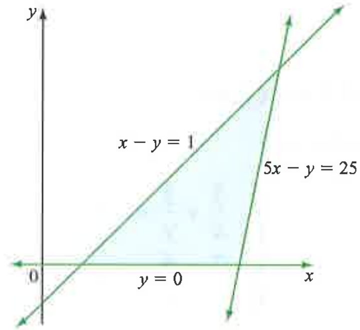
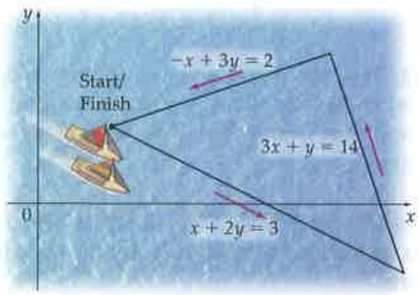
In Exercises 43 to 64, solve by using a system of equations.

43. Area of a Triangle Determine the area of the triangle shown in blue. Assume that x and y are measured in miles.



44. A Sailboat Racecourse Determine the distance around the racecourse shown by the triangle in the following diagram. Assume that x and y are measured in kilometers. Round your answer to the nearest tenth of a kilometer.



45. Supply–Demand The number x of smartphones a manufacturer is willing to sell is given by $x = 25p - 3400$, where p is the price, in dollars, per smartphone. The number x of smartphones a wholesale distributor is willing to purchase is given by $x = -5p + 2000$, where p is the price per smartphone. Find the equilibrium price.

46. Supply–Demand The number x of digital cameras a manufacturer is willing to sell is given by $x = 25p - 500$, where p is the price, in dollars, per digital camera. The number x of digital cameras a store is willing to purchase is given by $x = -7p + 1100$, where p is the price per digital camera. Find the equilibrium price.

47. Rate of Wind Flying with the wind, a plane traveled 450 miles in 3 hours. Flying against the wind, the plane traveled the same distance in 5 hours. Find the rate of the plane in calm air and the rate of the wind.

48. Rate of Wind A plane flew 800 miles in 4 hours while flying with the wind. Against the wind, it took the plane 5 hours to travel 800 miles. Find the rate of the plane in calm air and the rate of the wind.

49. Rate of Current A motorboat traveled a distance of 120 miles in 4 hours while traveling with the current. Against the current, the same trip took 6 hours. Find the rate of the boat in calm water and the rate of the current.

50. Rate of Current A canoeist can row 12 miles with the current in 2 hours. Rowing against the current, it takes the canoeist 4 hours to travel the same distance. Find the rate of the canoeist in calm water and the rate of the current.

51. Metallurgy A metallurgist made two purchases. The first purchase, which cost \$1080, included 30 kilograms of an iron alloy and 45 kilograms of a lead alloy. The second purchase, at the same prices, cost \$372 and included 15 kilograms of the iron alloy and 12 kilograms of the lead alloy. Find the cost per kilogram of the iron and lead alloys.

52. Chemistry For \$14.10, a chemist purchased 10 liters of hydrochloric acid and 15 liters of silver nitrate. A second purchase, at the same prices, cost \$18.16 and included 12 liters of hydrochloric acid and 20 liters of silver nitrate. Find the cost per liter of each of the two chemicals.

53. Chemistry A goldsmith has two gold alloys. The first alloy is 40% gold; the second alloy is 60% gold. How many grams of each should be mixed to produce 20 grams of an alloy that is 52% gold?

54. Chemistry One acetic acid solution is 70% water, and another is 30% water. How many liters of each solution should be mixed to produce 20 liters of a solution that is 40% water?

55. Geometry A right triangle in the first quadrant is bounded by the lines $y = 0$, $y = \frac{1}{2}x$, and $y = -2x + 6$. Find its area.

56. Geometry The lines whose equations are $2x + 3y = 1$, $3x - 4y = 10$, and $4x + ky = 5$ all intersect at the same point. What is the value of k?

57. Number Theory Adding a three-digit number $5Z7$ to 256 gives $XY3$. If $XY3$ is divisible by 3, then what is the largest possible value of Z?

58. Number Theory Find the value of k if
$2x + 5 = 6x + k = 4x - 7$.

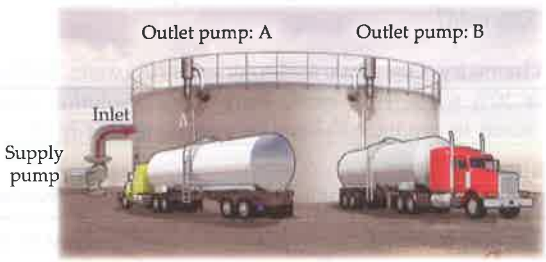
59. Number Theory A Pythagorean triple is three positive integers—a, b, and c—for which $a^2 + b^2 = c^2$. Given $a = 42$, find all the values of b and c such that a, b, and c form a Pythagorean triple. *Suggestion:* If $a = 42$, then $1764 + b^2 = c^2$ or $1764 = c^2 - b^2 = (c - b)(c + b)$. Because the product $(c - b)(c + b) = 1764$, $c - b$ and $c + b$ must be factors of 1764. For instance, one possibility is $2 = c - b$ and $882 = c + b$. Solving this system of equations yields one set of Pythagorean triples. Now repeat the process for other possible factors of 1764. Remember that answers must be positive integers.

60. Number Theory Given $a = 30$, find all the values of b and c such that a, b, and c form a Pythagorean triple. (See the preceding exercise.)

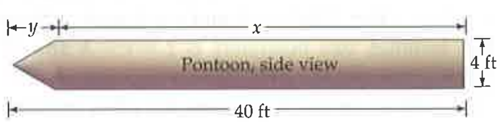
61. Marketing A marketing company asked 100 people whether they liked a new skin cream and lip balm. The company found that 80% of the people who liked the new skin cream also liked the new lip balm and that 50% of the people who did not like the new skin cream liked the new lip balm. If 77 people liked the lip balm, how many people liked the skin cream?

62. Fire Science An analysis of 200 scores on a firefighter qualifying exam found that 75% of those who passed the basic fire science exam also passed the exam on containing chemical fires. Of those who did not pass the basic fire science exam, 25% passed the exam on containing chemical fires. If 120 people passed the exam on containing chemical fires, how many people passed the basic fire science exam?

63. Inlet and Outlet Pump Rates A fuel storage tank has one supply pump and two identical outlet pumps. With one outlet pump running, the supply pump can increase the fuel level in the storage tank by 8750 gallons in 30 minutes. With both outlet pumps running, the supply pump can increase the fuel level in the storage tank by 11,250 gallons in 45 minutes. Find the pumping rate, in gallons per hour, for each of the pumps.



64. Dimensions of a Pontoon The pontoons on a boat are cylinders with conical tips. The length of a pontoon is 40 feet, and its diameter is 4 feet. The volume of each pontoon is 477.5 cubic feet.



a. Write a system of equations that describes the relationships between x and y. See the accompanying figure.

b. Find x and y. Round to the nearest tenth of a foot.

Enrichment Exercises

The system of equations

$$\begin{cases} \dfrac{3}{x} + \dfrac{2}{y} = 1 \\ -\dfrac{7}{x} + \dfrac{6}{y} = -1 \end{cases}$$

is not a linear system of equations. However, the system can be written in the form of a linear system involving the variables u and v, by using the substitutions $u = 1/x$ and $v = 1/y$ to produce

$$\begin{cases} 3u + 2v = 1 \\ -7u + 6v = -1 \end{cases}$$

Solving this system yields $u = 1/4$ and $v = 1/8$, which allows us to determine that $x = 4$ and $y = 8$, since x and u are reciprocals, and y and v are reciprocals. Thus (4, 8) is the solution of the original system.

In Exercises 65 and 66, use the substitutions $u = 1/x$ and $v = 1/y$, as explained above, to solve the given systems of equations.

65. $\begin{cases} \dfrac{2}{x} - \dfrac{1}{y} = \dfrac{2}{3} \\ \dfrac{5}{x} + \dfrac{3}{y} = -\dfrac{1}{6} \end{cases}$

66. $\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} = 1 \\ \dfrac{2}{x} - \dfrac{1}{y} = 0 \end{cases}$

67. Use a graphing utility to graph each of the equations given in Exercise 65.

a. Is the point (6, −3) an intersection point of the graphs?

b. The point (0, 0) appears to be an intersection point for the graphs. Explain how you know that the graphs do not intersect at (0, 0).

SECTION 6.2

Systems of Linear Equations in Three Variables
Triangular Form
Nonsquare Systems of Equations
Homogeneous Systems of Equations
Applications of Systems of Equations

Systems of Linear Equations in Three Variables

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A38.

PS1. Solve $2x - 5y = 15$ for y. [1.1]

PS2. If $x = 2c + 1$, $y = -c + 3$, and $z = 2x + 5y - 4$, write z in terms of c. [P.1]

PS3. Solve: $\begin{cases} 5x - 2y = 10 \\ 2y = 8 \end{cases}$ [6.1]

PS4. Solve: $\begin{cases} 3x - y = 11 \\ 2x + 3y = -11 \end{cases}$ [6.1]

PS5. Solve: $\begin{cases} y = 3x - 4 \\ y = 4x - 2 \end{cases}$ [6.1]

PS6. Solve: $\begin{cases} 4x + y = 9 \\ -8x - 2y = -18 \end{cases}$ [6.1]

Systems of Linear Equations in Three Variables

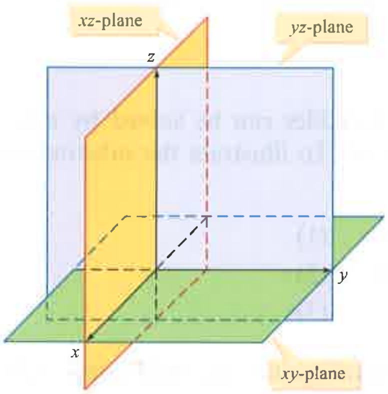


Figure 6.9

An equation of the form $Ax + By + Cz = D$, with A, B, and C not all zero, is a linear equation in three variables. A solution of an equation in three variables is an **ordered triple** (x, y, z).

The ordered triple $(2, -1, -3)$ is one of the solutions of the equation $2x - 3y + z = 4$. The ordered triple $(3, 1, 1)$ is another solution. In fact, an infinite number of ordered triples are solutions of the equation.

Graphing an equation in three variables requires a third coordinate axis perpendicular to the xy-plane. This third axis is commonly called the **z-axis.** The result is a three-dimensional coordinate system called the xyz-coordinate system (see Figure 6.9). To visualize a three-dimensional coordinate system, think of a corner of a room: the floor is the xy-plane, one wall is the yz-plane, and the other wall is the xz-plane.

Graphing an ordered triple requires three moves: the first along the x-axis, the second along the y-axis, and the third along the z-axis. Figure 6.10 is the graph of the points $(-5, -4, 3)$ and $(4, 5, -2)$.

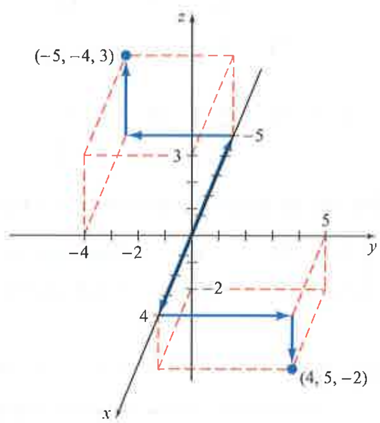


Figure 6.10

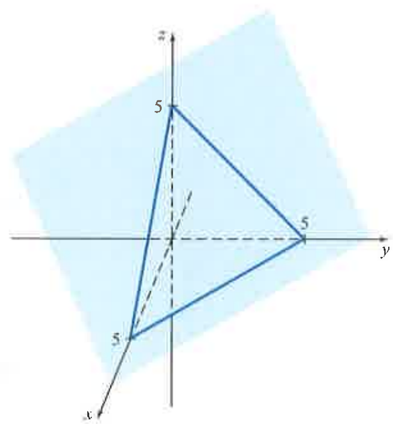


Figure 6.11

The graph of a linear equation in three variables is a plane. That is, if all the solutions of a linear equation in three variables were plotted in an xyz-coordinate system, the graph would look like a large, flat piece of paper with infinite extent. Figure 6.11 is a portion of the graph of $x + y + z = 5$.

There are different ways in which three planes can be oriented in an xyz-coordinate system. Figure 6.12 on the next page illustrates several ways.

For a system of linear equations in three variables to have a solution, the graphs of the equations must be three planes that intersect at a point, be three planes that intersect along a common line, or all be the same plane. In Figure 6.12, the graphs in **a.**, **b.**, and **c.** represent systems of equations that have a solution. The system of equations represented in Figure 6.12a is a consistent system of equations. Figure 6.12b and Figure 6.12c are graphs of dependent systems of equations. The remaining graphs are the graphs of inconsistent systems of equations.

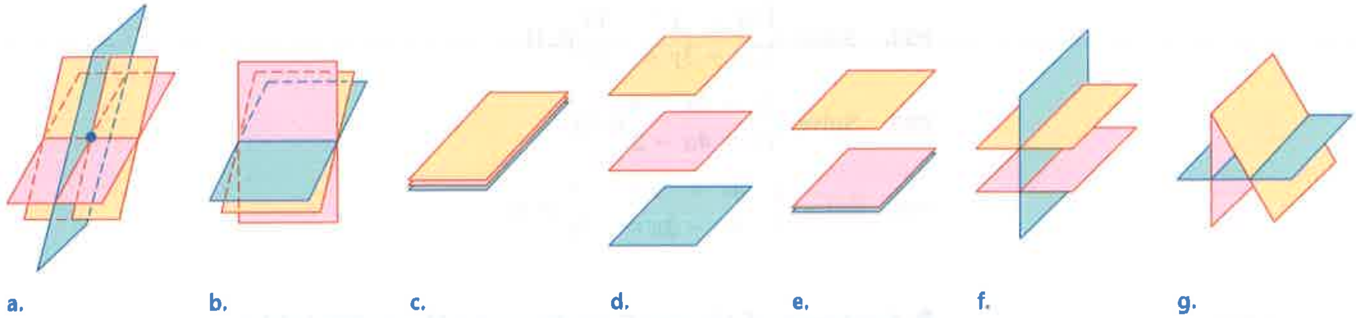


Figure 6.12

A system of equations in more than two variables can be solved by using the substitution method or the elimination method. To illustrate the substitution method, consider the system of equations

$$\begin{cases} x - 2y + z = 7 & (1) \\ 2x + y - z = 0 & (2) \\ 3x + 2y - 2z = -2 & (3) \end{cases}$$

Solve Equation (1) for x and substitute the result into Equation (2) and Equation (3).

$$x = 2y - z + 7 \quad (4)$$

$$2(2y - z + 7) + y - z = 0$$ • Substitute $2y - z + 7$ for x in Equation (2).

$$4y - 2z + 14 + y - z = 0$$ • Simplify.

$$5y - 3z = -14 \quad (5)$$

$$3(2y - z + 7) + 2y - 2z = -2$$ • Substitute $2y - z + 7$ for x in Equation (3).

$$6y - 3z + 21 + 2y - 2z = -2$$ • Simplify.

$$8y - 5z = -23 \quad (6)$$

Now solve the system of equations formed from Equations (5) and (6).

$$\begin{cases} 5y - 3z = -14 \\ 8y - 5z = -23 \end{cases}$$ Multiply by 8. → $40y - 24z = -112$; Multiply by −5. → $-40y + 25z = 115$

$$z = 3$$

Substitute 3 for z into Equation (5) and solve for y.

$$5y - 3z = -14$$ • Equation (5)

$$5y - 3(3) = -14$$

$$5y - 9 = -14$$

$$5y = -5$$

$$y = -1$$

Substitute -1 for y and 3 for z into Equation (4) and solve for x.

$$x = 2y - z + 7 = 2(-1) - (3) + 7 = 2$$

The ordered-triple solution is $(2, -1, 3)$. The graphs of the three planes intersect at a single point.

Triangular Form

There are many approaches we can take to determine the solution of a system of equations by the elimination method. For consistency, we will generally follow a plan that produces an equivalent system of equations in **triangular form.** Three examples of systems of equations in triangular form are

$$\begin{cases} 2x - 3y + z = -4 \\ 2y + 3z = 9 \\ -2z = -2 \end{cases} \quad \begin{cases} w + 3x - 2y + 3z = 0 \\ 2x - y + 4z = 8 \\ -3y - 2z = -1 \\ 3z = 9 \end{cases} \quad \begin{cases} 3x - 4y + z = 1 \\ 3y + 2z = 3 \end{cases}$$

Once a system of equations is written in triangular form, the solution can be found by *back substitution*—that is, by solving the last equation of the system and substituting *back* into the previous equation. This process is continued until the value of each variable has been found.

As an example of solving a system of equations by back substitution, consider the following system of equations in triangular form.

$$\begin{cases} 2x - 4y + z = -3 & (1) \\ 3y - 2z = 9 & (2) \\ 3z = -9 & (3) \end{cases}$$

Solve Equation (3) for z. Substitute the value of z into Equation (2) and solve for y.

$$3z = -9 \quad \text{• Equation (3)}$$
$$z = -3$$

$$3y - 2z = 9 \quad \text{• Equation (2)}$$
$$3y - 2(-3) = 9 \quad \text{• } z = -3$$
$$3y = 3$$
$$y = 1$$

Replace z with -3 and y with 1 in Equation (1) and then solve for x.

$$2x - 4y + z = -3 \quad \text{• Equation (1)}$$
$$2x - 4(1) + (-3) = -3$$
$$2x - 7 = -3$$
$$x = 2$$

The solution is the ordered triple $(2, 1, -3)$.

Question • What is the solution of the system of equations at the right? $\begin{cases} x + 2y + z = 2 \\ y - z = 3 \\ z = 2 \end{cases}$

EXAMPLE 1 Solve an Independent System of Equations

Solve: $\begin{cases} x + 2y - z = 1 & (1) \\ 2x - y + z = 6 & (2) \\ 2x - y - z = 0 & (3) \end{cases}$

Solution

Eliminate x from Equation (2) by multiplying Equation (1) by -2 and then adding it to Equation (2). Replace Equation (2) with the new equation.

$$-2x - 4y + 2z = -2 \quad \text{• } -2 \text{ times Equation (1)}$$
$$2x - y + z = 6 \quad \text{• Equation (2)}$$
$$-5y + 3z = 4 \quad \text{• Add the equations.}$$

$$\begin{cases} x + 2y - z = 1 & (1) \\ -5y + 3z = 4 & (4) \\ 2x - y - z = 0 & (3) \end{cases} \quad \text{• Replace Equation (2).}$$

(continued)

Answer • $(-10, 5, 2)$

Eliminate x from Equation (3) by multiplying Equation (1) by -2 and adding it to Equation (3). Replace Equation (3) with the new equation.

$$\begin{aligned} -2x - 4y + 2z &= -2 && \bullet\ -2 \text{ times Equation (1)} \\ 2x - y - z &= 0 && \bullet\ \text{Equation (3)} \\ \hline -5y + z &= -2 && \bullet\ \text{Add the equations.} \end{aligned}$$

$$\begin{cases} x + 2y - z = 1 & (1) \\ -5y + 3z = 4 & (4) \\ -5y + z = -2 & (5) \end{cases} \quad \bullet\ \text{Replace Equation (3).}$$

Eliminate y from Equation (5) by multiplying Equation (4) by -1 and then adding it to Equation (5). Replace Equation (5) with the new equation.

$$\begin{aligned} 5y - 3z &= -4 && \bullet\ -1 \text{ times Equation (4)} \\ -5y + z &= -2 && \bullet\ \text{Equation (5)} \\ \hline -2z &= -6 && \bullet\ \text{Add the equations.} \end{aligned}$$

$$\begin{cases} x + 2y - z = 1 & (1) \\ -5y + 3z = 4 & (4) \\ -2z = -6 & (6) \end{cases} \quad \bullet\ \text{Replace Equation (5).}$$

The system of equations is now in triangular form. Solve the system of equations by back substitution.

Solve Equation (6) for z. Substitute the value into Equation (4), and then solve for y.

$$\begin{aligned} -2z &= -6 && \bullet\ \text{Equation (6)} \\ z &= 3 \end{aligned}$$

$$\begin{aligned} -5y + 3z &= 4 && \bullet\ \text{Equation (4)} \\ -5y + 3(3) &= 4 && \bullet\ \text{Replace } z \text{ with } 3. \\ -5y &= -5 && \bullet\ \text{Solve for } y. \\ y &= 1 \end{aligned}$$

Replace z with 3 and y with 1 in Equation (1), and then solve for x.

$$\begin{aligned} x + 2y - z &= 1 && \bullet\ \text{Equation (1)} \\ x + 2(1) - 3 &= 1 && \bullet\ \text{Replace } y \text{ with } 1;\ \text{replace } z \text{ with } 3. \\ x &= 2 \end{aligned}$$

The system of equations is consistent. The solution is the ordered triple $(2, 1, 3)$. See Figure 6.13.

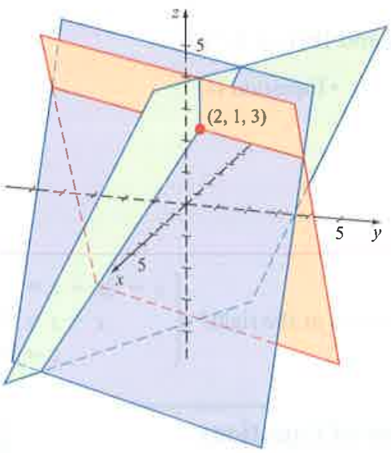


Figure 6.13

▶ **Try Exercise 16, page 498**

EXAMPLE 2 Solve a Dependent System of Equations

Solve: $\begin{cases} 2x - y - z = -1 & (1) \\ -x + 3y - z = -3 & (2) \\ -5x + 5y + z = -1 & (3) \end{cases}$

Solution

Eliminate x from Equation (2) by multiplying Equation (2) by 2 and then adding it to Equation (1). Replace Equation (2) with the new equation.

$$\begin{aligned} 2x - y - z &= -1 && \bullet\ \text{Equation (1)} \\ -2x + 6y - 2z &= -6 && \bullet\ 2 \text{ times Equation (2)} \\ \hline 5y - 3z &= -7 && \bullet\ \text{Add the equations.} \end{aligned}$$

$$\begin{cases} 2x - y - z = -1 & (1) \\ 5y - 3z = -7 & (4) \\ -5x + 5y + z = -1 & (3) \end{cases}$$ • Replace Equation (2).

Eliminate x from Equation (3) by multiplying Equation (1) by 5 and multiplying Equation (3) by 2. Then add. Replace Equation (3) with the new equation.

$$\begin{aligned} 10x - 5y - 5z &= -5 \\ -10x + 10y + 2z &= -2 \\ \hline 5y - 3z &= -7 \end{aligned}$$

• 5 times Equation (1)
• 2 times Equation (3)
• Add the equations.

$$\begin{cases} 2x - y - z = -1 & (1) \\ 5y - 3z = -7 & (4) \\ 5y - 3z = -7 & (5) \end{cases}$$ • Replace Equation (3).

Eliminate y from Equation (5) by multiplying Equation (4) by -1 and then adding it to Equation (5). Replace Equation (5) with the new equation.

$$\begin{aligned} -5y + 3z &= 7 \\ 5y - 3z &= -7 \\ \hline 0 &= 0 \end{aligned}$$

• -1 times Equation (4)
• Equation (5)
• Add the equations.

$$\begin{cases} 2x - y - z = -1 & (1) \\ 5y - 3z = -7 & (4) \\ 0 = 0 & (6) \end{cases}$$ • Replace Equation (5).

Because any ordered triple (x, y, z) is a solution of Equation (6), the solutions of the system of equations will be the ordered triples that are solutions of Equations (1) and (4).

Solve Equation (4) for y.

$$\begin{aligned} 5y - 3z &= -7 \\ 5y &= 3z - 7 \\ y &= \frac{3}{5}z - \frac{7}{5} \end{aligned}$$

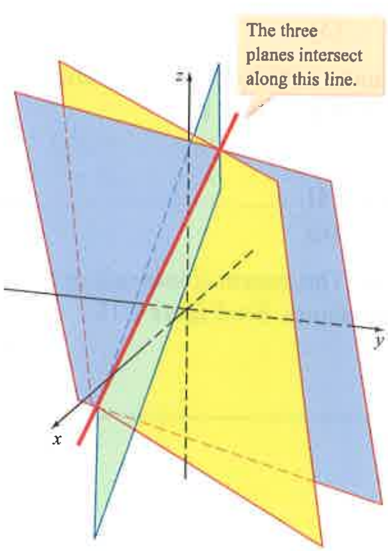


Figure 6.14

Substitute $\frac{3}{5}z - \frac{7}{5}$ for y in Equation (1) and solve for x.

$$\begin{aligned} 2x - y - z &= -1 \\ 2x - \left(\frac{3}{5}z - \frac{7}{5}\right) - z &= -1 \\ 2x - \frac{8}{5}z + \frac{7}{5} &= -1 \\ 2x &= \frac{8}{5}z - \frac{12}{5} \\ x &= \frac{4}{5}z - \frac{6}{5} \end{aligned}$$

• Equation (1)
• Replace y with $\frac{3}{5}z - \frac{7}{5}$.
• Simplify and solve for x.

By choosing any real number c for z, we have $y = \frac{3}{5}c - \frac{7}{5}$ and $x = \frac{4}{5}c - \frac{6}{5}$. For any real number c, the ordered-triple solutions of the system of equations are $\left(\frac{4}{5}c - \frac{6}{5}, \frac{3}{5}c - \frac{7}{5}, c\right)$. The solid red line shown in Figure 6.14 is a graph of the solutions.

▶ Try Exercise 22, page 498

Study tip

Although the ordered triples

$$\left(\frac{4}{5}c - \frac{6}{5}, \frac{3}{5}c - \frac{7}{5}, c\right)$$

and

$$\left(a, \frac{3}{4}a - \frac{1}{2}, \frac{5}{4}a + \frac{3}{2}\right)$$

appear to be different, they represent exactly the same solutions. For instance, choosing $c = -1$, we have $(-2, -2, -1)$. Choosing $a = -2$ results in the same ordered triple, $(-2, -2, -1)$.

As in the case of a dependent system of equations in two variables, there is more than one way to represent the solutions of a dependent system of equations in three variables. For instance, from Example 2, let $a = \frac{4}{5}c - \frac{6}{5}$, the x-coordinate of the ordered triple $\left(\frac{4}{5}c - \frac{6}{5}, \frac{3}{5}c - \frac{7}{5}, c\right)$, and solve for c.

$$a = \frac{4}{5}c - \frac{6}{5} \quad \rightarrow \quad c = \frac{5}{4}a + \frac{3}{2}$$

Substitute this value of c into each component of the ordered triple.

$$\left(\frac{4}{5}\left(\frac{5}{4}a + \frac{3}{2}\right) - \frac{6}{5}, \frac{3}{5}\left(\frac{5}{4}a + \frac{3}{2}\right) - \frac{7}{5}, \frac{5}{4}a + \frac{3}{2}\right) = \left(a, \frac{3}{4}a - \frac{1}{2}, \frac{5}{4}a + \frac{3}{2}\right)$$

Thus the solutions of the system of equations can also be written as

$$\left(a, \frac{3}{4}a - \frac{1}{2}, \frac{5}{4}a + \frac{3}{2}\right)$$

EXAMPLE 3 Identify an Inconsistent System of Equations

Solve: $\begin{cases} x + 2y + 3z = 4 & (1) \\ 2x - y - z = 3 & (2) \\ 3x + y + 2z = 5 & (3) \end{cases}$

Solution

Eliminate x from Equation (2) by multiplying Equation (1) by -2 and then adding it to Equation (2). Replace Equation (2). Eliminate x from Equation (3) by multiplying Equation (1) by -3 and adding it to Equation (3). Replace Equation (3). The equivalent system is

$$\begin{cases} x + 2y + 3z = 4 & (1) \\ -5y - 7z = -5 & (4) \\ -5y - 7z = -7 & (5) \end{cases}$$

Eliminate y from Equation (5) by multiplying Equation (4) by -1 and adding it to Equation (5). Replace Equation (5). The equivalent system is

$$\begin{cases} x + 2y + 3z = 4 & (1) \\ -5y - 7z = -5 & (4) \\ 0 = -2 & (6) \end{cases}$$

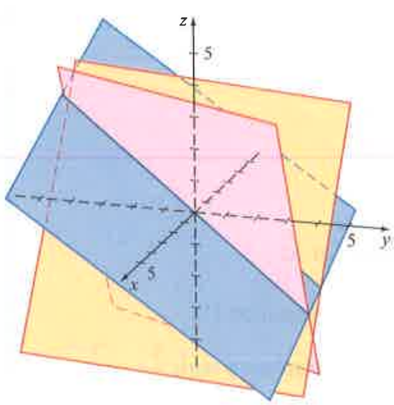


Figure 6.15

This system of equations contains a false equation. The system is inconsistent and has no solution. There is no point on all three planes. See Figure 6.15.

Try Exercise 20, page 498

Nonsquare Systems of Equations

The systems of linear equations we have solved so far contain the same number of variables as equations. These are *square systems of equations.* If there are fewer equations than variables—a *nonsquare system of equations*—the system has either no solution or an infinite number of solutions.

EXAMPLE 4 Solve a Nonsquare System of Equations

Solve: $\begin{cases} x - 2y + 2z = 3 & (1) \\ 2x - y - 2z = 15 & (2) \end{cases}$

Solution

Eliminate x from Equation (2) by multiplying Equation (1) by -2 and adding it to Equation (2). Replace Equation (2).

$$\begin{cases} x - 2y + 2z = 3 & (1) \\ 3y - 6z = 9 & (3) \end{cases}$$

Solve Equation (3) for y.

$$3y - 6z = 9$$
$$y = 2z + 3$$

Substitute $2z + 3$ for y in Equation (1) and solve for x.

$$x - 2y + 2z = 3$$
$$x - 2(2z + 3) + 2z = 3 \qquad \bullet\ y = 2z + 3$$
$$x = 2z + 9$$

For each value of z selected, there are corresponding values for x and y. If z is any real number c, then the solutions of the system are the ordered triples $(2c + 9, 2c + 3, c)$.

Try Exercise 24, page 498

Homogeneous Systems of Equations

A system of linear equations in which the constant term is zero for all equations is called a **homogeneous system of equations.** Two examples of homogeneous systems of equations are

$$\begin{cases} 3x + 4y = 0 \\ 2x + 3y = 0 \end{cases} \qquad \begin{cases} 2x - 3y + 5z = 0 \\ 3x + 2y + z = 0 \\ x - 4y + 5z = 0 \end{cases}$$

The ordered pair $(0, 0)$ is always a solution of a homogeneous system of equations in two variables, and the ordered triple $(0, 0, 0)$ is always a solution of a homogeneous system of equations in three variables. This solution is called the **trivial solution.**

Sometimes a homogeneous system of equations may have solutions other than the trivial solution. For example, $(1, -1, -1)$ is a solution of the homogeneous system of three equations in three variables given above.

If a homogeneous system of equations has a unique solution, the graphs intersect only at the origin. Solutions of a homogeneous system of equations can be found by using the substitution method or the elimination method.

EXAMPLE 5 Solve a Homogeneous System of Equations

Solve: $\begin{cases} x + 2y - 3z = 0 & (1) \\ 2x - y + z = 0 & (2) \\ 3x + y - 2z = 0 & (3) \end{cases}$

Solution

Eliminate x from Equations (2) and (3) and replace these equations with the new equations.

$$\begin{cases} x + 2y - 3z = 0 & (1) \\ -5y + 7z = 0 & (4) \\ -5y + 7z = 0 & (5) \end{cases}$$

(continued)

Eliminate y from Equation (5). Replace Equation (5).

$$\begin{cases} x + 2y - 3z = 0 & (1) \\ -5y + 7z = 0 & (4) \\ 0 = 0 & (6) \end{cases}$$

Because Equation (6) is an identity, the solutions of the system are the solutions of Equations (1) and (4).

Solve Equation (4) for y.

$$y = \frac{7}{5}z$$

Substitute the expression for y into Equation (1) and solve for x.

$$x + 2y - 3z = 0 \quad \text{• Equation (1)}$$

$$x + 2\left(\frac{7}{5}z\right) - 3z = 0 \quad \text{• } y = \frac{7}{5}z$$

$$x = \frac{1}{5}z$$

Letting z be any real number c, we find that the solutions of the system are the ordered triples

$$\left(\frac{1}{5}c, \frac{7}{5}c, c\right)$$

▶ Try Exercise 36, page 498

Applications of Systems of Equations

One application of a system of equations is *curve fitting*. Given a set of points in the plane, we can try to find an equation whose graph passes through, or fits, all of the points.

EXAMPLE 6 Solve an Application of a System of Equations to Curve Fitting

Find an equation of the form $y = ax^2 + bx + c$ whose graph passes through the points located at $(1, 4)$, $(-1, 6)$, and $(2, 9)$.

Solution

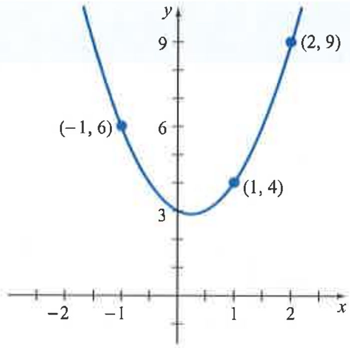
Substitute each of the given ordered pairs into the equation $y = ax^2 + bx + c$. Write the resulting system of equations.

$$\begin{cases} 4 = a(1)^2 + b(1) + c \\ 6 = a(-1)^2 + b(-1) + c \\ 9 = a(2)^2 + b(2) + c \end{cases} \quad \text{or} \quad \begin{cases} a + b + c = 4 & (1) \\ a - b + c = 6 & (2) \\ 4a + 2b + c = 9 & (3) \end{cases}$$

Solve the resulting system of equations for a, b, and c.

Eliminate a from Equation (2) by multiplying Equation (1) by -1 and then adding it to Equation (2). Now eliminate a from Equation (3) by multiplying Equation (1) by -4 and adding it to Equation (3). The result is

$$\begin{cases} a + b + c = 4 \\ -2b = 2 \\ -2b - 3c = -7 \end{cases}$$



$y = 2x^2 - x + 3$

Figure 6.16

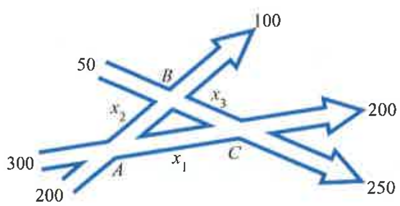
Although this system of equations is not in triangular form, we can solve the second equation for b and use this value to find a and c.

Solving by substitution, we obtain $a = 2$, $b = -1$, and $c = 3$. The equation of the form $y = ax^2 + bx + c$ whose graph passes through $(1, 4)$, $(-1, 6)$, and $(2, 9)$ is $y = 2x^2 - x + 3$. See Figure 6.16.

▶ **Try Exercise 40, page 498**

Traffic engineers use systems of equations to study the flow of traffic. The analysis of traffic flow is based on the principle that the numbers of cars that enter and leave an intersection must be equal.

EXAMPLE 7 Traffic Flow



Suppose the traffic flow for some one-way streets can be modeled by the diagram at the left, where the numbers and the variables represent the numbers of cars entering or leaving an intersection per hour.

If the street connecting intersections A and C has a traffic flow of 100 to 200 cars per hour, what is the traffic flow between A and B (which is x_2) and between B and C (which is x_3)?

Solution

Let x_1, x_2, and x_3 represent the numbers of cars per hour that are traveling on AC, AB, and BC, respectively. Now consider intersection A. There are $300 + 200 = 500$ cars per hour entering A and $x_1 + x_2$ cars leaving A. Therefore, $x_1 + x_2 = 500$. For intersection B, we have $50 + x_2$ cars per hour entering the intersection and $100 + x_3$ cars leaving the intersection. Thus $50 + x_2 = 100 + x_3$, or $x_2 - x_3 = 50$. Applying the same reasoning to C, we have $x_1 + x_3 = 450$. These equations result in the system of equations

$$\begin{cases} x_1 + x_2 = 500 & (1) \\ x_2 - x_3 = 50 & (2) \\ x_1 + x_3 = 450 & (3) \end{cases}$$

Subtracting Equation (2) from Equation (1) gives

$$\begin{array}{rl} x_1 + x_2 = 500 & (1) \\ \underline{x_2 - x_3 = 50} & (2) \\ x_1 + x_3 = 450 & (4) \end{array}$$

Subtracting Equation (4) from Equation (3) gives

$$\begin{array}{rl} x_1 + x_3 = 450 & (3) \\ \underline{x_1 + x_3 = 450} & (4) \\ 0 = 0 & \end{array}$$

This indicates that the system of equations is dependent. Because we are given that 100 to 200 cars per hour flow between A and C (the value of x_1), we will solve each equation in terms of x_1. From Equation (1) we have $x_2 = -x_1 + 500$, and from Equation (3) we have $x_3 = -x_1 + 450$. Because $100 \le x_1 \le 200$, we have, by substituting for x_1, $300 \le x_2 \le 400$ and $250 \le x_3 \le 350$. The traffic flow between A and B is 300 to 400 cars per hour, and the traffic flow between B and C is 250 to 350 cars per hour.

▶ **Try Exercise 46, page 499**

EXERCISE SET 6.2

Concept Check

1. Determine which of the following ordered triples are solutions of

$$3x - 2y + z = 6$$

a. (3, 4, 5) b. (5, 3, −3) c. (0, 3, 11)

2. What is a homogeneous system of equations?

3. Consider the following nonsquare system of equations.

$$\begin{cases} x + y + z = 6 \\ 2x - y - 3z = 9 \end{cases}$$

Describe the graph of the solutions of this system.

4. Is the following system an independent system, a dependent system, or an inconsistent system of equations?

$$\begin{cases} x + y + z = 9 \\ x - y + z = 1 \end{cases}$$

In Exercises 5 to 28, solve each system of equations.

5. $\begin{cases} 2x - y + z = 8 \\ 2y - 3z = -11 \\ 3y + 2z = 3 \end{cases}$

6. $\begin{cases} 3x + y + 2z = -4 \\ -3y - 2z = -5 \\ 2y + 5z = -4 \end{cases}$

7. $\begin{cases} 2x + 2y - 3z = 18 \\ x - 3y + 4z = -6 \\ 3x - y + 2z = 10 \end{cases}$

8. $\begin{cases} x - 2y + 3z = 5 \\ 3x - 3y + z = 9 \\ 5x + y - 3z = 3 \end{cases}$

9. $\begin{cases} 3x + 4y - z = -7 \\ x - 5y + 2z = 19 \\ 5x + y - 2z = 5 \end{cases}$

10. $\begin{cases} 2x - 3y - 2z = 12 \\ x + 4y + z = -9 \\ 4x + 2y - 3z = 6 \end{cases}$

11. $\begin{cases} 3x - 2y + 3z = 8 \\ 2x + 3y - 4z = -28 \\ x - y + 2z = 9 \end{cases}$

12. $\begin{cases} 4x - y + 2z = -1 \\ 2x + 3y - 3z = -13 \\ x + 5y + z = 7 \end{cases}$

13. $\begin{cases} 3x - 2y + 4z = 16 \\ 2x + 3y + 2z = 16 \\ 5x + 2y + 4z = 24 \end{cases}$

14. $\begin{cases} x - 3y + 2z = -11 \\ 3x + y + 4z = 4 \\ 5x - 5y + 8z = -18 \end{cases}$

15. $\begin{cases} 2x - 5y + 2z = -4 \\ 3x + 2y + 3z = 13 \\ 5x - 3y - 4z = -18 \end{cases}$

16. $\begin{cases} 3x + 2y - 5z = 6 \\ 5x - 4y + 3z = -12 \\ 4x + 5y - 2z = 15 \end{cases}$

17. $\begin{cases} 2x + y - z = -2 \\ 3x + 2y + 3z = 21 \\ 7x + 4y + z = 17 \end{cases}$

18. $\begin{cases} 3x + y + 2z = 2 \\ 4x - 2y + z = -4 \\ 11x - 3y + 4z = -6 \end{cases}$

19. $\begin{cases} 6x - 7y + 12z = 81 \\ 4x + 3y + 2z = 7 \\ 3x - 5y + 11z = 65 \end{cases}$

20. $\begin{cases} x + 3y - 2z = 2 \\ -2x - 4y + z = 0 \\ -3x - 7y + 3z = -1 \end{cases}$

21. $\begin{cases} 2x - 3y + 6z = 3 \\ x + 2y - 4z = 5 \\ 3x + 4y - 8z = 7 \end{cases}$

22. $\begin{cases} 2x + 3y - 6z = 4 \\ 3x - 2y - 9z = -7 \\ 2x + 5y - 6z = 8 \end{cases}$

23. $\begin{cases} 2x - 3y + 5z = 14 \\ x + 4y - 3z = -2 \end{cases}$

24. $\begin{cases} x - 3y + 4z = 9 \\ 3x - 8y - 2z = 4 \end{cases}$

25. $\begin{cases} 6x - 9y + 6z = 7 \\ 4x - 6y + 4z = 9 \end{cases}$

26. $\begin{cases} 4x - 2y + 6z = 5 \\ 2x - y + 3z = 2 \end{cases}$

27. $\begin{cases} 5x + 3y + 2z = 10 \\ 3x - 4y - 4z = -5 \end{cases}$

28. $\begin{cases} 3x - 4y - 7z = -5 \\ 2x + 3y - 5z = 2 \end{cases}$

In Exercises 29 to 36, solve each homogeneous system of equations.

29. $\begin{cases} x + 3y - 4z = 0 \\ 2x + 7y + z = 0 \\ 3x - 5y - 2z = 0 \end{cases}$

30. $\begin{cases} x - 2y + 3z = 0 \\ 3x - 7y - 4z = 0 \\ 4x - 4y + z = 0 \end{cases}$

31. $\begin{cases} 2x - 3y + z = 0 \\ 2x + 4y - 3z = 0 \\ 6x - 2y - z = 0 \end{cases}$

32. $\begin{cases} 5x - 4y - 3z = 0 \\ 2x + y + 2z = 0 \\ x - 6y - 7z = 0 \end{cases}$

33. $\begin{cases} 3x - 5y + 3z = 0 \\ 2x - 3y + 4z = 0 \\ 7x - 11y + 11z = 0 \end{cases}$

34. $\begin{cases} 5x - 2y - 3z = 0 \\ 3x - y - 4z = 0 \\ 4x - y - 9z = 0 \end{cases}$

35. $\begin{cases} 4x - 7y - 2z = 0 \\ 2x + 4y + 3z = 0 \\ 3x - 2y - 5z = 0 \end{cases}$

36. $\begin{cases} 5x + 2y + 3z = 0 \\ 3x + y - 2z = 0 \\ 4x - 7y + 5z = 0 \end{cases}$

In Exercises 37 to 50, solve each exercise by solving a system of equations.

37. **Curve Fitting** Find the quadratic equation of the form $y = ax^2 + bx + c$ whose graph passes through the points (2, 3), (−2, 7), and (1, −2).

38. **Curve Fitting** Find the quadratic equation of the form $y = ax^2 + bx + c$ whose graph passes through the points (1, −2), (3, −4), and (2, −2).

39. **Curve Fitting** Find the equation of the circle whose graph passes through the points (5, 3), (−1, −5), and (−2, 2). (*Hint:* Use the equation $x^2 + y^2 + ax + by + c = 0$.)

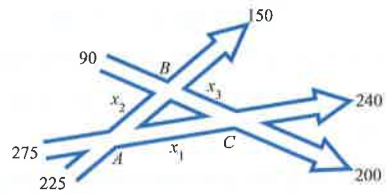
40. **Curve Fitting** Find the equation of the circle whose graph passes through the points (0, 6), (1, 5), and (−7, −1). (*Hint:* See Exercise 39.)

Indicates Try It Exercises

41. **Curve Fitting** Find the center and radius of the circle whose graph passes through the points $(-2, 10)$, $(-12, -14)$, and $(5, 3)$. (*Hint:* See Exercise 39.)

42. **Curve Fitting** Find the center and radius of the circle whose graph passes through the points $(2, 5)$, $(-4, -3)$, and $(3, 4)$. (*Hint:* See Exercise 39.)

43. **Traffic Flow** Suppose that the traffic flow for some one-way streets can be modeled by the diagram below, where each number or variable represents the number of cars entering or leaving an intersection per hour.

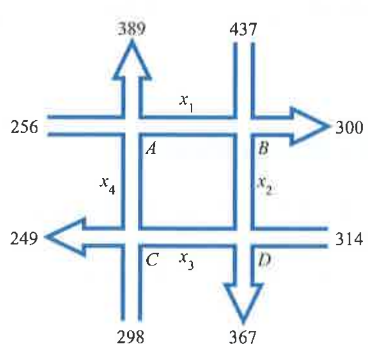


If the street connecting intersections A and B has a traffic flow of 150 to 250 cars per hour, what is the traffic flow between B and C?

44. **Traffic Flow** A roundabout is a type of intersection that accommodates traffic flow in one direction, around a circular island. The graphic model below shows the numbers of cars per hour that are entering or leaving a roundabout. The variables x_1, x_2, and x_3 represent the traffic flow per hour along the three portions of the roundabout.

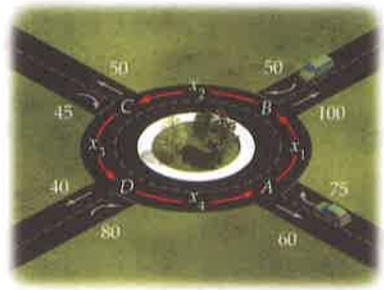
If the portion of the roundabout between A and B has a traffic flow of from 60 to 80 cars per hour, what is the traffic flow between C and A and between B and C?

45. **Traffic Flow** Suppose that the traffic flow for some one-way streets can be modeled by the accompanying diagram, where each number or variable represents the number of cars entering or leaving an intersection per hour.



If the street connecting intersections A and B has an estimated traffic flow of from 125 to 175 cars per hour, what is the estimated traffic flow between C and A, D and C, and B and D?

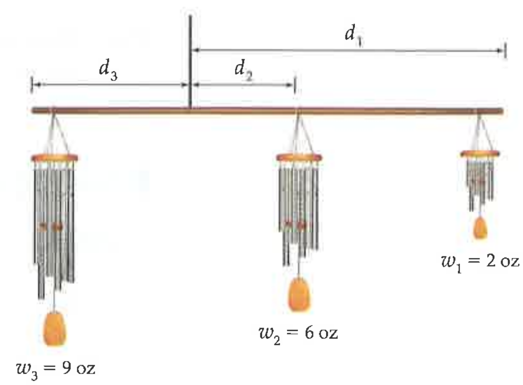
46. **Traffic Flow** The graphic model below shows the numbers of cars per hour that are entering and leaving a roundabout. What is the minimum number of cars per hour that can travel between B and C?



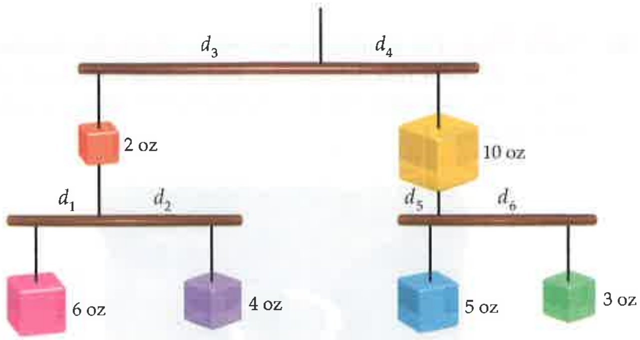
47. **Coin Exchange** Every day Emily puts her quarters, dimes, and nickels into a shoe box. After several weeks, she decides to take the coins to a machine that counts coins and provides a voucher that she can exchange for dollars. The coin machine charges a fee equal to 10% of the total value of the coins. After the machine has counted her coins, she receives a voucher for $139.50. The voucher also shows that she had a total of 1025 coins and twice as many quarters as nickels. How many coins of each denomination did Emily have?

48. **Hotel Data** A hotel resources report listed the total number of hotels in the Holiday Inn, Hilton, and Hyatt hotel chains as 2238. The number of Holiday Inn hotels was 400 more than the total number of Hilton and Hyatt hotels. There were 121 more Hilton hotels than Hyatt hotels. Determine the number of hotels in each of these hotel chains.

49. **Art** A sculptor is creating a wind chime consisting of three chimes that will be suspended from a rod 13 inches long. The weights, in ounces, of the chimes are shown in the diagram. For the rod to remain horizontal, the chimes must be positioned so that $w_1d_1 + w_2d_2 = w_3d_3$. If the sculptor wants d_2 to be one-third of d_1, find the position of the middle chime that will make the wind chime balance.



50. Art A designer wants to create a mobile of colored blocks as shown in the diagram below. The weight, in ounces, of each of the blocks is shown next to the block.



Given that $d_3 + d_4 = 20$ inches, $d_1 + d_2 = 10$ inches, and $d_5 + d_6 = 8$ inches, find the values of d_1 through d_6 so that each bar is horizontal. (A bar is horizontal when the value of weight times distance on each side of a vertical support is equal. For instance, for the diagram above, $6d_1$ must equal $4d_2$. Because there are six variables, the resulting system of equations must contain six equations.)

In Exercises 51 and 52, find an equation of the plane that contains the given points. (*Hint:* The equation of a plane can be written as $z = ax + by + c$.)

51. $(1, -1, 5), (2, -2, 9), (-3, -1, -1)$

52. $(2, 1, 1), (-1, 2, 12), (3, 2, 0)$

Enrichment Exercises

In Exercises 53 and 54, use the system of equations

$$\begin{cases} x - 3y - 2z = A^2 \\ 2x - 5y + Az = 9 \\ 2x - 8y + z = 18 \end{cases}$$

53. Find all values of A for which the system has no solution.

54. Find all values of A for which the system has a unique solution.

In Exercises 55 and 56, use the system of equations

$$\begin{cases} x + 2y + z = A^2 \\ -2x - 3y + Az = 1 \\ 7x + 12y + A^2z = 4A^2 - 3 \end{cases}$$

55. Find all values of A for which the system has a unique solution.

56. Find all values of A for which the system has an infinite number of solutions.

SECTION 6.3

Solving Nonlinear Systems of Equations

Nonlinear Systems of Equations

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A39.

PS1. Solve $x^2 + 2x - 2 = 0$ for x. [1.3]

PS2. Solve: $\begin{cases} x + 4y = -11 \\ 3x - 2y = 9 \end{cases}$ [6.1]

PS3. Name the graph of $(y + 3)^2 = 8x$. [5.1]

PS4. Name the graph of $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$. [5.3]

PS5. How many times do the graphs of $y = 2x - 1$ and $x^2 + y^2 = 4$ intersect? [2.1/2.2]

PS6. How many times do the graphs of $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ and $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ intersect? [5.2]

Solving Nonlinear Systems of Equations

A **nonlinear system of equations** is one in which one or more equations of the system are not linear equations. In this section, we will consider only solutions whose coordinates are real numbers. Therefore, if a system of equations does not have any solution in which both coordinates are real numbers, we will simply state that the system has no solution.