5.4 Solving Radical Equations and Inequalities

Learning Target Solve equations and inequalities containing radicals and rational exponents.

Success Criteria
- I can identify radical equations and inequalities.
- I can solve radical equations and inequalities.
- I can identify extraneous solutions of radical equations.
- I can solve real-life problems involving radical equations.

EXPLORE IT! Solving Radical Equations

Work with a partner.

a. Two students solve the equation $x + 2 = \sqrt{5x + 16}$ as shown. Justify each solution step in the first student's solution. Then describe each student's method. Are the methods valid? Explain.

Math Practice

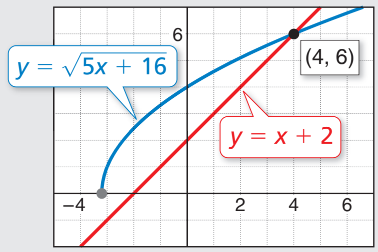
Understand Mathematical Terms

The solution $x = -3$ is called an *extraneous solution*. Why is it called *extraneous*?

Student 1

$x + 2 = \sqrt{5x + 16}$	Write the equation.
$(x + 2)^2 = \left(\sqrt{5x + 16}\right)^2$	______________
$x^2 + 4x + 4 = 5x + 16$	______________
$x^2 - x - 12 = 0$	______________
$(x - 4)(x + 3) = 0$	______________
$x - 4 = 0$ or $x + 3 = 0$	______________
$x = 4$ or $x = -3$	______________

Student 2



The graphs intersect at the point (4, 6). So, the only solution is $x = 4$.

b. Which student is correct? Explain why the other student's solution is incorrect and how the student arrived at an incorrect answer.

c. Explain how you might solve the equation $(9n)^{3/2} - 7 = 20$.

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Vocabulary

radical equation, *p. 254*
extraneous solution, *p. 255*

Solving Equations

A **radical equation** contains radicals that have variables in the radicands. An example of a radical equation is $2\sqrt{x+1} = 4$.

KEY IDEA

Solving Radical Equations

Step 1 Isolate the radical on one side of the equation, if necessary.

Step 2 Raise each side of the equation to the same exponent to eliminate the radical and obtain a linear, quadratic, or other polynomial equation.

Step 3 Solve the resulting equation using techniques you learned in previous chapters. Check your solution.

EXAMPLE 1 Solving Radical Equations

WATCH

Solve each equation.

a. $2\sqrt{x+1} = 4$ **b.** $\sqrt[3]{2x-9} - 1 = 2$

SOLUTION

a.

$2\sqrt{x+1} = 4$ Write the equation.

$\sqrt{x+1} = 2$ Divide each side by 2.

$\left(\sqrt{x+1}\right)^2 = 2^2$ Square each side to eliminate the radical.

$x + 1 = 4$ Simplify.

$x = 3$ Subtract 1 from each side.

The solution is $x = 3$.

Check

$2\sqrt{3+1} \stackrel{?}{=} 4$

$2\sqrt{4} \stackrel{?}{=} 4$

$4 = 4$ ✓

b.

$\sqrt[3]{2x-9} - 1 = 2$ Write the equation.

$\sqrt[3]{2x-9} = 3$ Add 1 to each side.

$\left(\sqrt[3]{2x-9}\right)^3 = 3^3$ Cube each side to eliminate the radical.

$2x - 9 = 27$ Simplify.

$2x = 36$ Add 9 to each side.

$x = 18$ Divide each side by 2.

The solution is $x = 18$.

Check

$\sqrt[3]{2(18) - 9} - 1 \stackrel{?}{=} 2$

$\sqrt[3]{27} - 1 \stackrel{?}{=} 2$

$2 = 2$ ✓

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the equation. Check your solution.

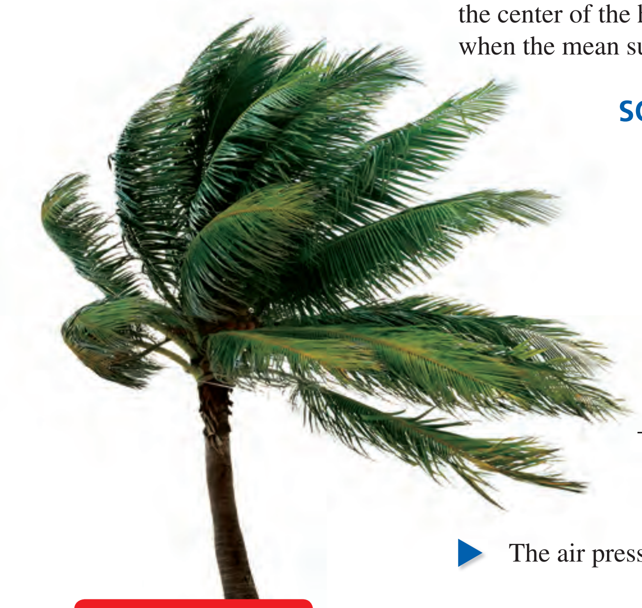
1. $\sqrt[3]{x} - 9 = -6$ **2.** $\sqrt{x+25} = 2$ **3.** $2\sqrt[3]{x-3} = 4$

4. WHICH ONE DOESN'T BELONG? Which equation does *not* belong with the other three? Explain your reasoning.

$\sqrt[3]{x} + 7 = 11$ $3\sqrt{x} + 5 = 21$ $3x - \sqrt{2} = 6$ $\sqrt[3]{x-1} = 5$

EXAMPLE 2 Modeling Real Life

The mean sustained wind velocity (in meters per second) of a hurricane is modeled by $v(p) = 6.3\sqrt{1013 - p}$, where p is the air pressure (in millibars) at the center of the hurricane. Estimate the air pressure at the center of the hurricane when the mean sustained wind velocity is 54.5 meters per second.



SOLUTION

$v(p) = 6.3\sqrt{1013 - p}$	Write the function.
$54.5 = 6.3\sqrt{1013 - p}$	Substitute 54.5 for $v(p)$.
$8.65 \approx \sqrt{1013 - p}$	Divide each side by 6.3.
$8.65^2 \approx \left(\sqrt{1013 - p}\right)^2$	Square each side.
$74.8 \approx 1013 - p$	Simplify.
$-938.2 \approx -p$	Subtract 1013 from each side.
$938.2 \approx p$	Divide each side by -1.

▶ The air pressure at the center of the hurricane is about 938 millibars.

Math Practice

Explain the Meaning
To understand how extraneous solutions can be introduced, consider the equation $\sqrt{x} = -3$. This equation has no real solution, however, you obtain $x = 9$ after squaring each side.

Raising each side of an equation to the same exponent may introduce solutions that are *not* solutions of the original equation. These solutions are called **extraneous solutions**. When you use this procedure, you should always check each apparent solution in the *original* equation.

EXAMPLE 3 Solving an Equation with an Extraneous Solution

Solve $x + 1 = \sqrt{7x + 15}$.

SOLUTION

$x + 1 = \sqrt{7x + 15}$	Write the equation.
$(x + 1)^2 = \left(\sqrt{7x + 15}\right)^2$	Square each side.
$x^2 + 2x + 1 = 7x + 15$	Expand left side and simplify right side.
$x^2 - 5x - 14 = 0$	Write in standard form.
$(x - 7)(x + 2) = 0$	Factor.
$x - 7 = 0$ or $x + 2 = 0$	Zero-Product Property
$x = 7$ or $x = -2$	Solve for x.

Check

$$7 + 1 \stackrel{?}{=} \sqrt{7(7) + 15} \qquad -2 + 1 \stackrel{?}{=} \sqrt{7(-2) + 15}$$

$$8 \stackrel{?}{=} \sqrt{64} \qquad -1 \stackrel{?}{=} \sqrt{1}$$

$$8 = 8 \ ✓ \qquad -1 \neq 1 \ ✗$$

▶ The apparent solution $x = -2$ is extraneous. So, the only solution is $x = 7$.

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EXAMPLE 4 Solving an Equation with Two Radicals

Solve $\sqrt{x+2} + 1 = \sqrt{3-x}$.

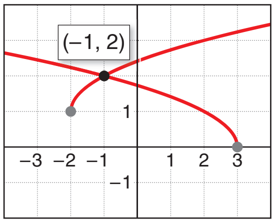
WATCH

SOLUTION

$\sqrt{x+2} + 1 = \sqrt{3-x}$	Write the equation.
$(\sqrt{x+2} + 1)^2 = (\sqrt{3-x})^2$	Square each side.
$x + 2 + 2\sqrt{x+2} + 1 = 3 - x$	Expand left side and simplify right side.
$2\sqrt{x+2} = -2x$	Isolate radical expression.
$\sqrt{x+2} = -x$	Divide each side by 2.
$(\sqrt{x+2})^2 = (-x)^2$	Square each side.
$x + 2 = x^2$	Simplify.
$0 = x^2 - x - 2$	Write in standard form.
$0 = (x-2)(x+1)$	Factor.
$x - 2 = 0$ or $x + 1 = 0$	Zero-Product Property
$x = 2$ or $x = -1$	Solve for x.

ANOTHER WAY

You can also graph each side of the equation and find the x-value where the graphs intersect.



Check $\sqrt{2+2} + 1 \stackrel{?}{=} \sqrt{3-2}$ $\qquad$ $\sqrt{-1+2} + 1 \stackrel{?}{=} \sqrt{3-(-1)}$

$\sqrt{4} + 1 \stackrel{?}{=} \sqrt{1}$ $\qquad$ $\sqrt{1} + 1 \stackrel{?}{=} \sqrt{4}$

$3 \neq 1$ ✗ $\qquad$ $2 = 2$ ✓

▶ The apparent solution $x = 2$ is extraneous. So, the only solution is $x = -1$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

5. **WHAT IF?** In Example 2, estimate the air pressure at the center of the hurricane when the mean sustained wind velocity is 48.3 meters per second.

Solve the equation. Check your solution(s).

6. $\sqrt{10x+9} = x + 3$ $\qquad$ **7.** $\sqrt{2x+5} = \sqrt{x+7}$ $\qquad$ **8.** $\sqrt{x+6} - 2 = \sqrt{x-2}$

When an equation contains a power with a rational exponent, you can solve the equation using a procedure similar to the one for solving radical equations. In this case, first isolate the power and then raise each side of the equation to the reciprocal of the rational exponent.

EXAMPLE 5 Solving an Equation with a Rational Exponent

Solve $(2x)^{3/4} + 2 = 10$.

WATCH

SOLUTION

$(2x)^{3/4} + 2 = 10$	Write the equation.
$(2x)^{3/4} = 8$	Subtract 2 from each side.
$[(2x)^{3/4}]^{4/3} = 8^{4/3}$	Raise each side to the four-thirds.
$2x = 16$	Simplify.
$x = 8$	Divide each side by 2.

Check

$(2 \cdot 8)^{3/4} + 2 \stackrel{?}{=} 10$

$16^{3/4} + 2 \stackrel{?}{=} 10$

$10 = 10$ ✓

▶ The solution is $x = 8$.

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EXAMPLE 6 Solving an Equation with a Rational Exponent

WATCH

Solve $(x + 30)^{1/2} = x$.

SOLUTION

$(x + 30)^{1/2} = x$	Write the equation.
$[(x + 30)^{1/2}]^2 = x^2$	Square each side.
$x + 30 = x^2$	Simplify.
$0 = x^2 - x - 30$	Write in standard form.
$0 = (x - 6)(x + 5)$	Factor.
$x - 6 = 0$ or $x + 5 = 0$	Zero-Product Property
$x = 6$ or $x = -5$	Solve for x.

▶ The apparent solution $x = -5$ is extraneous. So, the only solution is $x = 6$.

Check

$$(6 + 30)^{1/2} \stackrel{?}{=} 6$$
$$36^{1/2} \stackrel{?}{=} 6$$
$$6 = 6 \checkmark$$

$$(-5 + 30)^{1/2} \stackrel{?}{=} -5$$
$$25^{1/2} \stackrel{?}{=} -5$$
$$5 \neq -5 \text{ ✗}$$

Solving Radical Inequalities

To solve a simple radical inequality of the form $\sqrt[n]{u} < d$, where u is an algebraic expression and d is a nonnegative number, raise each side to the exponent n. This procedure also works for $>$, $\leq$, and $\geq$. Be sure to consider the possible values of the radicand.

EXAMPLE 7 Solving a Radical Inequality

WATCH

Solve $3\sqrt{x - 1} \leq 12$.

SOLUTION

Step 1 Solve for x.

$3\sqrt{x - 1} \leq 12$	Write the inequality.
$\sqrt{x - 1} \leq 4$	Divide each side by 3.
$x - 1 \leq 16$	Square each side.
$x \leq 17$	Add 1 to each side.

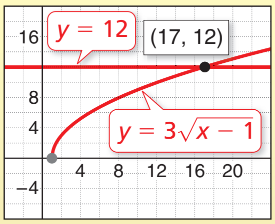
Step 2 Consider the radicand.

$x - 1 \geq 0$	The radicand cannot be negative.
$x \geq 1$	Add 1 to each side.

▶ So, the solution is $1 \leq x \leq 17$.

Check

The graph of $y = 3\sqrt{x - 1}$ is on or below the graph of $y = 12$ when $1 \leq x \leq 17$.



SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the equation. Check your solution(s).

9. $(3x)^{1/3} = -3$ **10.** $(x + 20)^{1/2} = x$ **11.** $(x + 2)^{3/4} = 8$

Solve the inequality.

12. $2\sqrt{x} - 3 \geq 3$ **13.** $4\sqrt[3]{x + 1} < 8$ **14.** $\frac{1}{2}\sqrt{6 - x} \leq 5$

15. MP **REASONING** How does changing 8 to -8 change the solution in Exercise 13? cut.

5.4 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–10, solve the equation. Check your solution. *Example 1*

1. $\sqrt{5x+1} = 6$

2. $\sqrt{3x+10} = 8$

3. $\sqrt[3]{x-16} = 2$

4. $\sqrt[3]{x} - 10 = -7$

5. $-2\sqrt{24x} + 13 = -11$

6. $\sqrt{2x} - \frac{2}{3} = 0$

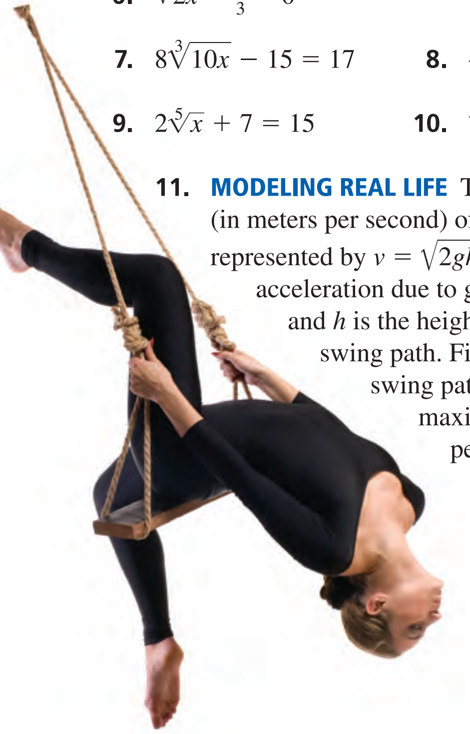
7. $8\sqrt[3]{10x} - 15 = 17$

8. $\frac{1}{5}\sqrt[3]{3x} + 10 = 8$

9. $2\sqrt[5]{x} + 7 = 15$

10. $\sqrt[4]{4x} - 13 = -15$

11. MODELING REAL LIFE The maximum speed v (in meters per second) of a trapeze artist is represented by $v = \sqrt{2gh}$, where g is the acceleration due to gravity ($g \approx 9.8$ m/sec^2) and h is the height (in meters) of the swing path. Find the height of the swing path for a performer whose maximum speed is 7 meters per second. *Example 2*



12. MODELING REAL LIFE The shoulder height h (in centimeters) of a male Asian elephant can be modeled by $h = 62.5\sqrt[3]{t} + 75.8$, where t is the age (in years) of the elephant. Determine the age of an elephant with a shoulder height of 250 centimeters.

In Exercises 13–22, solve the equation. Check your solution(s). *Examples 3 and 4*

13. $x - 6 = \sqrt{3x}$

14. $x - 10 = \sqrt{9x}$

15. $\sqrt{44-2x} = x - 10$

16. $\sqrt{2x+30} = x + 3$

17. $\sqrt[3]{2x^3 - 1} = x$

18. $\sqrt[3]{3 - 8x^2} = 2x$

19. $\sqrt{4x+1} = \sqrt{x+10}$

20. $\sqrt{3x-3} = \sqrt{x+12}$

21. $\sqrt[3]{2x-5} - \sqrt[3]{8x+1} = 0$

22. $\sqrt[3]{x+5} - 2\sqrt[3]{2x+6} = 0$

In Exercises 23–30, solve the equation. Check your solution(s). *Examples 5 and 6*

23. $2x^{2/3} = 8$

24. $4x^{3/2} = 32$

25. $x^{1/4} + 3 = 0$

26. $2x^{3/4} - 14 = 40$

27. $(x+6)^{1/2} = x$

28. $(5-x)^{1/2} - 2x = 0$

29. $2(x+11)^{1/2} = x + 3$

30. $(5x^2 - 4)^{1/4} = x$

ERROR ANALYSIS In Exercises 31 and 32, describe and correct the error in solving the equation.

31.

✗

$$\sqrt[3]{3x-8} = 4$$
$$\left(\sqrt[3]{3x-8}\right)^3 = 4$$
$$3x - 8 = 4$$
$$3x = 12$$
$$x = 4$$

32.

✗

$$8x^{3/2} = 1000$$
$$8(x^{3/2})^{2/3} = 1000^{2/3}$$
$$8x = 100$$
$$x = \frac{25}{2}$$

In Exercises 33–40, solve the inequality. *Example 7*

33. $4\sqrt{x} - 2 > 18$

34. $7\sqrt{x} + 1 < 9$

35. $\sqrt[3]{x-5} \geq 3$

36. $\sqrt[3]{x-4} \leq 5$

37. $4\sqrt[3]{x+7} \geq 8$

38. $-2\sqrt[3]{x+4} < 12$

39. $2\sqrt{x} + 3 \leq 8$

40. $-0.25\sqrt{x} - 6 \leq -3$

41. MODELING REAL LIFE The least possible frequency of a string is its *fundamental frequency*. The fundamental frequency n (in hertz) of a certain string on a violin is represented by $n = \sqrt{\frac{T}{0.0054}}$, where T is the tension (in newtons). The fundamental frequency of the string is 196 hertz. What is the tension of the string?

42. MP REASONING A company finds that the function $p = 70 - \sqrt{0.02x + 1}$ relates the price p of an item and the number x of units demanded per day. Explain how changing the price affects the number of units demanded.

MP USING TOOLS **In Exercises 43–46, solve the nonlinear system. Justify your answer with a graph.**

43. $y = \sqrt{x - 3}$
$y = x - 3$

44. $y = \sqrt{4x + 17}$
$y = x + 5$

45. $y = \pm\sqrt{-x^2 + 1}$
$y = \frac{1}{2}x^2 - 1$

46. $x^2 + y^2 = 4$
$y = \pm\sqrt{x + 2}$

47. **MP PROBLEM SOLVING** The speed s (in miles per hour) of a car is given by $s = \sqrt{30fd}$, where f is the coefficient of friction and d is the stopping distance (in feet). The table shows the coefficient of friction for different surfaces.

Surface	Coefficient of friction, f
dry asphalt	0.75
wet asphalt	0.60
snow	0.30
ice	0.15

a. Compare the stopping distances of a car traveling 45 miles per hour on the surfaces given in the table.

b. You are driving 35 miles per hour on an icy road when a deer jumps in front of your car. How far away must you begin to brake to avoid hitting the deer? Justify your answer.

48. **MODELING REAL LIFE** The Beaufort wind scale was devised to measure wind speed. The Beaufort numbers B, which range from 0 to 12, can be modeled by $B = 1.69\sqrt{s + 4.25} - 3.55$, where s is the wind speed (in miles per hour).

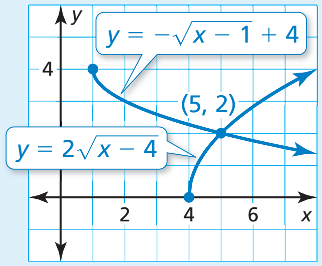
Beaufort number	Force of wind
0	calm
3	gentle breeze
6	strong breeze
9	strong gale
12	hurricane

a. What is the wind speed for $B = 0$? $B = 3$?

b. Write an inequality that describes the range of wind speeds represented by the Beaufort model.

49. **MP STRUCTURE** Without performing any calculations, explain how you know that the radical equation $\sqrt{x + 4} = -5$ has no real solution.

50. **HOW DO YOU SEE IT?**
Use the graph to find the solution of the equation $2\sqrt{x - 4} = -\sqrt{x - 1} + 4$. Justify your answer.



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51. **MODELING REAL LIFE** The Moeraki Boulders are stone spheres along the coast of New Zealand. A formula for the radius of a sphere is $r = \frac{1}{2}\sqrt{\frac{S}{\pi}}$ where S is the surface area of the sphere. Find the surface area of a Moeraki Boulder with a radius of 3 feet.

52. **DRAWING CONCLUSIONS** "Hang time" is the time you are suspended in the air during a jump. Your hang time t (in seconds) is given by the function $t = 0.5\sqrt{h}$, where h is the height (in feet) of the jump. Suppose a wallaby and a skier jump with the hang times shown.

a. Find the heights that the wallaby and the skier jump.

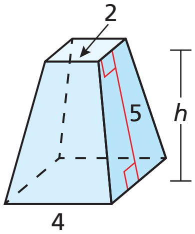
b. If the hang time doubles, does the height of the jump double? Justify your answer.

53. **MAKING AN ARGUMENT** Is it possible for a radical equation to have two extraneous solutions? Justify your answer.

54. **THOUGHT PROVOKING**
City officials rope off a circular area to prepare for a concert in a park. They estimate that each person occupies 6 square feet. Describe how you can use a radical inequality to determine the possible radius of the region when P people are expected to attend the concert.

55. **MP PROBLEM SOLVING** The height h and slant height ℓ of a truncated square pyramid are related by the formula shown.

$$\ell = \sqrt{h^2 + \frac{1}{4}(b_2 - b_1)^2}$$



In the given formula, b_1 and b_2 are the side lengths of the upper and lower bases of the pyramid, respectively. What is the height of the truncated square pyramid shown?

56. **DIG DEEPER** A burning candle has a radius of r inches and was initially h_0 inches tall. After t minutes, the height of the candle has been reduced to h inches. These quantities are related by the formula

$$r = \sqrt{\frac{kt}{\pi(h_0 - h)}}$$

where k is a constant. Suppose the radius of a candle is 0.875 inch, its initial height is 6.5 inches, and $k = 0.04$.

a. Rewrite the formula, solving for h in terms of t.

b. Use your formula in part (a) to determine the height of the candle after it burns for 45 minutes.

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REVIEW & REFRESH

WATCH

In Exercises 57–60, perform the operation.

57. $(x^3 - 2x^2 + 3x + 1) + (x^4 - 7x)$

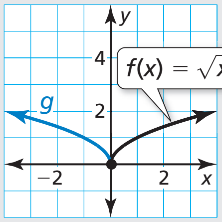
58. $(2x^5 + x^4 - 4x^2) - (x^5 - 3)$

59. $(x^3 + 2x^2 + 1)(x^2 + 5)$

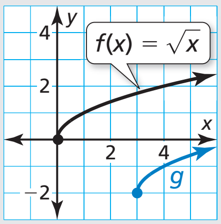
60. $(x^4 + 2x^3 + 11x^2 + 14x - 16) \div (x + 2)$

In Exercises 61 and 62, write a rule for g.

61.



62.



In Exercises 63–66, simplify the expression.

63. $\sqrt[3]{64p^9}$

64. $\sqrt[4]{81m^4n^8}$

65. $\sqrt[4]{\frac{y^{16}}{z^4}}$

66. $\sqrt[7]{\frac{g^5k^{17}}{g^{-2}k^3}}$

67. **MP REASONING** The graph of f is a parabola with axis of symmetry $x = -3$ that passes through the point $(-7, 12)$. Solve $f(x) = 12$. Explain your reasoning.

In Exercises 68–71, let $f(x) = x^3 - 4x^2 + 6$. Write a rule for g. Describe the graph of g as a transformation of the graph of f.

68. $g(x) = f(-x) + 4$

69. $g(x) = \frac{1}{2}f(x) - 3$

70. $g(x) = -f(x - 1)$

71. $g(x) = f\left(\frac{1}{4}x\right) + 5$

In Exercises 72–75, solve the inequality.

72. $6\sqrt{x - 1} \leq 18$

73. $4\sqrt[3]{x} + 7 > 23$

74. $-\sqrt[3]{x} + 6 < 11$

75. $-4\sqrt{x - 1} \geq -3$

76. **MODELING REAL LIFE** Some countries use the Fujita scale to describe the potential damage inflicted by tornados. The number for the rating on the scale can be found using the equation $y = \left(\frac{w}{14.1}\right)^{2/3} - 2$, where w is the wind speed (in miles per hour). What is the rating for a tornado with wind speeds of 200 miles per hour?

y	Rating
$y < 1$	F0
$1 \leq y < 2$	F1
$2 \leq y < 3$	F2
$3 \leq y < 4$	F3
$4 \leq y < 5$	F4
$5 \leq y$	F5

In Exercises 77–80, solve the equation by completing the square.

77. $x^2 - 8x = 6$

78. $2x^2 + 4x = 16$

79. $x^2 + 4x + 11 = 0$

80. $-x^2 + 3x + 1 = 4$

81. Solve the system using any method. Explain your choice of method.

$$3x - y + 4z = 14$$
$$-x - 2y + 3z = 25$$
$$-5x + 3y - 2z = 24$$