5.3 Graphing Radical Functions

GO DIGITAL

Learning Target Describe and graph transformations of radical functions.

Success Criteria
- I can graph radical functions.
- I can describe transformations of radical functions.
- I can write functions that represent transformations of radical functions.

EXPLORE IT! Graphing Radical Functions

Work with a partner.

a. In your own words, define a *radical* function. Give several examples.

b. MP **CHOOSE TOOLS** Graph each function. How are the graphs alike? How are they different?

i. $f(x) = \sqrt{x}$ **ii.** $f(x) = \sqrt[3]{x}$ **iii.** $f(x) = \sqrt[4]{x}$ **iv.** $f(x) = \sqrt[5]{x}$

Math Practice

Use Technology to Explore

How are the domain and range of a radical function related to the index of the radical?

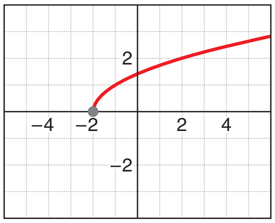
c. Match each function with its graph. Explain your reasoning. Then describe g as a transformation of its parent function f.

i. $g(x) = \sqrt{x+2}$ **ii.** $g(x) = \sqrt{x} - 2$

iii. $g(x) = \sqrt[3]{x} + 2$ **iv.** $g(x) = \sqrt[4]{x} - 2$

v. $g(x) = \sqrt[3]{x+2} - 2$ **vi.** $g(x) = -\sqrt[5]{x+2}$

A.



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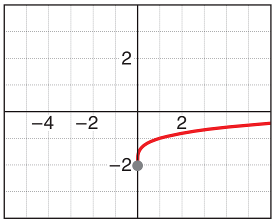

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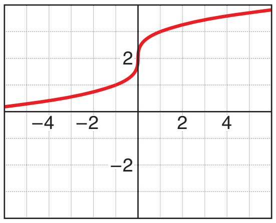

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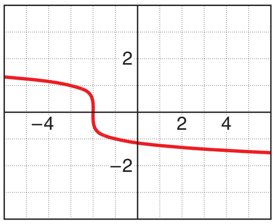

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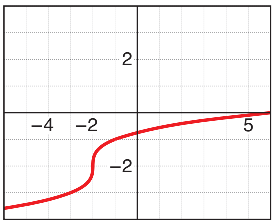

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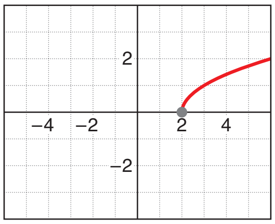


d. Describe the transformation of $f(x) = \sqrt{x}$ represented by $g(x) = -\sqrt{x} + 1$. Then graph each function.

Graphing Radical Functions

Vocabulary

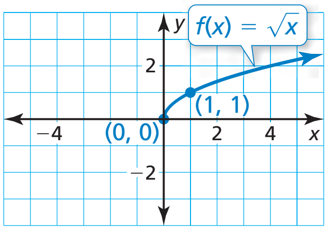
radical function, *p. 246*

A **radical function** contains a radical expression with the independent variable in the radicand. When the radical is a square root, the function is called a *square root function*. When the radical is a cube root, the function is called a *cube root function*.

KEY IDEA

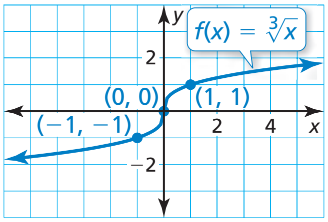
Parent Functions for Square Root and Cube Root Functions

The parent function for the family of square root functions is $f(x) = \sqrt{x}$.



Domain: $x \geq 0$, Range: $y \geq 0$

The parent function for the family of cube root functions is $f(x) = \sqrt[3]{x}$.



Domain and range: All real numbers

STUDY TIP

A *power function* has the form $y = ax^b$, where a is a real number and b is a rational number. Notice that the parent square root function is a power function, where $a = 1$ and $b = \frac{1}{2}$.

EXAMPLE 1 Graphing Radical Functions

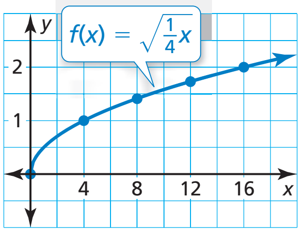
Graph each function. Find the domain and range of each function.

a. $f(x) = \sqrt{\frac{1}{4}x}$ **b.** $g(x) = -3\sqrt[3]{x}$

SOLUTION

a. Make a table of values and sketch the graph.

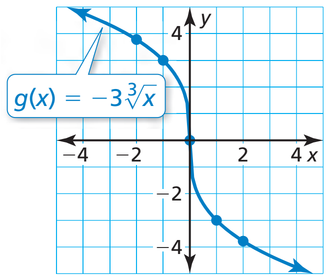
x	0	4	8	12	16
y	0	1	1.41	1.73	2



▶ The radicand of a square root must be nonnegative. So, the domain is $x \geq 0$. The range is $y \geq 0$.

b. Make a table of values and sketch the graph.

x	−2	−1	0	1	2
y	3.78	3	0	−3	−3.78



▶ The radicand of a cube root can be any real number. So, the domain and range are all real numbers.

Math Practice

Look for Structure

How can you choose convenient *x*-values when making a table for a radical function?

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Graph the function. Find the domain and range of the function.

1. $g(x) = \sqrt{x} + 1$ **2.** $f(x) = \sqrt[3]{2x}$ **3.** $h(x) = -\frac{1}{2}\sqrt{x} - 3$

Graphing Transformations of Radical Functions

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In Example 1, notice that the graph of f is a horizontal stretch of the graph of the parent square root function. The graph of g is a vertical stretch and a reflection in the x-axis of the graph of the parent cube root function. You can transform graphs of radical functions in the same way you transformed graphs of functions previously.

KEY IDEAS

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = \sqrt{x - 2}$ $g(x) = \sqrt{x + 3}$	2 units right 3 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = \sqrt{x} + 7$ $g(x) = \sqrt{x} - 1$	7 units up 1 unit down
Reflection Graph flips over a line.	$f(-x)$ $-f(x)$	$g(x) = \sqrt{-x}$ $g(x) = -\sqrt{x}$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis by a factor of $\frac{1}{a}$.	$f(ax)$	$g(x) = \sqrt{3x}$ $g(x) = \sqrt{\frac{1}{2}x}$	shrink by a factor of $\frac{1}{3}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis by a factor of a.	$a \cdot f(x)$	$g(x) = 4\sqrt{x}$ $g(x) = \frac{1}{5}\sqrt{x}$	stretch by a factor of 4 shrink by a factor of $\frac{1}{5}$

EXAMPLE 2 Transforming Radical Functions

WATCH

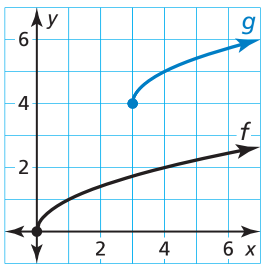
Describe the transformation of f represented by g. Then graph each function.

a. $f(x) = \sqrt{x}$, $g(x) = \sqrt{x - 3} + 4$ **b.** $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{-8x}$

SOLUTION

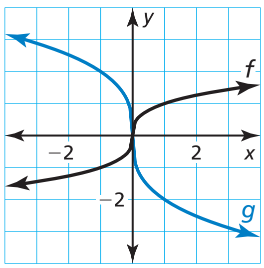
a. Notice that the function is of the form $g(x) = \sqrt{x - h} + k$, where $h = 3$ and $k = 4$.

▶ So, the graph of g is a translation 3 units right and 4 units up of the graph of f.



b. Notice that the function is of the form $g(x) = \sqrt[3]{ax}$, where $a = -8$.

▶ So, the graph of g is a horizontal shrink by a factor of $\frac{1}{8}$ and a reflection in the y-axis of the graph of f.



Math Practice

Look for Structure

In Example 2(b), how can you rewrite g to describe the transformation a different way?

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

4. Describe the transformation of $f(x) = \sqrt[3]{x}$ represented by $g(x) = -\sqrt[3]{x} - 2$. Then graph each function.

Writing Transformations of Radical Functions

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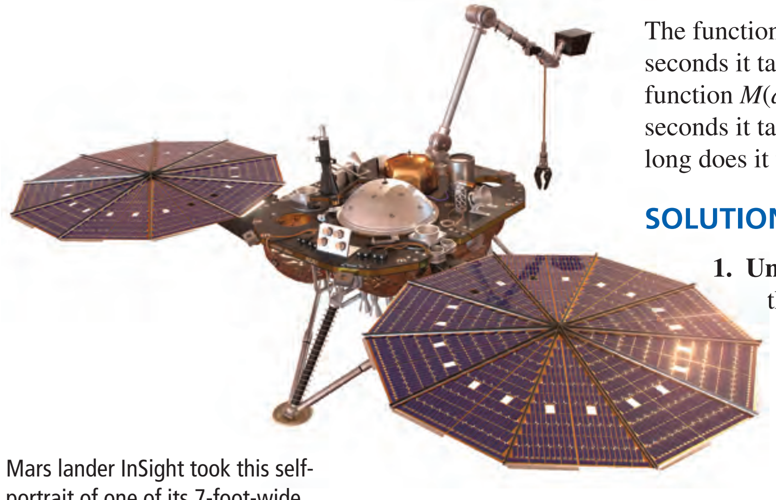
EXAMPLE 3 Modeling Real Life

WATCH



Mars lander InSight took this self-portrait of one of its 7-foot-wide solar panels in December 2018.

The function $E(d) = 0.25\sqrt{d}$ approximates the number of seconds it takes a dropped object to fall d feet on Earth. The function $M(d) = 1.6 \bullet E(d)$ approximates the number of seconds it takes a dropped object to fall d feet on Mars. How long does it take a dropped object to fall 64 feet on Mars?

SOLUTION

1. **Understand the Problem** You are given functions that represent the number of seconds it takes a dropped object to fall d feet on Earth and on Mars. You are asked how long it takes a dropped object to fall a given distance on Mars.
2. **Make a Plan** Multiply $E(d)$ by 1.6 to write a rule for M. Then find $M(64)$.
3. **Solve and Check** $M(d) = 1.6 \bullet E(d)$

$= 1.6 \bullet 0.25\sqrt{d}$ — Substitute $0.25\sqrt{d}$ for $E(d)$.

$= 0.4\sqrt{d}$ — Simplify.

$M(64) = 0.4\sqrt{64} = 0.4(8) = 3.2$

▶ It takes a dropped object about 3.2 seconds to fall 64 feet on Mars.

Check
Use the original functions to check your solution.

$E(64) = 0.25\sqrt{64} = 2$ ✓

$M(64) = 1.6 \bullet E(64)$

$= 1.6 \bullet 2 = 3.2$ ✓

EXAMPLE 4 Writing a Transformed Radical Function

WATCH

Let the graph of g be a horizontal shrink by a factor of $\frac{1}{6}$, followed by a translation 3 units left of the graph of $f(x) = \sqrt[3]{x}$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the horizontal shrink of f.

$h(x) = f(6x)$ — Multiply the input by $1 \div \frac{1}{6} = 6$.

$= \sqrt[3]{6x}$ — Replace x with $6x$ in $f(x)$.

Step 2 Then write a function g that represents the translation of h.

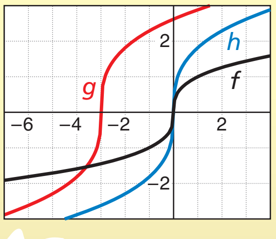
$g(x) = h(x + 3)$ — Subtract -3, or add 3, to the input.

$= \sqrt[3]{6(x + 3)}$ — Replace x with $x + 3$ in $h(x)$.

$= \sqrt[3]{6x + 18}$ — Distributive Property

▶ The transformed function is $g(x) = \sqrt[3]{6x + 18}$.

Check



SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

5. **WHAT IF?** In Example 3, the function $N(d) = 2.4 \bullet E(d)$ approximates the number of seconds it takes a dropped object to fall d feet on the Moon. How long does it take a dropped object to fall 25 feet on the Moon?

6. **WRITING** In Example 4, is the transformed function the same when you perform the translation followed by the horizontal shrink? Explain your reasoning.

Graphing Parabolas and Circles

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You can use radical functions to graph circles and parabolas that open left or right.

EXAMPLE 5 Graphing a Parabola (Horizontal Axis of Symmetry)

WATCH

Use radical functions to graph $\frac{1}{2}y^2 = x$. Identify the vertex and the direction that the parabola opens.

SOLUTION

Step 1 Solve for y.

$\frac{1}{2}y^2 = x$ Write the original equation.

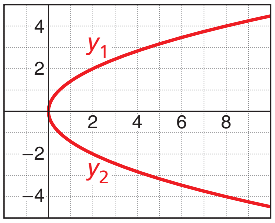
$y^2 = 2x$ Multiply each side by 2.

$y = \pm\sqrt{2x}$ Take square root of each side.

Step 2 Graph both radical functions.

$y_1 = \sqrt{2x}$

$y_2 = -\sqrt{2x}$



▶ The vertex is (0, 0) and the parabola opens right.

STUDY TIP

Notice that y_1 is a function and y_2 is a function, but $\frac{1}{2}y^2 = x$ is not a function.

EXAMPLE 6 Graphing a Circle (Center at the Origin)

WATCH

Use radical functions to graph $x^2 + y^2 = 16$. Identify the radius and the intercepts.

SOLUTION

Step 1 Solve for y.

$x^2 + y^2 = 16$ Write the original equation.

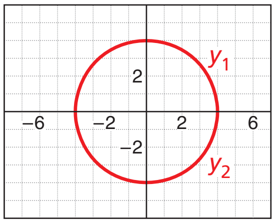
$y^2 = 16 - x^2$ Subtract x^2 from each side.

$y = \pm\sqrt{16 - x^2}$ Take square root of each side.

Step 2 Graph both radical functions.

$y_1 = \sqrt{16 - x^2}$

$y_2 = -\sqrt{16 - x^2}$



▶ The radius is 4 units. The x-intercepts are ± 4. The y-intercepts are also ± 4.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Use radical functions to graph the equation of the parabola. Identify the vertex and the direction that the parabola opens.

7. $x = 2y^2$ **8.** $-4y^2 = x + 1$ **9.** $\frac{1}{3}x = 1 - y^2$

Use radical functions to graph the equation of the circle. Identify the radius and the intercepts.

10. $x^2 + y^2 = 25$ **11.** $y^2 = 49 - x^2$ **12.** $4x^2 + 4y^2 = 1$

5.3 Practice WITH CalcChat® AND CalcView®

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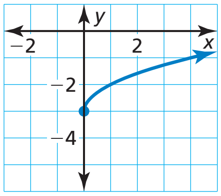
In Exercises 1–10, graph the function. Find the domain and range of the function. *Example 1*

1. $h(x) = \sqrt{x} + 4$
2. $g(x) = \sqrt{x} - 5$
3. $g(x) = -\sqrt[3]{2x}$
4. $f(x) = \sqrt[3]{-5x}$
5. $g(x) = \frac{1}{5}\sqrt{x-3}$
6. $f(x) = \frac{1}{2}\sqrt[3]{x+6}$
7. $f(x) = (6x)^{1/2} + 3$
8. $g(x) = -3(x+1)^{1/3}$
9. $h(x) = -\sqrt[4]{x}$
10. $h(x) = \sqrt[5]{2x}$

In Exercises 11–16, match the function with its graph.

11. $f(x) = \sqrt{x+3}$
12. $h(x) = \sqrt{x} + 3$
13. $f(x) = \sqrt{x-3}$
14. $g(x) = \sqrt{x} - 3$
15. $h(x) = \sqrt{x+3} - 3$
16. $f(x) = \sqrt{x-3} + 3$

A.



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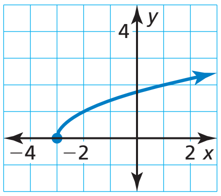

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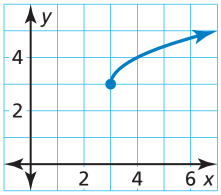

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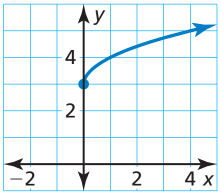

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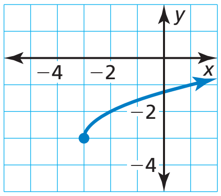

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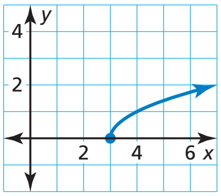
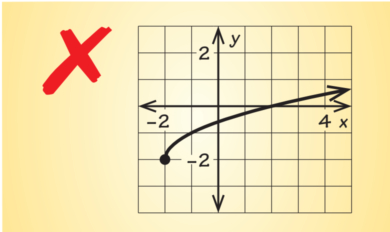


In Exercises 17–24, describe the transformation of *f* represented by *g*. Then graph each function. *Example 2*

17. $f(x) = \sqrt{x}$, $g(x) = \sqrt{x+1} + 8$
18. $f(x) = \sqrt{x}$, $g(x) = 2\sqrt{x-1}$
19. $f(x) = \sqrt[3]{x}$, $g(x) = -\sqrt[3]{x} - 1$
20. $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{x+4} - 5$
21. $f(x) = x^{1/2}$, $g(x) = \frac{1}{4}(-x)^{1/2}$
22. $f(x) = x^{1/3}$, $g(x) = \frac{1}{3}x^{1/3} + 6$
23. $f(x) = \sqrt[4]{x}$, $g(x) = 2\sqrt[4]{x+5} - 4$
24. $f(x) = \sqrt[5]{x}$, $g(x) = \sqrt[5]{-32x} + 3$

25. **ERROR ANALYSIS** Describe and correct the error in graphing $f(x) = \sqrt{x-2} - 2$.



26. **ERROR ANALYSIS** Describe and correct the error in describing the transformation of $f(x) = \sqrt{x}$ represented by $g(x) = \sqrt{\frac{1}{2}x} + 3$.

> ✗ The graph of g is a horizontal shrink by a factor of $\frac{1}{2}$ and a translation 3 units up of the graph of f.

MP USING TOOLS **In Exercises 27–32, use technology to graph the function. Then find the domain and range of the function.**

27. $g(x) = \sqrt{x^2 + x}$
28. $h(x) = \sqrt{x^2 - 2x}$
29. $f(x) = \sqrt[3]{x^2 + 2x}$
30. $f(x) = \sqrt[3]{2x^2 - x}$
31. $f(x) = \sqrt{x^2 - 4x + 6}$
32. $h(x) = \sqrt[3]{\frac{1}{2}x^2 - 3x + 4}$

33. **MODELING REAL LIFE** The functions approximate the velocity (in feet per second) of an object dropped from a height of x feet right before it hits the ground on Earth and on Mars.

Earth: $E(x) = 8\sqrt{x}$

Mars: $M(x) = 0.6 \cdot E(x)$

What is the velocity of an object dropped from a height of 25 feet right before it hits the ground on Mars? *Example 3*

34. **MODELING REAL LIFE** The speed (in knots) of sound waves in air can be modeled by

$$v(K) = 643.855\sqrt{\frac{K}{273.15}}$$

where K is the air temperature (in Kelvin). The speed (in meters per second) of sound waves in air can be modeled by

$$s(K) = \frac{v(K)}{1.944}.$$

What is the speed (in meters per second) of sound waves when the air temperature is 305 Kelvin?

In Exercises 35–38, write a rule for g described by the transformations of the graph of f. *Example 4*

35. Let g be a vertical stretch by a factor of 2, followed by a translation 2 units up of the graph of $f(x) = \sqrt{x} + 3$.

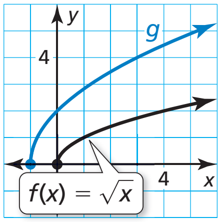
36. Let g be a reflection in the y-axis, followed by a translation 1 unit right of the graph of $f(x) = 2\sqrt[3]{x - 1}$.

37. Let g be a horizontal shrink by a factor of $\frac{2}{3}$, followed by a translation 4 units left of the graph of $f(x) = \sqrt{6x}$.

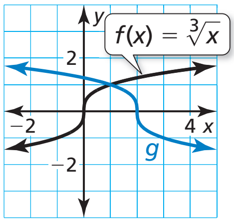
38. Let g be a translation 1 unit down and 5 units right, followed by a reflection in the x-axis of the graph of $f(x) = -\frac{1}{2}\sqrt[4]{x} + \frac{3}{2}$.

In Exercises 39 and 40, write a rule for g.

39.



40.



In Exercises 41–44, write a rule for g that represents the indicated transformation of the graph of f.

41. $f(x) = 2\sqrt{x}$, $g(x) = f(x + 3)$

42. $f(x) = \frac{1}{3}\sqrt{x - 1}$, $g(x) = -f(x) + 9$

43. $f(x) = -\sqrt{x^2 - 2}$, $g(x) = -2f(x + 5)$

44. $f(x) = \sqrt[3]{x^2 + 10x}$, $g(x) = \frac{1}{4}f(-x) + 6$

In Exercises 45–50, use radical functions to graph the equation of the parabola. Identify the vertex and the direction that the parabola opens. *Example 5*

45. $\frac{1}{4}y^2 = x$

46. $3y^2 = x$

47. $-8y^2 + 2 = x$

48. $2y^2 = x - 4$

49. $x + 8 = \frac{1}{5}y^2$

50. $\frac{1}{2}x = y^2 - 4$

In Exercises 51–56, use radical functions to graph the equation of the circle. Identify the radius and the intercepts. *Example 6*

51. $x^2 + y^2 = 9$

52. $x^2 + y^2 = 4$

53. $1 - y^2 = x^2$

54. $64 - x^2 = y^2$

55. $-y^2 = x^2 - 36$

56. $x^2 = 100 - y^2$

ABSTRACT REASONING **In Exercises 57–60, complete the statement with *sometimes*, *always*, or *never*.**

57. The domain of the function $y = a\sqrt{x}$ is ______ $x \geq 0$.

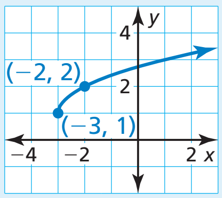
58. The range of the function $y = a\sqrt{x}$ is ______ $y \geq 0$.

59. The domain and range of the function $y = \sqrt[3]{x - h} + k$ are ________ all real numbers.

60. The domain of the function $y = a\sqrt{-x} + k$ is ________ $x \geq 0$.

61. **MODELING REAL LIFE** The *period* of a pendulum is the time the pendulum takes to complete one back-and-forth swing. The period T (in seconds) can be modeled by the function $T = 1.11\sqrt{\ell}$, where ℓ is the length (in feet) of the pendulum. Estimate the length of a pendulum with a period of 2 seconds.

62. **HOW DO YOU SEE IT?** Does the graph represent a square root function or a cube root function? Explain. What are the domain and range of the function?



63. **MP** **PROBLEM SOLVING** For a drag race car with a total weight of 3500 pounds, the speed s (in miles per hour) at the end of a race can be modeled by $s = 14.8\sqrt[3]{p}$, where p is the power (in horsepower).

a. Determine the power of a 3500-pound car that reaches a speed of 200 miles per hour.

b. What is the average rate of change in speed as the power changes from 1000 horsepower to 1500 horsepower?

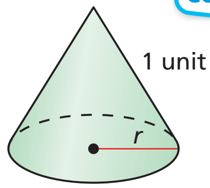
64. MULTIPLE REPRESENTATIONS The terminal velocity v_t (in feet per second) of a skydiver who weighs 140 pounds is given by

$$v_t = 33.7\sqrt{\frac{140}{A}}$$

where A is the cross-sectional surface area (in square feet) of the skydiver. The table shows the terminal velocities (in feet per second) for various surface areas (in square feet) of a skydiver who weighs 165 pounds. Which skydiver has a greater terminal velocity for each value of A given in the table? How is it possible for the value of A to vary for one skydiver?

Cross-sectional surface area, A	Terminal velocity, v_t
1	432.9
3	249.9
5	193.6
7	163.6

65. CONNECTING CONCEPTS The surface area S of a right circular cone with a slant height of 1 unit is given by $S = \pi r + \pi r^2$, where r is the radius of the cone.



a. Use completing the square to show that

$$r = \frac{1}{\sqrt{\pi}}\sqrt{S + \frac{\pi}{4}} - \frac{1}{2}.$$

b. Use technology to graph the equation in part (a). Then find the radius of a right circular cone with a slant height of 1 unit and a surface area of $\frac{3\pi}{4}$ square units.

66. THOUGHT PROVOKING
The graph of a radical function f passes through the points (3, 1) and (4, 0). Write two different functions that can represent $f(x + 2) + 1$.

REVIEW & REFRESH

In Exercises 67 and 68, solve the inequality.

67. $x^2 + 7x + 12 < 0$ **68.** $x^2 - 10x + 25 \geq 4$

In Exercises 69–72, write the expression in simplest form. Assume all variables are positive.

69. $\sqrt[3]{216p^9}$ **70.** $\dfrac{\sqrt[5]{32}}{\sqrt[5]{m^3}}$

71. $\sqrt[4]{n^4q} + 7n\sqrt[4]{q}$ **72.** $\dfrac{21ab^{3/2}}{3a^{1/3}b^{1/2}c^{-1/4}}$

73. Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

x	−3	−2	−1	0	1	2
$f(x)$	−7	−3	−2	−1	3	13

In Exercises 74 and 75, graph the function. Label the vertex and axis of symmetry.

74. $g(x) = -(x + 4)^2 - 1$ **75.** $h(x) = 4x^2 + 8x - 5$

76. Evaluate $10^{2/3}$ using technology. Round your answer to two decimal places.

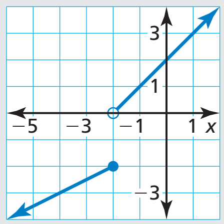
77. Graph $f(x) = -2\sqrt{x + 3}$. Find the domain and range of the function.

In Exercises 78–81, solve the equation.

78. $|3x + 2| = 5$ **79.** $|4x + 9| = -7$

80. $|x - 9| = 2x$ **81.** $|x + 8| = |2x + 2|$

82. Write a piecewise function represented by the graph.



83. MODELING REAL LIFE The prices of smartphone cases at a store have a median of \$29.99 and a range of \$40. The manager considers decreasing all prices by either \$5 or 15%. Which decrease results in a lesser median price? a lesser range of prices?

84. Solve the system.

$$3x + 2y - z = -11$$
$$2x + y + 2z = 3$$
$$4x - 5y + z = -13$$

85. Describe the transformation of $f(x) = \sqrt[3]{x}$ represented by $g(x) = \sqrt[3]{x - 2} - 4$. Then graph each function.