5.7 Inverse of a Function

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Learning Target Understand the relationship between inverse functions.

Success Criteria
- I can explain what inverse functions are.
- I can find inverses of linear and nonlinear functions.
- I can determine whether a pair of functions are inverses.

EXPLORE IT! Describing Functions and Their Inverses

Work with a partner.

a. Consider each pair of functions, f and g, below. For each pair, create an input-output table of values for each function. Use the outputs of f as the inputs of g. What do you notice about the relationship between the equations of f and g?

i. $f(x) = 4x + 3$

$g(x) = \dfrac{x - 3}{4}$

ii. $f(x) = x^3 + 1$

$g(x) = \sqrt[3]{x - 1}$

iii. $f(x) = \sqrt{x - 3}$

$g(x) = x^2 + 3, x \geq 0$

b. What do you notice about the graphs of each pair of functions in part (a)?

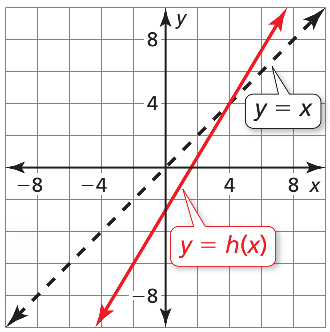
c. For each pair of functions in part (a), find $f(g(x))$ and $g(f(x))$. What do you notice?

Math Practice

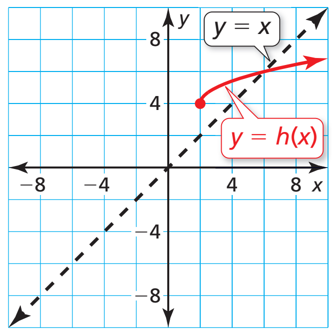
Build Arguments
In part (c), why do you think this occurs when you find the compositions of these functions?

d. The functions h and j are inverses of each other. Use the graph of h to find the given value. Explain how you found your answers.

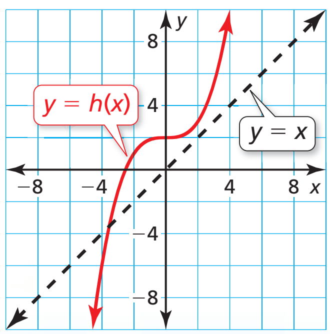
i. $j(-6)$



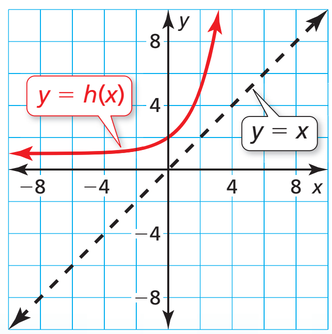
ii. $j(4)$



iii. $j(-6)$



iv. $j(2)$



Exploring Inverses of Functions

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You can solve equations of the form $y = f(x)$ for x to obtain an equation that gives the input for a specific output of f.

Vocabulary

VOCAB

Inverse functions, *p. 274*

EXAMPLE 1 Writing a Formula for the Input of a Function

Let $f(x) = 2x + 3$. Solve $y = f(x)$ for x. Then find the input when the output is -7.

WATCH

SOLUTION

$y = 2x + 3$ — Set y equal to $f(x)$.

$y - 3 = 2x$ — Subtract 3 from each side.

$\dfrac{y-3}{2} = x$ — Divide each side by 2.

Find the input when $y = -7$.

$x = \dfrac{-7-3}{2}$ — Substitute -7 for y.

$= \dfrac{-10}{2}$ — Subtract.

$= -5$ — Divide.

Check

$f(-5) = 2(-5) + 3$
$= -10 + 3$
$= -7$ ✓

▶ So, the input is -5 when the output is -7.

In Example 1, notice the operations in the equations $y = 2x + 3$ and $x = \dfrac{y-3}{2}$.

$y = 2x + 3$		$x = \dfrac{y-3}{2}$
Multiply by 2.		Subtract 3.
Add 3.	inverse operations in the reverse order	Divide by 2.

Math Practice

Communicate Precisely

The term *inverse functions* does not refer to a new type of function. The term describes any pair of functions that are inverses.

These operations *undo* each other. **Inverse functions** are functions that undo each other. In Example 1, use the equation solved for x to write the inverse of f by switching x and y.

$$x = \frac{y-3}{2} \xrightarrow{\text{switch } x \text{ and } y} y = \frac{x-3}{2}$$

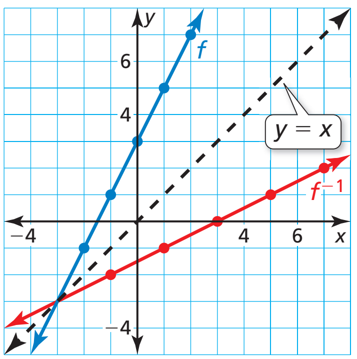
An inverse function can be denoted by f^{-1}, read as "f inverse." Because an inverse function switches the input and output values of the original function, the domain and range are also switched.

Original function: $f(x) = 2x + 3$

x	-2	-1	0	1	2
y	-1	1	3	5	7

Inverse function: $f^{-1}(x) = \dfrac{x-3}{2}$

x	-1	1	3	5	7
y	-2	-1	0	1	2



The graph of f^{-1} is a *reflection* of the graph of f. The *line of reflection* is $y = x$. This is true for all inverses.

To find the inverse of a function algebraically, switch the roles of x and y, and then solve for y.

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EXAMPLE 2 Finding the Inverse of a Linear Function

Find the inverse of $f(x) = 3x - 1$.

WATCH

SOLUTION

Method 1 Use inverse operations in the reverse order.

$$f(x) = 3x - 1 \qquad \text{Multiply the input } x \text{ by 3 and then subtract 1.}$$

To find the inverse, apply inverse operations in the reverse order.

$$f^{-1}(x) = \frac{x+1}{3} \qquad \text{Add 1 to the input } x \text{ and then divide by 3.}$$

▶ The inverse of f is $f^{-1}(x) = \frac{x+1}{3}$.

READING

The -1 in f^{-1} is not an exponent. It indicates that the function is an inverse, *not* that it is equal to $\frac{1}{f(x)}$.

Method 2 Set y equal to $f(x)$. Switch the roles of x and y and solve for y.

$$y = 3x - 1 \qquad \text{Set } y \text{ equal to } f(x).$$

$$x = 3y - 1 \qquad \text{Switch } x \text{ and } y.$$

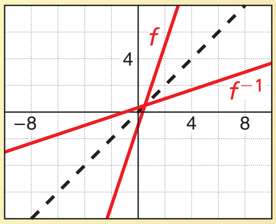
$$x + 1 = 3y \qquad \text{Add 1 to each side.}$$

$$\frac{x+1}{3} = y \qquad \text{Divide each side by 3.}$$

▶ The inverse of f is $f^{-1}(x) = \frac{x+1}{3}$.

Check

Use technology to graph f and f^{-1}.



The graph of f^{-1} appears to be a reflection of the graph of f in the line $y = x$. ✓

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve $y = f(x)$ for x. Then find the input(s) when the output is 2.

1. $f(x) = x - 2$

2. $f(x) = 2x^2$

3. $f(x) = -x^3 + 3$

4. VOCABULARY In your own words, state the definition of inverse functions.

Find the inverse of the function. Then graph the function and its inverse.

5. $f(x) = 2x$

6. $f(x) = -x + 1$

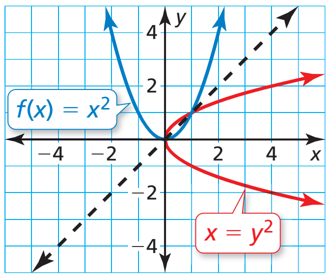
7. $f(x) = \frac{1}{3}x - 2$

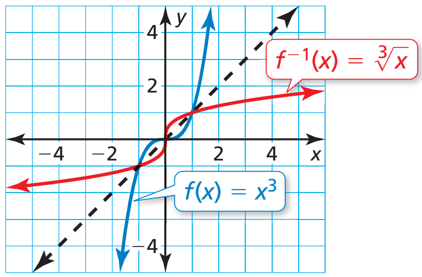
Inverses of Nonlinear Functions

In the previous examples, the inverses of the linear functions were also functions. However, inverses of functions are *not* always functions. The graphs of $f(x) = x^2$ and $f(x) = x^3$ are shown along with their reflections in the line $y = x$. Notice that the inverse of $f(x) = x^3$ is a function, but the inverse of $f(x) = x^2$ is *not* a function.

REMEMBER

You can use the Vertical Line Test to check whether the inverse is a function.





When the domain of $f(x) = x^2$ is *restricted* to only nonnegative real numbers, the inverse of f *is* a function, as shown in the next example.

EXAMPLE 3 Finding the Inverse of a Quadratic Function

WATCH

Find the inverse of $f(x) = x^2$, $x \geq 0$. Then graph the function and its inverse.

SOLUTION

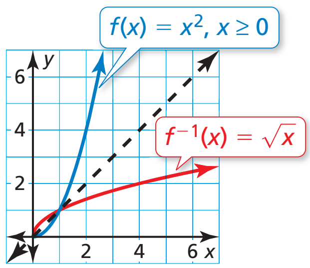
$f(x) = x^2$	Write the original function.
$y = x^2$	Set y equal to $f(x)$.
$x = y^2$	Switch x and y.
$\pm\sqrt{x} = y$	Take square root of each side.

STUDY TIP

If the domain of f is instead restricted to $x \leq 0$, then the inverse is $f^{-1}(x) = -\sqrt{x}$.

The domain of f is restricted to nonnegative values of x. So, the range of the inverse must also be restricted to nonnegative values.

► So, the inverse of f is $f^{-1}(x) = \sqrt{x}$.



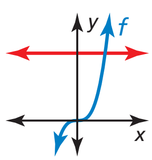
You can use the graph of a function f to determine whether the inverse of f is a function by applying the *Horizontal Line Test*.

KEY IDEA

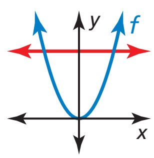
Horizontal Line Test

The inverse of a function f is also a function if and only if no horizontal line intersects the graph of f more than once.

Inverse is a function



Inverse is not a function

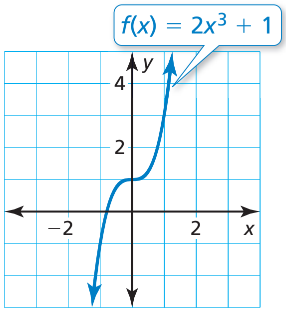


EXAMPLE 4 Finding the Inverse of a Cubic Function

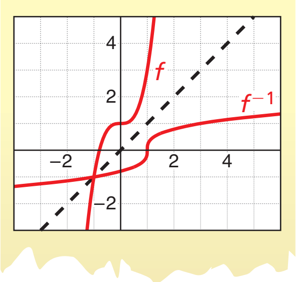
Consider the function $f(x) = 2x^3 + 1$. Determine whether the inverse of f is a function. Then find the inverse.

SOLUTION

Graph the function f. Notice that no horizontal line intersects the graph more than once. So, the inverse of f is a function. Find the inverse.



$y = 2x^3 + 1$	Set y equal to $f(x)$.
$x = 2y^3 + 1$	Switch x and y.
$x - 1 = 2y^3$	Subtract 1 from each side.
$\dfrac{x-1}{2} = y^3$	Divide each side by 2.
$\sqrt[3]{\dfrac{x-1}{2}} = y$	Take cube root of each side.



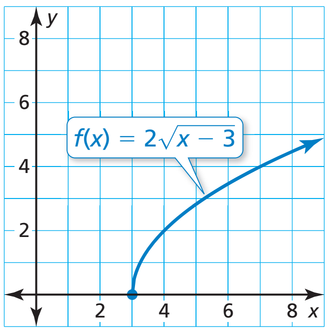
▶ So, the inverse of f is $f^{-1}(x) = \sqrt[3]{\dfrac{x-1}{2}}$.

EXAMPLE 5 Finding the Inverse of a Radical Function

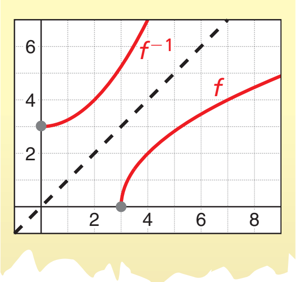
Consider the function $f(x) = 2\sqrt{x-3}$. Determine whether the inverse of f is a function. Then find the inverse.

SOLUTION

Graph the function f. Notice that no horizontal line intersects the graph more than once. So, the inverse of f is a function. Find the inverse.



$y = 2\sqrt{x-3}$	Set y equal to $f(x)$.
$x = 2\sqrt{y-3}$	Switch x and y.
$x^2 = \left(2\sqrt{y-3}\right)^2$	Square each side.
$x^2 = 4(y-3)$	Simplify.
$\frac{1}{4}x^2 = y - 3$	Divide each side by 4.
$\frac{1}{4}x^2 + 3 = y$	Add 3 to each side.



Because the range of f is $y \geq 0$, the domain of the inverse must be restricted to $x \geq 0$.

▶ So, the inverse of f is $f^{-1}(x) = \frac{1}{4}x^2 + 3, x \geq 0$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

8. Find the inverse of $f(x) = -x^2, x \leq 0$. Then graph the function and its inverse.

Determine whether the inverse of f is a function. Then find the inverse.

9. $f(x) = -x^3 + 4$

10. $f(x) = \dfrac{1}{x^2 + 1}$

11. $f(x) = \sqrt{x + 2}$

12. WRITING Explain why you can use horizontal lines to determine whether the inverse of a function is also a function.

Math Practice

Communicate Precisely
Inverse functions *undo* each other. In your own words, explain what this means.

Let f and g be inverse functions. If $f(a) = b$, then $g(b) = a$. So, in general,

$$f(g(x)) = x \qquad \text{and} \qquad g(f(x)) = x.$$

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EXAMPLE 6 Determining Whether Functions Are Inverses

WATCH

Determine whether $f(x) = 3x - 1$ and $g(x) = \dfrac{x+1}{3}$ are inverse functions.

SOLUTION

Use compositions to determine whether f and g are inverse functions.

Step 1 Find $f(g(x))$.

$$\begin{aligned} f(g(x)) &= f\left(\frac{x+1}{3}\right) \\ &= 3\left(\frac{x+1}{3}\right) - 1 \\ &= x + 1 - 1 \\ &= x \checkmark \end{aligned}$$

Step 2 Find $g(f(x))$.

$$\begin{aligned} g(f(x)) &= g(3x - 1) \\ &= \frac{3x - 1 + 1}{3} \\ &= \frac{3x}{3} \\ &= x \checkmark \end{aligned}$$

So, f and g are inverse functions.

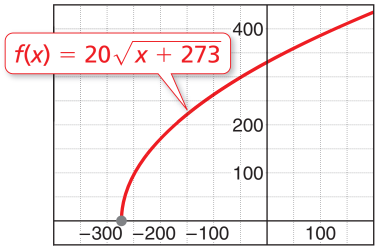
EXAMPLE 7 Modeling Real Life

WATCH

The speed of sound (in meters per second) through air is approximated by $f(x) = 20\sqrt{x + 273}$ where x is the temperature in degrees Celsius. Find and interpret $f^{-1}(340)$.

SOLUTION

Graph the function f. Notice that no horizontal line intersects the graph more than once. So, the inverse of f is a function. Find the inverse.



$y = 20\sqrt{x + 273}$	Set y equal to $f(x)$.
$x = 20\sqrt{y + 273}$	Switch x and y.
$x^2 = \left(20\sqrt{y + 273}\right)^2$	Square each side.
$x^2 = 400(y + 273)$	Simplify.
$\frac{1}{400}x^2 = y + 273$	Divide each side by 400.
$\frac{1}{400}x^2 - 273 = y$	Subtract 273 from each side.

Because the range of f is $y \geq 0$, the domain of the inverse must be restricted to $x \geq 0$. The inverse of f is $f^{-1}(x) = \frac{1}{400}x^2 - 273$, $x \geq 0$.

Using $f^{-1}(x)$, you obtain $f^{-1}(340) = 16$. This represents that the temperature is 16 degrees Celsius when the speed of sound through air is 340 meters per second.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Determine whether the functions are inverse functions.

13. $f(x) = x + 5$, $g(x) = x - 5$

14. $f(x) = 8x^3$, $g(x) = \sqrt[3]{2x}$

15. **WHAT IF?** In Example 7, find and interpret $f^{-1}(350)$.

5.7 Practice WITH CalcChat® AND CalcView®

In Exercises 1–8, solve $y = f(x)$ for x. Then find the input(s) when the output is -3. ▶ *Example 1*

1. $f(x) = 3x + 5$

2. $f(x) = -7x - 2$

3. $f(x) = \frac{1}{2}x - 3$

4. $f(x) = -\frac{2}{3}x + 1$

5. $f(x) = 3x^3$

6. $f(x) = 2x^4 - 5$

7. $f(x) = (x - 2)^2 - 7$

8. $f(x) = (x - 5)^3 - 1$

In Exercises 9–16, find the inverse of the function. Then graph the function and its inverse. ▶ *Example 2*

9. $f(x) = 6x$

10. $f(x) = -3x$

11. $f(x) = -2x + 5$

12. $f(x) = 6x - 3$

13. $f(x) = -\frac{1}{2}x + 4$

14. $f(x) = \frac{1}{3}x - 1$

15. $f(x) = \frac{2}{3}x - \frac{1}{3}$

16. $f(x) = -\frac{4}{5}x + \frac{1}{5}$

MP **REASONING** In Exercises 17 and 18, determine whether functions f and g are inverses. Explain your reasoning.

17.

x	-2	-1	0	1	2
$f(x)$	-2	1	4	7	10

x	-2	1	4	7	10
$g(x)$	-2	-1	0	1	2

18.

x	2	3	4	5	6
$f(x)$	8	6	4	2	0

x	2	3	4	5	6
$g(x)$	-8	-6	-4	-2	0

In Exercises 19–24, find the inverse of the function. Then graph the function and its inverse. ▶ *Example 3*

19. $f(x) = 4x^2,\ x \le 0$

20. $f(x) = 9x^2,\ x \le 0$

21. $f(x) = (x - 3)^2,\ x \ge 3$

22. $f(x) = (x + 4)^2,\ x \ge -4$

23. $f(x) = -(x - 1)^2 + 6,\ x \ge 1$

24. $f(x) = 2(x + 5)^2 - 2,\ x \le -5$

ERROR ANALYSIS In Exercises 25 and 26, describe and correct the error in finding the inverse of the function.

25.

✗

$f(x) = -x + 3$

$y = -x + 3$

$-x = y + 3$

$-x - 3 = y$

So, $f^{-1}(x) = -x - 3$.

26.

✗

$f(x) = \frac{1}{7}x^2, x \ge 0$

$y = \frac{1}{7}x^2$

$x = \frac{1}{7}y^2$

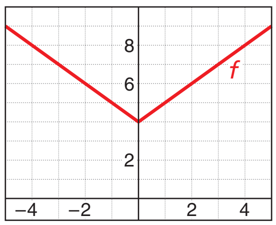
$7x = y^2$

$\pm\sqrt{7x} = y$

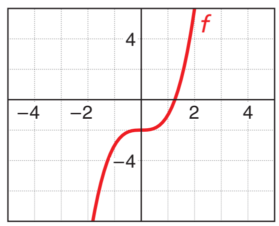
So, $f^{-1}(x) = \pm\sqrt{7x}$.

MP **USING TOOLS** In Exercises 27 and 28, use the graph to determine whether the inverse of f is a function. Explain your reasoning.

27.



28.



In Exercises 29–40, find the inverse of the function. Then graph the function and its inverse. ▶ *Examples 4 and 5*

29. $f(x) = x^3 - 1$

30. $f(x) = -x^3 + 3$

31. $f(x) = -x^3 + 2$

32. $f(x) = 2x^3 - 5$

33. $f(x) = \sqrt{x + 4}$

34. $f(x) = \sqrt{x - 6}$

35. $f(x) = 2\sqrt[3]{x - 5}$

36. $f(x) = 3\sqrt[3]{x + 1}$

37. $f(x) = \frac{2}{3}(x + 1)^3 + 8$

38. $f(x) = -\frac{2}{5}(x - 2)^3 - 4$

39. $f(x) = -\sqrt[3]{\dfrac{2x + 4}{3}}$

40. $f(x) = -3\sqrt{\dfrac{4x - 7}{3}}$

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41. **COLLEGE PREP** What is the inverse of $f(x) = -\frac{1}{64}x^3$?

Ⓐ $g(x) = -4x^3$ Ⓑ $g(x) = 4\sqrt[3]{x}$

Ⓒ $g(x) = -4\sqrt[3]{x}$ Ⓓ $g(x) = \sqrt[3]{-4x}$

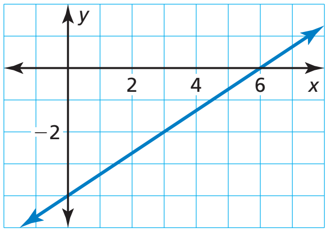
42. **COLLEGE PREP** What is the inverse of the function whose graph is shown?

Ⓐ $g(x) = \frac{3}{2}x - 6$

Ⓑ $g(x) = \frac{3}{2}x + 6$

Ⓒ $g(x) = \frac{2}{3}x + \frac{8}{3}$

Ⓓ $g(x) = \frac{3}{2}x + 4$



In Exercises 43–46, determine whether the functions are inverse functions. *Example 6*

43. $f(x) = 2x - 9$, $g(x) = \frac{x}{2} + 9$

44. $f(x) = \frac{x-1}{5}$, $g(x) = 5x + 1$

45. $f(x) = \sqrt[5]{\frac{x+9}{5}}$, $g(x) = 5x^5 - 9$

46. $f(x) = 7x^{3/2} - 4$, $g(x) = \left(\frac{x+4}{7}\right)^{3/2}$

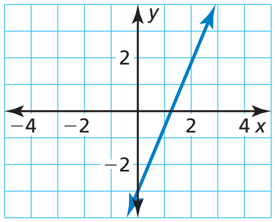
47. **MODELING REAL LIFE** The maximum hull speed (in knots) of a boat with a displacement hull can be approximated by $f(x) = 1.34\sqrt{x}$, where x is the waterline length (in feet) of the boat. Find and interpret $f^{-1}(7.5)$. *Example 7*

48. **MODELING REAL LIFE** Elastic bands can be used for exercising to provide a range of resistance. The resistance (in pounds) of a band can be modeled by $r(x) = \frac{3}{8}x - 5$, where x is the total length (in inches) of the stretched band. Find and interpret $r^{-1}(19)$.

ANALYZING RELATIONSHIPS

In Exercises 49–52, match the graph of the function with the graph of its inverse.

49.



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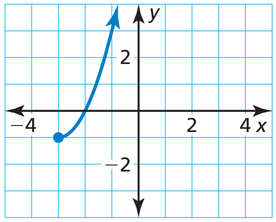

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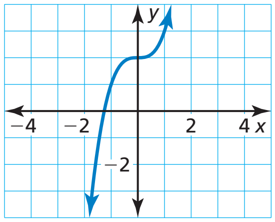

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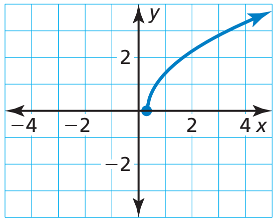

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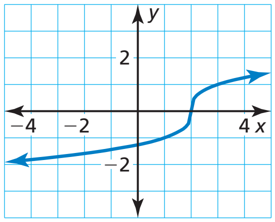

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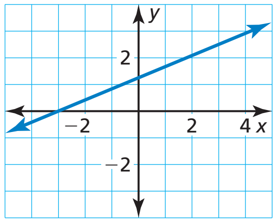
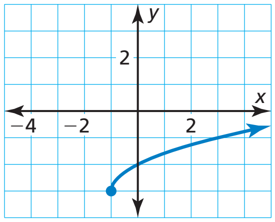


C.



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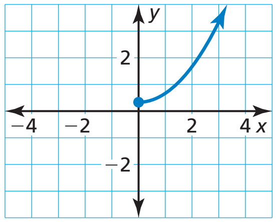
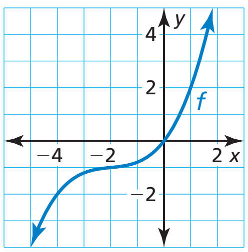


In Exercises 53 and 54, use the table or graph to find $f^{-1}(-2)$. Explain your reasoning.

53.

x	−2	−1	0	1	2	3
$f(x)$	−1	−2	1	4	7	10

54.



55. **COMPARING METHODS** Find the inverse of $f(x) = -3x + 4$ by switching the roles of x and y and solving for y. Then find the inverse of f by using inverse operations in the reverse order. Which method do you prefer? Explain.

56. **MP REASONING** The graph of a function passes through the points $(-2, 5)$, $(0, 1)$, $(3, -6)$, and $(7, n)$. For what values of n is the inverse a function? Explain your reasoning.

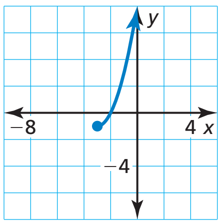
MP STRUCTURE In Exercises 57–60, match the function with the graph of its inverse.

57. $f(x) = \sqrt[3]{x - 4}$

58. $f(x) = \sqrt[3]{x + 4}$

59. $f(x) = \sqrt{x + 1} - 3$

60. $f(x) = \sqrt{x - 1} + 3$

A.


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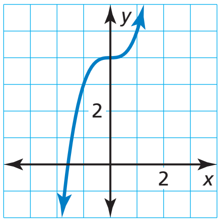

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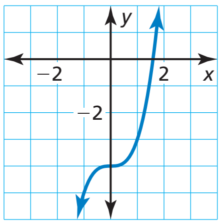

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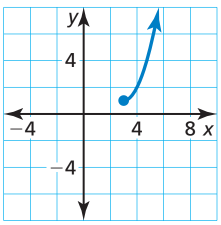
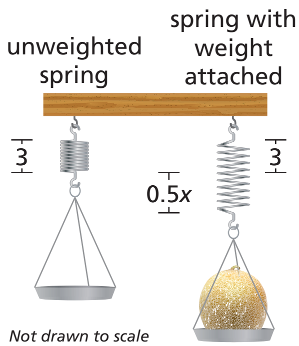


61. **MP PROBLEM SOLVING** When calibrating a spring scale, you need to know how far the spring stretches for various weights. Hooke's Law states that the distance a spring stretches is proportional to the weight attached to it. The length (in inches) of the spring on a certain scale is represented by $h(x) = 0.5x + 3$, where x is the weight (in pounds) of the object.



Not drawn to scale

a. Find the inverse function. Describe what it represents.

b. You place a melon on the scale, and the spring stretches to a total length of 5.5 inches. Determine the weight of the melon.

c. Verify that h and the function you found in part (a) are inverse functions.

62. **MP PROBLEM SOLVING** The surface area (in square meters) of a person with a mass of 60 kilograms can be approximated by $s(x) = 0.2195x^{0.3964}$, where x is the height (in centimeters) of the person.

GO DIGITAL

a. Find the inverse function. Then estimate the height of a 60-kilogram person who has a body surface area of 1.6 square meters.

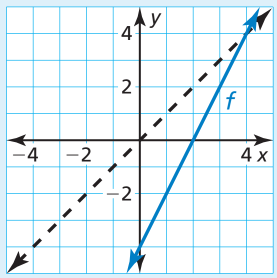
b. Verify that s and the function you found in part (a) are inverse functions.

63. **MODELING REAL LIFE** At the start of a dog sled race in Anchorage, Alaska, the temperature was 5°C. By the end of the race, the temperature was −10°C. The temperature in degrees Celsius is represented by $C(x) = \frac{5}{9}(x - 32)$, where x is the temperature in degrees Fahrenheit.

a. Find the inverse function. Describe what it represents.

b. Find the Fahrenheit temperatures at the start and end of the race.

64. **HOW DO YOU SEE IT?** The graph of the function f is shown. Name three points that lie on the graph of the inverse of f. Explain your reasoning.



65. **MAKING AN ARGUMENT** Does every quadratic function whose domain is restricted to nonnegative values have an inverse function? Explain your reasoning.

66. **THOUGHT PROVOKING** Do functions of the form $y = \sqrt[n]{x^m}$, where m and n are positive integers, have inverse functions? Justify your answer with examples.

67. **ABSTRACT REASONING** Compare the slope and the y-intercept of a linear function with the slope and the y-intercept of its inverse. Is the inverse of any linear function also a linear function? Explain.

68. PERFORMANCE TASK When communicating by a secret code, the sender and the receiver of a message each use the same *key*. The sender uses the key to encode the message, and the receiver uses the key to decipher the message. This process is called *cryptography*. Work with a partner to write a function that can be used as the key for a secret code. Each of you encode a message and then decipher your partner's message. Explain how inverse functions are used in this process.

69. DRAWING CONCLUSIONS Determine whether the statement is *true* or *false*. Explain your reasoning.

a. If $f(x) = x^n$ and n is a positive even integer, then the inverse of f is a function.

b. If $f(x) = x^n$ and n is a positive odd integer, then the inverse of f is a function.

c. If $f(x) = x^n$, where $x \le 0$ and n is a positive even integer, then the inverse of f is a function.

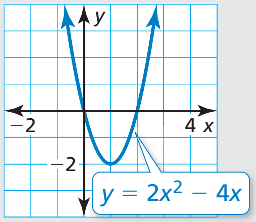
GO DIGITAL

REVIEW & REFRESH

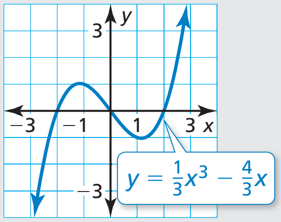
WATCH

In Exercises 70 and 71, describe the x-values for which the function is increasing, decreasing, positive, and negative.

70.



71.



In Exercises 72–75, find the inverse of the function. Then graph the function and its inverse.

72. $f(x) = -4x + 7$

73. $f(x) = -3x^2 - 9,\ x \ge 0$

74. $f(x) = 2x^3 - 10$

75. $f(x) = 5\sqrt{x + 3}$

In Exercises 76–79, solve the equation. Check your solution(s).

76. $3\sqrt{4x - 3} = 15$

77. $x + 3 = \sqrt{4x + 17}$

78. $\sqrt{x - 8} = \sqrt{x + 3} - 1$

79. $(3x)^{2/3} - 6 = 3$

80. Write an equation that represents the data in the table.

x	1	2	3	4	5	6
y	12	10	0	−18	−44	−78

81. Write a quadratic equation that has the given solutions.

$$x = \frac{-5 \pm \sqrt{89}}{4}$$

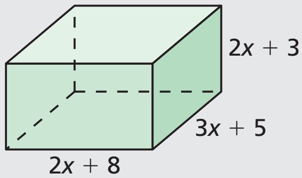
82. Find the values of x and y that satisfy the equation $7yi + 3 = 18x + 14i$.

In Exercises 83 and 84, find $(f + g)(x)$ and $(f - g)(x)$ and state the domain of each. Then evaluate $f + g$ and $f - g$ for the given value of x.

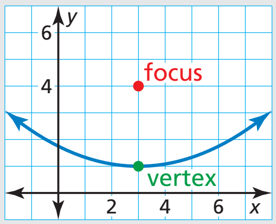
83. $f(x) = 3\sqrt[3]{x},\ g(x) = -12\sqrt[3]{x};\ x = 64$

84. $f(x) = 2x^2 - 3 + 7x,\ g(x) = 11 + 4x^2;\ x = 3$

85. Write an expression for the volume of the figure as a polynomial in standard form.



86. Write an equation of the parabola.



In Exercises 87–90, let $f(x) = 6x - 2$, $g(x) = 2x^{-1}$, and $h(x) = 1.5x + 3$. Perform the indicated operation and state the domain.

87. $f(h(x))$

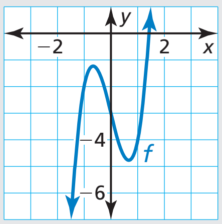
88. $h(f(x))$

89. $g(f(x))$

90. $f(g(x))$

In Exercises 91 and 92, determine the least possible degree of f.

91.



92.