5.1 *n*th Roots and Rational Exponents

Learning Target Evaluate expressions and solve equations containing *n*th roots and rational exponents.

Success Criteria
- I can explain the meaning of a rational exponent.
- I can evaluate expressions with rational exponents.
- I can solve equations using *n*th roots.

Previously, you learned that the *n*th root of a is

$$\sqrt[n]{a} = a^{1/n} \qquad \text{Definition of rational exponent}$$

for any real number a and integer n greater than 1.

EXPLORE IT! Writing Expressions in Different Forms

Work with a partner.

a. Use the definition of a rational exponent and the properties of exponents to write each expression as a base with a single rational exponent or as a radical raised to an exponent.

	Radical raised to an exponent	Base with a single rational exponent
i.	$(\sqrt{5})^3$	
ii.	$(\sqrt[4]{4})^2$	
iii.	$(\sqrt[3]{9})^2$	
iv.	$(\sqrt[5]{10})^4$	
v.	$(\sqrt{15})^3$	
vi.	$(\sqrt[3]{27})^4$	
vii.		$8^{2/3}$
viii.		$6^{5/2}$
ix.		$12^{3/4}$
x.		$10^{3/2}$
xi.		$16^{3/2}$
xii.		$20^{6/5}$

Math Practice

Look for Structure
How do you know when the value of $a^{1/n}$ is an integer?

b. Use technology to evaluate each expression in part (a). Round your answer to two decimal places, if necessary.

c. Simplify $\sqrt[n]{a^n}$. What does this imply about the relationship between raising an expression to the *n*th power and taking the *n*th root? How can you use this result to solve the equation $x^4 = 6$?

Finding nth Roots

GO DIGITAL

Vocabulary

VOCAB

nth root of a, *p. 232*
index of a radical, *p. 232*

You can extend the concept of a square root to other types of roots. For example, 2 is a cube root of 8 because $2^3 = 8$. In general, for an integer n greater than 1, if $b^n = a$, then b is an **nth root of a**. An nth root of a is written as $\sqrt[n]{a}$, where n is the **index of the radical**.

You can also write an nth root of a as a power of a. If you assume the Power of a Power Property applies to rational exponents, then the following is true.

$$(a^{1/2})^2 = a^{(1/2) \cdot 2} = a^1 = a$$

$$(a^{1/3})^3 = a^{(1/3) \cdot 3} = a^1 = a$$

$$(a^{1/4})^4 = a^{(1/4) \cdot 4} = a^1 = a$$

Because $a^{1/2}$ is a number whose square is a, you can write $\sqrt{a} = a^{1/2}$. Similarly, $\sqrt[3]{a} = a^{1/3}$ and $\sqrt[4]{a} = a^{1/4}$. In general, $\sqrt[n]{a} = a^{1/n}$ for any integer n greater than 1.

STUDY TIP

When n is even and $a > 0$, there are two real roots. The positive root is called the *principal root*.

KEY IDEA

Real nth Roots of a

Let n be an integer greater than 1 and let a be a real number.

	n is an even integer.		**n is an odd integer.**
$a < 0$	No real nth roots	$a < 0$	One real nth root: $\sqrt[n]{a} = a^{1/n}$
$a = 0$	One real nth root: $\sqrt[n]{0} = 0$	$a = 0$	One real nth root: $\sqrt[n]{0} = 0$
$a > 0$	Two real nth roots: $\pm\sqrt[n]{a} = \pm a^{1/n}$	$a > 0$	One real nth root: $\sqrt[n]{a} = a^{1/n}$

EXAMPLE 1 Finding nth Roots

WATCH

Find the indicated real nth root(s) of a.

a. $n = 3$, $a = -216$

b. $n = 4$, $a = 81$

SOLUTION

a. Because $n = 3$ is odd and $a = -216 < 0$, -216 has one real cube root. Because $(-6)^3 = -216$, you can write

$$\sqrt[3]{-216} = -6 \quad \text{or} \quad (-216)^{1/3} = -6.$$

b. Because $n = 4$ is even and $a = 81 > 0$, 81 has two real fourth roots. Because $3^4 = 81$ and $(-3)^4 = 81$, you can write

$$\pm\sqrt[4]{81} = \pm 3 \quad \text{or} \quad \pm 81^{1/4} = \pm 3.$$

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Find the indicated real nth root(s) of a.

1. $n = 4$, $a = 16$

2. $n = 2$, $a = -49$

3. $n = 3$, $a = -125$

4. $n = 5$, $a = 243$

5. **COMPLETE THE SENTENCE** For an integer n greater than 1, if $b^n = a$, then b is a(n) ____________ of a.

6. **WRITING** Explain how to use the sign of a to determine the number of real fourth roots of a and the number of real fifth roots of a.

GO DIGITAL

Rational Exponents

A rational exponent does not have to be of the form $1/n$. Other rational numbers, such as $3/2$ and $-1/2$, can also be exponents. Two properties of rational exponents are shown below.

KEY IDEA

Rational Exponents

Let $a^{1/n}$ be an nth root of a, and let m be a positive integer.

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m \quad \text{or} \quad a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$$

$$a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{(a^{1/n})^m} = \frac{1}{(\sqrt[n]{a})^m},\ a \neq 0 \quad \text{or} \quad a^{-m/n} = \frac{1}{(a^m)^{1/n}} = \frac{1}{\sqrt[n]{a^m}},\ a \neq 0$$

EXAMPLE 2 Evaluating Expressions with Rational Exponents

WATCH

Evaluate (a) $16^{3/2}$ and (b) $32^{-3/5}$.

SOLUTION

	Rational Exponent Form	Radical Form
a.	$16^{3/2} = (16^{1/2})^3 = 4^3 = 64$	$16^{3/2} = (\sqrt{16})^3 = 4^3 = 64$
b.	$32^{-3/5} = \frac{1}{32^{3/5}} = \frac{1}{(32^{1/5})^3} = \frac{1}{2^3} = \frac{1}{8}$	$32^{-3/5} = \frac{1}{32^{3/5}} = \frac{1}{(\sqrt[5]{32})^3} = \frac{1}{2^3} = \frac{1}{8}$

EXAMPLE 3 Approximating Expressions with Rational Exponents

WATCH

Evaluate each expression using technology. Round your answer to two decimal places.

a. $9^{1/5}$ **b.** $12^{3/8}$ **c.** $(\sqrt[4]{7})^3$

SOLUTION

a. $9^{1/5} \approx 1.55$

b. $12^{3/8} \approx 2.54$

c. Before evaluating $(\sqrt[4]{7})^3$, rewrite the expression in rational exponent form.

$$(\sqrt[4]{7})^3 = 7^{3/4} \approx 4.30$$

$9^{\frac{1}{5}}$	= 1.55184557392
$12^{\frac{3}{8}}$	= 2.53917695148
$7^{\frac{3}{4}}$	= 4.30351707066

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Evaluate the expression without using technology.

7. $4^{5/2}$ **8.** $9^{-1/2}$ **9.** $81^{3/4}$ **10.** $1^{7/8}$

Evaluate the expression using technology. Round your answer to two decimal places, if necessary.

11. $6^{2/5}$ **12.** $64^{-2/3}$ **13.** $(\sqrt[4]{16})^5$ **14.** $(\sqrt[3]{-30})^2$

15. WHICH ONE DOESN'T BELONG? Which expression does *not* belong with the other three? Explain your reasoning.

$(a^{1/n})^m$ $(\sqrt[n]{a})^m$ $(\sqrt[m]{a})^{-n}$ $a^{m/n}$

Solving Equations Using nth Roots

GO DIGITAL

Raising to the nth power and taking the nth root are inverse operations.

$$\sqrt[n]{a^n} = (a^n)^{1/n} = a^{n \cdot 1/n} = a^1 = a$$

So, you can solve an equation of the form $u^n = d$, where u is an algebraic expression and d is a real number, by taking the nth root of each side.

EXAMPLE 4 Solving Equations Using nth Roots

WATCH

Find the real solution(s) of (a) $4x^5 = 128$ and (b) $(x - 3)^4 = 21$. Round your answer to two decimal places, if necessary.

SOLUTION

a. $4x^5 = 128$ — Write original equation.

$x^5 = 32$ — Divide each side by 4.

$x = \sqrt[5]{32}$ — Take fifth root of each side.

$x = 2$ — Simplify.

▶ The solution is $x = 2$.

b. $(x - 3)^4 = 21$ — Write original equation.

$x - 3 = \pm\sqrt[4]{21}$ — Take fourth root of each side.

$x = 3 \pm \sqrt[4]{21}$ — Add 3 to each side.

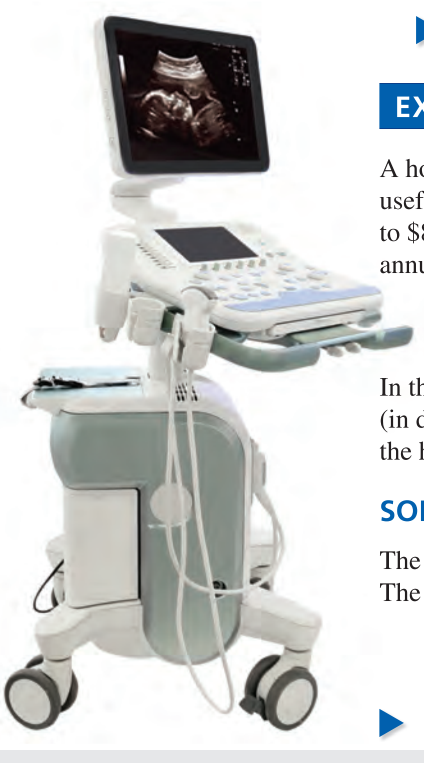
▶ The solutions are $x = 3 + \sqrt[4]{21} \approx 5.14$ and $x = 3 - \sqrt[4]{21} \approx 0.86$.

COMMON ERROR

When n is even and $a > 0$, be sure to consider both the positive and negative nth roots of a.

EXAMPLE 5 Modeling Real Life

WATCH INFO



A hospital purchases an ultrasound machine for \$50,000. The hospital expects the useful life of the machine to be 10 years, at which time its value will have depreciated to \$8000. The hospital uses the declining balances method for depreciation, so the annual depreciation rate r (in decimal form) is given by the formula

$$r = 1 - \left(\frac{S}{C}\right)^{1/n}.$$

In the formula, n is the useful life of the item (in years), S is the salvage value (in dollars), and C is the original cost (in dollars). What annual depreciation rate did the hospital use?

SOLUTION

The useful life is 10 years, so $n = 10$. The machine depreciates to \$8000, so $S = 8000$. The original cost is \$50,000, so $C = 50{,}000$. So, the annual depreciation rate is

$$r = 1 - \left(\frac{S}{C}\right)^{1/n} = 1 - \left(\frac{8000}{50{,}000}\right)^{1/10} = 1 - \left(\frac{4}{25}\right)^{1/10} \approx 0.167.$$

▶ The annual depreciation rate is about 0.167, or 16.7%.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Find the real solution(s) of the equation. Round your answer to two decimal places, if necessary.

16. $8x^3 = 64$

17. $\frac{1}{2}x^5 = 512$

18. $(x + 5)^4 = 16$

19. $(x - 2)^3 = -14$

20. **WHAT IF?** In Example 5, what is the annual depreciation rate when the salvage value is \$6000?

5.1 Practice WITH CalcChat® AND CalcView®

GO DIGITAL

In Exercises 1–6, find the indicated real *n*th root(s) of *a*. *Example 1*

1. $n = 3, a = 8$
2. $n = 5, a = -1$
3. $n = 2, a = 0$
4. $n = 4, a = 256$
5. $n = 5, a = -32$
6. $n = 6, a = -729$

In Exercises 7–14, evaluate the expression without using technology. *Example 2*

7. $64^{1/6}$
8. $8^{1/3}$
9. $25^{3/2}$
10. $32^{4/5}$
11. $(-243)^{1/5}$
12. $(-64)^{4/3}$
13. $8^{-2/3}$
14. $16^{-7/4}$

ERROR ANALYSIS In Exercises 15 and 16, describe and correct the error in evaluating the expression.

15. ✗
$$-27^{5/3} = (-27^{1/3})^5 = 3^5 = 243$$

16. ✗
$$64^{3/2} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16$$

MP STRUCTURE In Exercises 17–20, match the equivalent expressions. Explain your reasoning.

17. $\left(\sqrt[3]{5}\right)^4$
18. $\left(\sqrt[4]{5}\right)^3$
19. $\dfrac{1}{\sqrt[4]{5}}$
20. $-\sqrt[4]{5}$

A. $5^{-1/4}$
B. $5^{4/3}$
C. $-5^{1/4}$
D. $5^{3/4}$

In Exercises 21–28, evaluate the expression using technology. Round your answer to two decimal places, if necessary. *Example 3*

21. $\sqrt[5]{32{,}768}$
22. $\sqrt[7]{1695}$
23. $25^{-1/3}$
24. $85^{1/6}$
25. $20{,}736^{4/5}$
26. $86^{-5/6}$
27. $\left(\sqrt[4]{187}\right)^3$
28. $\left(\sqrt[5]{-8}\right)^8$

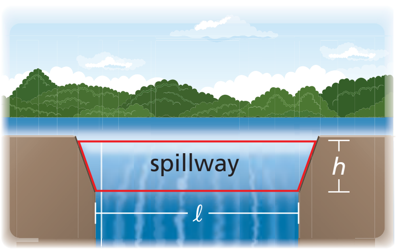
In Exercises 29–38, find the real solution(s) of the equation. Round your answer to two decimal places, if necessary. *Example 4*

29. $x^3 = 125$
30. $5x^3 = 1080$
31. $(x + 10)^5 = 70$
32. $(x - 5)^4 = 256$
33. $x^5 = -48$
34. $7x^4 = 56$
35. $x^6 + 36 = 100$
36. $x^3 + 40 = 25$
37. $\frac{1}{3}x^4 = 27$
38. $\frac{1}{6}x^3 = -36$

39. **MODELING REAL LIFE** When the average price of an item increases from p_1 to p_2 over a period of n years, the annual rate of inflation r (in decimal form) is given by $r = \left(\dfrac{p_2}{p_1}\right)^{1/n} - 1$. Find the rate of inflation for each item in the table. *Example 5*

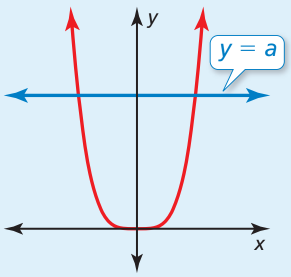
Item	Price per pound in 2009	Price per pound in 2019
Potatoes	\$0.620	\$0.749
Oranges	\$0.910	\$1.280
Ground beef	\$2.251	\$3.775

40. **MODELING REAL LIFE** A *weir* is a dam that is built across a river to regulate the flow of water. The flow rate Q (in cubic feet per second) can be calculated using the formula $Q = 3.367\ell h^{3/2}$, where ℓ is the length (in feet) of the bottom of the spillway and h is the depth (in feet) of the water on the spillway. Determine the flow rate of a weir with a spillway that is 20 feet long and has a water depth of 5 feet.



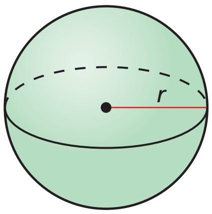
41. **MP NUMBER SENSE** Between which two consecutive integers does $\sqrt[4]{125}$ lie? Explain your reasoning.

42. HOW DO YOU SEE IT?
The graph of $y = x^n$ is shown in red. What can you conclude about the value of n? Determine the number of real nth roots of a. Explain your reasoning.

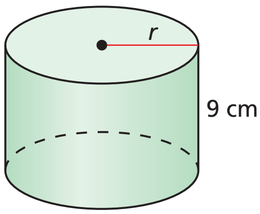


CONNECTING CONCEPTS In Exercises 43 and 44, find the radius of the figure with the given volume.

43. $V = 216\text{ ft}^3$



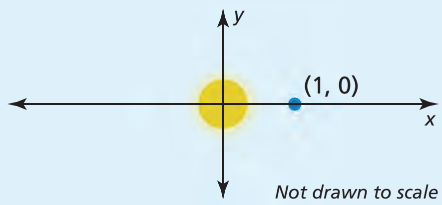
44. $V = 1332\text{ cm}^3$



45. MP **REPEATED REASONING** The mass of the particles that a river can transport is proportional to the sixth power of the speed of the river. A certain river normally flows at a speed of 1 meter per second. What must its speed be in order to transport particles that are twice as massive as usual? 10 times as massive? 100 times as massive?

GO DIGITAL

46. THOUGHT PROVOKING
In 1619, Johannes Kepler published his third law, which can be given by $d^3 = t^2$, where d is the mean distance (in astronomical units) of a planet from the Sun and t is the time (in years) it takes the planet to orbit the Sun. It takes Mars 1.88 years to orbit the Sun. Graph a possible location of Mars. Justify your answer. (The diagram shows the Sun at the origin of the xy-plane and a possible location of Earth.)



REVIEW & REFRESH

WATCH

In Exercises 47 and 48, graph the function.

47. $f(x) = (x + 1)(x - 2)(x - 4)$

48. $g(x) = 2(x + 2)^2(x - 1)(x + 5)$

49. Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function.

x	-2	-1	0	1	2	3
$f(x)$	1	9	1	1	9	1

50. Find all zeros of $f(x) = x^4 - 9x^3 + 19x^2 - 9x + 18$.

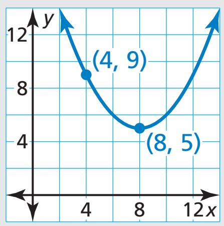
In Exercises 51 and 52, find the real solution(s) of the equation. Round your answer to two decimal places, if necessary.

51. $2x^4 = 1250$

52. $(x - 8)^3 = 144$

53. Let the graph of g be a translation 3 units left, followed by a vertical shrink by a factor of $\frac{1}{2}$ of the graph of $f(x) = x^3 - 4x$. Write a rule for g.

54. Write an equation of the parabola in vertex form.



55. Write $(1 + 7i) + (10 - 2i) - 3i(2 + 9i)$ as a complex number in standard form.

56. MODELING REAL LIFE The table shows the distances run by an athlete on a treadmill over time. What type of function can you use to model the data? Predict the distance traveled by the runner after 30 minutes.

Time (minutes), x	Distance (miles), y
5	0.5
10	1.0
15	1.5
20	2.0
25	2.5