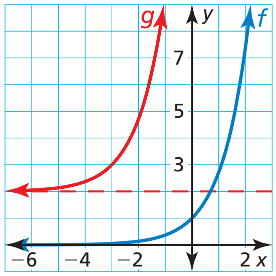
Describe the transformation of $f(x) = e^x$ represented by $g(x) = e^{x+3} + 2$. Then graph each function.

SOLUTION

Notice that the function is of the form $g(x) = e^{x-h} + k$. Rewrite the function to identify h and k.

$$g(x) = e^{x - \underset{h}{(-3)}} + \underset{k}{2}$$



▶ Because $h = -3$ and $k = 2$, the graph of g is a translation 3 units left and 2 units up of the graph of f.

STUDY TIP

Notice in the graph that the vertical translation also shifts the asymptote 2 units up, so the range of g is $y > 2$.

EXAMPLE 3 Transforming Exponential Functions

WATCH

Describe the transformation of f represented by g. Then graph each function.

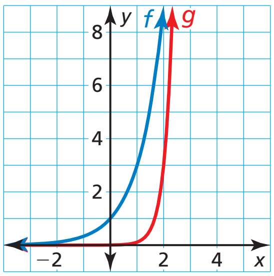
a. $f(x) = 3^x$, $g(x) = 3^{3x-5}$

b. $f(x) = e^{-x}$, $g(x) = -\frac{1}{8}e^{-x}$

SOLUTION

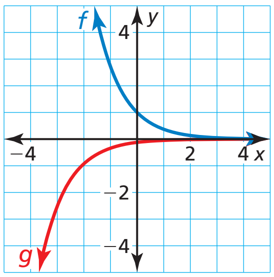
a. Notice that the function is of the form $g(x) = 3^{ax-h}$, where $a = 3$ and $h = 5$.

▶ So, the graph of g is a translation 5 units right, followed by a horizontal shrink by a factor of $\frac{1}{3}$ of the graph of f.



b. Notice that the function is of the form $g(x) = ae^{-x}$, where $a = -\frac{1}{8}$.

▶ So, the graph of g is a reflection in the x-axis and a vertical shrink by a factor of $\frac{1}{8}$ of the graph of f.



Math Practice

Look for Structure

Compare the transformation in Example 3(a) with the transformation of f represented by $h(x) = 3^{3(x-5)}$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Describe the transformation of f represented by g. Then graph each function.

5. $f(x) = e^x$, $g(x) = e^{x-3}$

6. $f(x) = e^{-x}$, $g(x) = e^{-x} - 5$

7. $f(x) = 0.4^x$, $g(x) = 0.4^{-2x}$

8. $f(x) = e^x$, $g(x) = -e^{x+6}$

9. **WRITING** Given the function $f(x) = ab^{x-h} + k$, describe the effects of a, h, and k on the graph of the function.

Transforming Graphs of Logarithmic Functions

Examples of transformations of the graph of $f(x) = \log x$ are shown below.

KEY IDEAS

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = \log(x - 4)$ $g(x) = \log(x + 7)$	4 units right 7 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = \log x + 3$ $g(x) = \log x - 1$	3 units up 1 unit down
Reflection Graph flips over a line.	$f(-x)$ $-f(x)$	$g(x) = \log(-x)$ $g(x) = -\log x$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis by a factor of $\frac{1}{a}$.	$f(ax)$	$g(x) = \log(4x)$ $g(x) = \log\left(\frac{1}{3}x\right)$	shrink by a factor of $\frac{1}{4}$ stretch by a factor of 3
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis by a factor of a.	$a \bullet f(x)$	$g(x) = 5 \log x$ $g(x) = \frac{2}{3} \log x$	stretch by a factor of 5 shrink by a factor of $\frac{2}{3}$

EXAMPLE 4 Transforming Logarithmic Functions

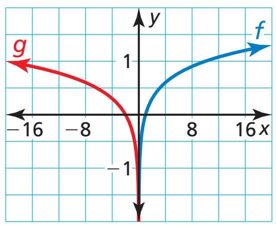
Describe the transformation of f represented by g. Then graph each function.

a. $f(x) = \log x$, $g(x) = \log\left(-\frac{1}{2}x\right)$ **b.** $f(x) = \log_{1/2} x$, $g(x) = 2 \log_{1/2}(x + 4)$

SOLUTION

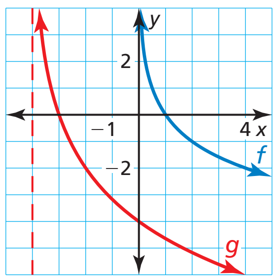
a. Notice that the function is of the form $g(x) = \log(ax)$, where $a = -\frac{1}{2}$.

▶ So, the graph of g is a reflection in the y-axis and a horizontal stretch by a factor of 2 of the graph of f.



b. Notice that the function is of the form $g(x) = a \log_{1/2}(x - h)$, where $a = 2$ and $h = -4$.

▶ So, the graph of g is a horizontal translation 4 units left and a vertical stretch by a factor of 2 of the graph of f.



STUDY TIP

In Example 4(b), notice in the graph that the horizontal translation also shifts the asymptote 4 units left, so the domain of g is $x > -4$.

Writing Transformations of Graphs of Functions

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EXAMPLE 5 Writing a Transformed Exponential Function

WATCH

Let the graph of g be a reflection in the x-axis, followed by a translation 4 units right of the graph of $f(x) = 2^x$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the reflection of f.

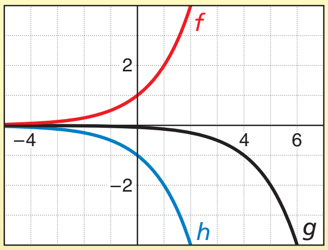
$$h(x) = -f(x) \qquad \text{Multiply the output by } -1.$$
$$= -2^x \qquad \text{Substitute } 2^x \text{ for } f(x).$$

Step 2 Then write a function g that represents the translation of h.

$$g(x) = h(x - 4) \qquad \text{Subtract 4 from the input.}$$
$$= -2^{x-4} \qquad \text{Replace } x \text{ with } x - 4 \text{ in } h(x).$$

▶ The transformed function is $g(x) = -2^{x-4}$.

Check



EXAMPLE 6 Writing a Transformed Logarithmic Function

WATCH

Let the graph of g be a translation 2 units up, followed by a vertical stretch by a factor of 2 of the graph of $f(x) = \log_{1/3} x$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the translation of f.

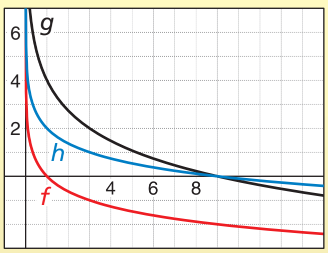
$$h(x) = f(x) + 2 \qquad \text{Add 2 to the output.}$$
$$= \log_{1/3} x + 2 \qquad \text{Substitute } \log_{1/3} x \text{ for } f(x).$$

Step 2 Then write a function g that represents the vertical stretch of h.

$$g(x) = 2 \cdot h(x) \qquad \text{Multiply the output by 2.}$$
$$= 2 \cdot (\log_{1/3} x + 2) \qquad \text{Substitute } \log_{1/3} x + 2 \text{ for } h(x).$$
$$= 2 \log_{1/3} x + 4 \qquad \text{Distributive Property}$$

▶ The transformed function is $g(x) = 2 \log_{1/3} x + 4$.

Check



SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Describe the transformation of f represented by g. Then graph each function.

10. $f(x) = \log_2 x$, $g(x) = -3 \log_2 x$

11. $f(x) = \log_{1/4} x$, $g(x) = \log_{1/4}(4x) - 5$

12. Let the graph of g be a horizontal stretch by a factor of 3, followed by a translation 2 units up of the graph of $f(x) = e^{-x}$. Write a rule for g.

13. Let the graph of g be a reflection in the y-axis, followed by a translation 4 units left of the graph of $f(x) = \log x$. Write a rule for g.

6.4 Practice WITH CalcChat® AND CalcView®

In Exercises 1–4, match the function with its graph. Explain your reasoning.

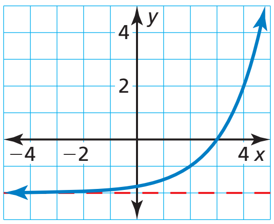
1. $y = 2^{x+2} - 2$

2. $y = 2^{x+2} + 2$

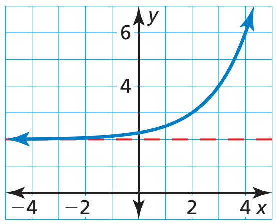
3. $y = 2^{x-2} - 2$

4. $y = 2^{x-2} + 2$

A.



B.



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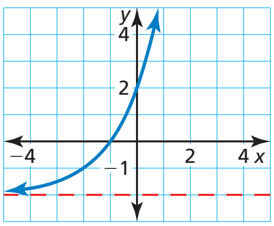

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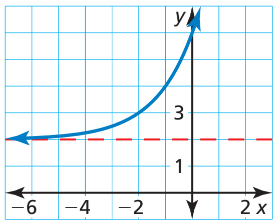


In Exercises 5–14, describe the transformation of *f* represented by *g*. Then graph each function.
Examples 1 and 2

5. $f(x) = 3^x,\ g(x) = 3^x + 5$

6. $f(x) = 4^x,\ g(x) = 4^x - 8$

7. $f(x) = e^x,\ g(x) = e^x - 1$

8. $f(x) = e^x,\ g(x) = e^x + 4$

9. $f(x) = 2^x,\ g(x) = 2^{x-7}$

10. $f(x) = 5^x,\ g(x) = 5^{x+1}$

11. $f(x) = e^{-x},\ g(x) = e^{-x} + 6$

12. $f(x) = e^{-x},\ g(x) = e^{-x} - 9$

13. $f(x) = 0.25^x,\ g(x) = 0.25^{x-3} + 12$

14. $f(x)= \left(\frac{1}{3}\right)^x,\ g(x) = \left(\frac{1}{3}\right)^{x+2} - \frac{2}{3}$

In Exercises 15–22, describe the transformation of *f* represented by *g*. Then graph each function.
Example 3

15. $f(x) = e^x,\ g(x) = e^{2x}$

16. $f(x) = e^x,\ g(x) = \frac{4}{3}e^x$

17. $f(x) = 2^x,\ g(x) = -2^{x-3}$

18. $f(x) = 4^x,\ g(x) = 4^{0.5x-5}$

19. $f(x) = e^{-x},\ g(x) = 3e^{-6x}$

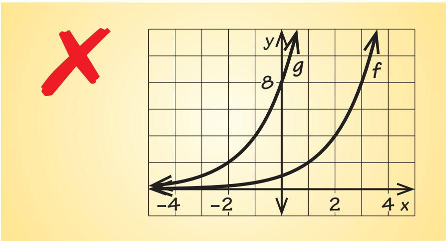
20. $f(x) = e^{-x},\ g(x) = e^{-5x} + 2$

21. $f(x) = 0.5^x,\ g(x) = 6(0.5)^{x+5} - 2$

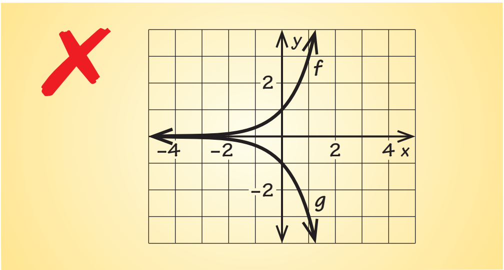
22. $f(x) = \left(\frac{3}{4}\right)^x,\ g(x) = -\left(\frac{3}{4}\right)^{x-7} + 1$

ERROR ANALYSIS **In Exercises 23 and 24, describe and correct the error in graphing the transformation of *f* represented by *g*.**

23. $g(x) = 2^x + 3$



24. $g(x) = 3^{-x}$



In Exercises 25–28, describe the transformation of *f* represented by *g*. Then graph each function.
Example 4

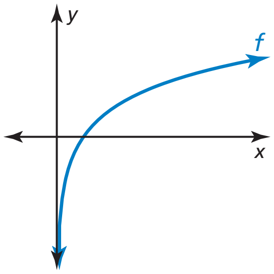
25. $f(x) = \log_4 x,\ g(x) = 3 \log_4 x - 5$

26. $f(x) = \log_{1/3} x,\ g(x) = \log_{1/3}(-x) + 6$

27. $f(x) = \log_{1/5} x,\ g(x) = -\log_{1/5}(x - 7)$

28. $f(x) = \log_2 x,\ g(x) = \log_2(x + 2) - 3$

ANALYZING RELATIONSHIPS **In Exercises 29–32, use the graph of *f* to match the transformation of *f* with its graph.**



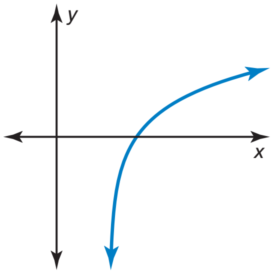
29. $y = f(x - 2)$

30. $y = f(x + 2)$

31. $y = 2f(x)$

32. $y = f(2x)$

A.



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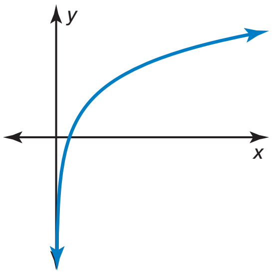

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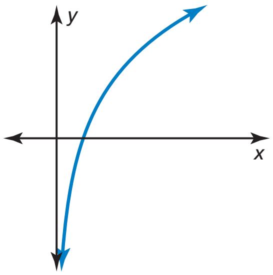

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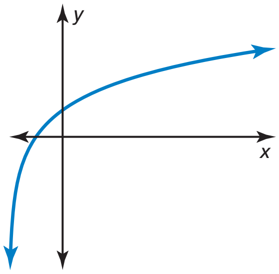


JUSTIFYING STEPS **In Exercises 33 and 34, justify each step in writing a rule for *g* that represents the indicated transformations of the graph of *f*.**

33. $f(x) = \log_7 x$; reflection in the x-axis, followed by a translation 6 units down

$h(x) = -f(x)$

$\quad = -\log_7 x$

$g(x) = h(x) - 6$

$\quad = -\log_7 x - 6$

34. $f(x) = 8^x$; vertical stretch by a factor of 4, followed by a translation 1 unit up and 3 units left

$h(x) = 4 \cdot f(x)$

$\quad = 4 \cdot 8^x$

$g(x) = h(x + 3) + 1$

$\quad = 4 \cdot 8^{x+3} + 1$

In Exercises 35–38, write a rule for *g* that represents the indicated transformations of the graph of *f*. *Example 5*

35. $f(x) = 5^x$; translation 2 units down, followed by a reflection in the y-axis

36. $f(x) = \left(\frac{2}{3}\right)^x$; reflection in the x-axis, followed by a vertical stretch by a factor of 6 and a translation 4 units left

37. $f(x) = e^x$; horizontal shrink by a factor of $\frac{1}{2}$, followed by a translation 5 units up

38. $f(x) = e^{-x}$; translation 4 units right and 1 unit down, followed by a vertical shrink by a factor of $\frac{1}{3}$

In Exercises 39–42, write a rule for *g* that represents the indicated transformations of the graph of *f*. *Example 6*

39. $f(x) = \log_6 x$; vertical stretch by a factor of 6, followed by a translation 5 units down

40. $f(x) = \log_5 x$; reflection in the x-axis, followed by a translation 9 units left

41. $f(x) = \log_{1/2} x$; translation 3 units left and 2 units up, followed by a reflection in the y-axis

42. $f(x) = \ln x$; translation 3 units right and 1 unit up, followed by a horizontal stretch by a factor of 8

MP **STRUCTURE** **In Exercises 43 and 44, describe the transformation of the graph of *f* represented by the graph of *g*. Then give an equation of the asymptote.**

43. $f(x) = 3^x$, $g(x) = 3^{x-9}$

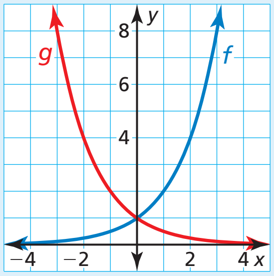
44. $f(x) = \log_{1/5} x$, $g(x) = \log_{1/5} x + 13$

45. **MODELING REAL LIFE** The speed (in miles per hour) of a hoverboard can be modeled by the function $g(t) = 9 - 9e^{-at}$, where t is the number of seconds since activation and $0 < a < 1$. Describe how changing the value of a affects the graph of g. What does this mean in terms of the speed of the hoverboard?

46. **MODELING REAL LIFE** Explain why the advertisement below is misleading for the hoverboard modeled in Exercise 45.

47. MAKING AN ARGUMENT Does a horizontal stretch by a factor of 4 have the same result as a vertical shrink by a factor of $\frac{1}{4}$ for $f(x) = \log_4 x$? Explain.

48. HOW DO YOU SEE IT? The graphs of $f(x) = b^x$ and $g(x) = \left(\frac{1}{b}\right)^x$ are shown for $b = 2$. Use the graphs to describe a transformation of the graph of f that results in the graph of g.



49. CRITICAL THINKING Consider the graph of the function $h(x) = e^{-x-2}$. Describe the transformation of the graph of $f(x) = e^{-x}$ represented by the graph of h. Then describe the transformation of the graph of $g(x) = e^x$ represented by the graph of h.

50. THOUGHT PROVOKING Consider the functions $f(x) = \log_a x$ and $g(x) = \dfrac{1}{\log a} \log x + k$. How are the graphs of f and g related?

GO DIGITAL

51. MP PROBLEM SOLVING The amount P (in grams) of 100 grams of plutonium-239 that remains after t years can be modeled by $P = 100(0.99997)^t$.

a. How much plutonium-239 remains after 12,000 years?

b. Describe the transformation of the function when the initial amount of plutonium-239 is 550 grams.

c. Does the transformation in part (b) affect the domain and range of the function? Explain.

52. OPEN-ENDED Write a function whose graph has a y-intercept of 5 and an asymptote of $y = 2$.

REVIEW & REFRESH

WATCH

In Exercises 53–56, simplify the expression.

53. $9^{\log_9 x}$

54. $e^{\ln 9}$

55. $\log_2 2^{2x}$

56. $10^{\log (x-3)}$

In Exercises 57 and 58, determine whether the data show a linear relationship. If so, write an equation of a line of fit. Then estimate y when $x = 20$ and explain its meaning in the context of the situation.

57.

Minutes gliding, x	1	4	8	12
Calories burned, y	8	32	64	96

58.

Months, x	3	6	9	12
Attendance (thousands), y	2	2.1	2.2	2.3

In Exercises 59 and 60, write a rule for g that represents the indicated transformations of the graph of f.

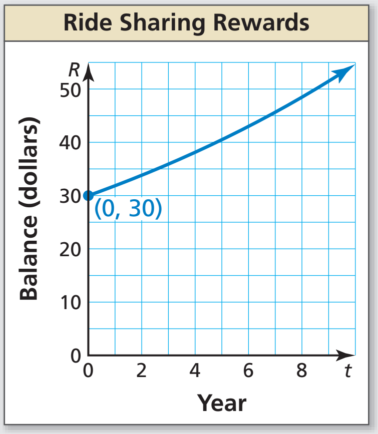
59. $f(x) = e^x$; horizontal shrink by a factor of $\frac{1}{3}$, followed by a translation 1 unit down

60. $f(x) = \log_2 x$; reflection in the y-axis, followed by a translation 3 units right

61. Find the zeros of $h(x) = x^4 + x^3 - 12x^2$. Then sketch the graph of h.

62. Solve $\sqrt{2x + 11} = 5$. Check your solution.

63. MODELING REAL LIFE Social media advertisement revenue and ride sharing rewards earn annual interest compounded continuously. The balance S (in dollars) of the social media advertisement revenue after t years is modeled by $S = 25e^{0.1t}$. The graph shows the balance R of the ride sharing rewards. Which account has a greater principal? Which account has a greater balance after 6 years?



In Exercises 64 and 65, rewrite the function in the form $y = a(1 + r)^t$ or $y = a(1 - r)^t$. State the growth or decay rate, and describe the end behavior of the function.

64. $y = a(2)^{t/4}$

65. $y = a\left(\frac{1}{3}\right)^{2t}$