4.8 Analyzing Graphs of Polynomial Functions

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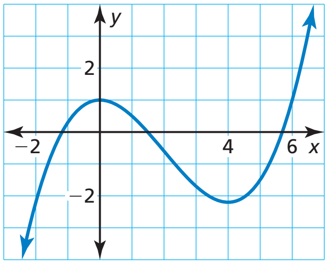
Learning Target Analyze graphs of polynomial functions.

Success Criteria
- I can identify a turning point of a polynomial function.
- I can analyze real zeros and turning points numerically.
- I can explain the relationship among the degree of a polynomial function, real zeros, and turning points.

EXPLORE IT! Approximating Turning Points

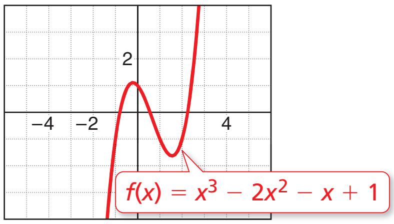


Work with a partner.

a. The graph of the function at the left has two *turning points*. What is meant by a turning point?

b. Use technology to approximate the coordinates of the turning points of the graph of each function. Round your answers to the nearest hundredth.

i.



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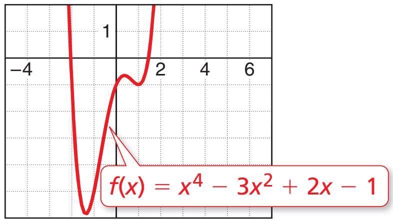

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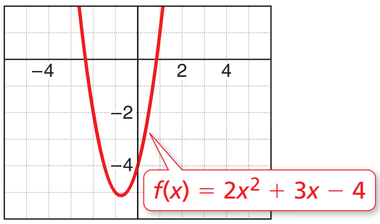

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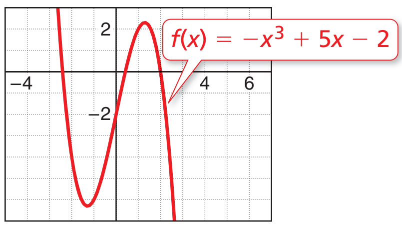

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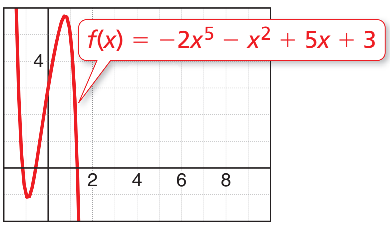

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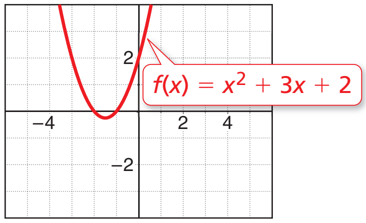

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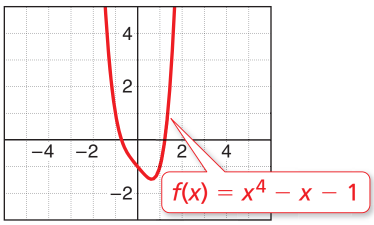

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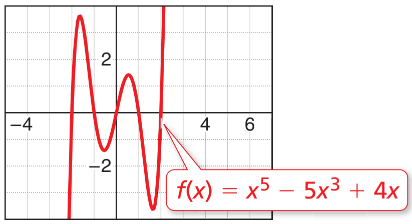

c. Make a conjecture about the number of turning points of the graph of a polynomial function of degree n. Explain your reasoning.

Math Practice

Use a Graph

How can turning points help you identify intervals on which a function is increasing or decreasing?

Graphing Polynomial Functions

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Vocabulary

VOCAB

local maximum, *p. 208*
local minimum, *p. 208*
even function, *p. 209*
odd function, *p. 209*

In this chapter, you have learned that zeros, factors, solutions, and x-intercepts are closely related concepts. Here is a summary of these relationships.

CONCEPT SUMMARY

Zeros, Factors, Solutions, and x-Intercepts

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial function. The following statements are equivalent.

Zero: k is a zero of the polynomial function f.

Factor: $x - k$ is a factor of the polynomial $f(x)$.

Solution: k is a solution (or root) of the polynomial equation $f(x) = 0$.

x-Intercept: If k is a real number, then k is an x-intercept of the graph of the polynomial function f. The graph of f passes through $(k, 0)$.

EXAMPLE 1 Using x-intercepts to Graph a Polynomial Function

Graph $f(x) = \frac{1}{6}(x + 3)(x - 2)^2$.

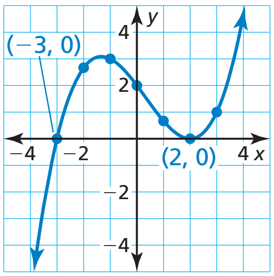
WATCH

SOLUTION

Step 1 Plot points corresponding to the x-intercepts. Because -3 and 2 are zeros of f, plot $(-3, 0)$ and $(2, 0)$.

Step 2 Plot points between and beyond the points in Step 1.

x	-2	-1	0	1	3
y	$\frac{8}{3}$	3	2	$\frac{2}{3}$	1



Step 3 Determine the end behavior. Because $f(x)$ has three factors of the form $x - k$ and a constant factor of $\frac{1}{6}$, f is a cubic function with a positive leading coefficient. So, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

Step 4 Draw the graph so that it passes through the plotted points and has the appropriate end behavior.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Graph the function.

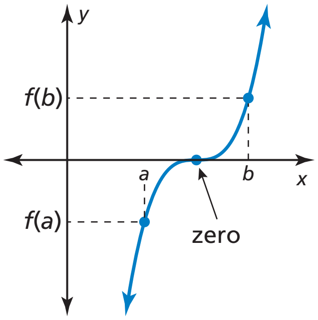
1. $f(x) = \frac{1}{2}(x + 1)(x - 4)^2$

2. $f(x) = \frac{1}{4}(x + 2)(x - 1)(x - 3)$

3. **MP REASONING** The graph of a function g crosses the x-axis at $(-1, 0)$ and touches but does not cross the x-axis at $(-10, 0)$. Can you determine any solutions of $g(x) = 0$? Can there be more solutions? Explain your reasoning.

The Location Principle

You can use the *Location Principle* to help you find real zeros of polynomial functions.



KEY IDEA

The Location Principle

If f is a polynomial function, and a and b are two real numbers such that $f(a) < 0$ and $f(b) > 0$, then f has at least one real zero between a and b.

To use this principle to locate real zeros of a polynomial function, find a value a at which the polynomial function is negative and another value b at which the function is positive. You can conclude that the function has *at least* one real zero between a and b.

EXAMPLE 2 Finding Real Zeros of a Polynomial Function

WATCH

Find all the real zeros of $f(x) = 6x^3 + 5x^2 - 17x - 6$.

SOLUTION

x	$f(x)$
0	−6
1	−12
2	28
3	150
4	390
5	784
6	1368

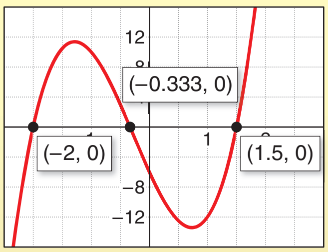
Step 1 Use technology to make a table.

Step 2 Use the Location Principle. From the table shown, you can see that $f(1) < 0$ and $f(2) > 0$. So, by the Location Principle, f has at least one real zero between 1 and 2. Because f is a polynomial function of degree 3, it has three zeros. By the Rational Root Theorem, the only possible *rational* zero between 1 and 2 is $\frac{3}{2}$. Use synthetic division to confirm that $\frac{3}{2}$ is a zero.

$$\begin{array}{r|rrrr} \frac{3}{2} & 6 & 5 & -17 & -6 \\ & & 9 & 21 & 6 \\ \hline & 6 & 14 & 4 & 0 \end{array}$$

The remainder is 0, so $x - \frac{3}{2}$ is a factor of $f(x)$.

Check



Step 3 Write $f(x)$ in factored form using its known factor $x - \frac{3}{2}$ and the quotient polynomial $6x^2 + 14x + 4$.

$$\begin{aligned} f(x) &= \left(x - \tfrac{3}{2}\right)(6x^2 + 14x + 4) \\ &= 2\left(x - \tfrac{3}{2}\right)(3x^2 + 7x + 2) \\ &= 2\left(x - \tfrac{3}{2}\right)(3x + 1)(x + 2) \end{aligned}$$

▶ From the factorization, there are three zeros. The zeros of f are $\frac{3}{2}$, $-\frac{1}{3}$, and -2.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Find all the real zeros of the function.

4. $f(x) = 18x^3 + 21x^2 - 13x - 6$

5. $f(x) = 2x^4 + x^3 - 9x^2 - 13x - 5$

6. MP **PRECISION** In your own words, explain why the Location Principle is true.

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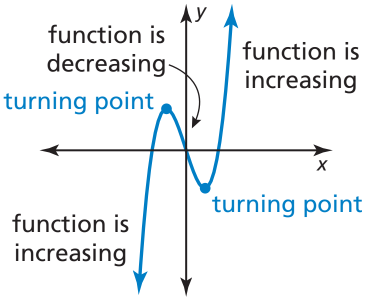
Turning Points

Another important characteristic of graphs of polynomial functions is that they have *turning points* corresponding to local maximum and minimum values.

- The y-coordinate of a turning point is a **local maximum** of the function when the point is higher than all nearby points.
- The y-coordinate of a turning point is a **local minimum** of the function when the point is lower than all nearby points.

A turning point of a graph of a function is a point on the graph at which the function changes from increasing to decreasing, or decreasing to increasing.



READING

Local maximum and local minimum are sometimes referred to as *relative maximum* and *relative minimum*.

KEY IDEA

Turning Points of Polynomial Functions

1. The graph of every polynomial function of degree n has *at most* $n - 1$ turning points.
2. If a polynomial function of degree n has n distinct real zeros, then its graph has *exactly* $n - 1$ turning points.

WORDS AND MATH

In everyday life, *turning point* refers to a point in time or to an event that changes the course of action.

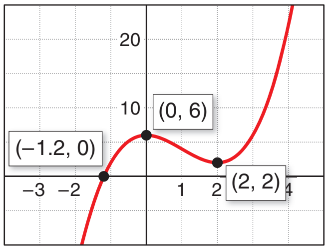
EXAMPLE 3 Finding Turning Points

WATCH

Graph each function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which each function is increasing or decreasing.

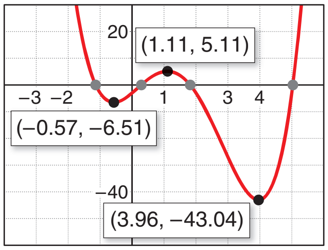
a. $f(x) = x^3 - 3x^2 + 6$ **b.** $g(x) = x^4 - 6x^3 + 3x^2 + 10x - 3$

SOLUTION



a. Use technology to graph the function. The graph of f has one x-intercept and two turning points. Approximate the x-intercept and the turning points.

▶ The x-intercept of the graph is $x \approx -1.20$. The function has a local maximum at $(0, 6)$ and a local minimum at $(2, 2)$. The function is increasing when $x < 0$ and $x > 2$ and decreasing when $0 < x < 2$.



b. Use technology to graph the function. The graph of g has four x-intercepts and three turning points. Approximate the x-intercepts and the turning points.

▶ The x-intercepts of the graph are $x \approx -1.14$, $x \approx 0.29$, $x \approx 1.82$, and $x \approx 5.03$. The function has a local maximum at $(1.11, 5.11)$ and local minimums at $(-0.57, -6.51)$ and $(3.96, -43.04)$. The function is increasing when $-0.57 < x < 1.11$ and $x > 3.96$ and decreasing when $x < -0.57$ and $1.11 < x < 3.96$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

7. Graph $f(x) = 0.5x^3 + x^2 - x + 2$. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.

8. **WRITING** Explain the local maximum of a function and how it may be different from the maximum value of the function.

Even and Odd Functions

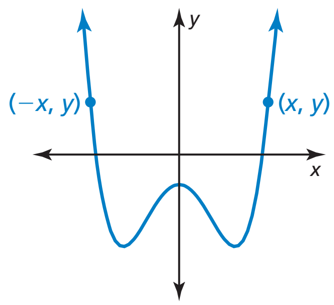
KEY IDEA

Even and Odd Functions

A function f is an **even function** when $f(-x) = f(x)$ for all x in its domain. The graph of an even function is *symmetric about the y-axis*.

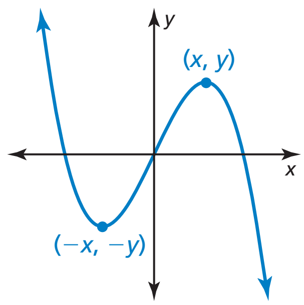
A function f is an **odd function** when $f(-x) = -f(x)$ for all x in its domain. The graph of an odd function is *symmetric about the origin*. One way to recognize a graph as symmetric about the origin is that it looks the same after a 180° rotation about the origin.

Even Function



For an even function, if (x, y) is on the graph, then $(-x, y)$ is also on the graph.

Odd Function



For an odd function, if (x, y) is on the graph, then $(-x, -y)$ is also on the graph.

EXAMPLE 4 Identifying Even and Odd Functions

Determine whether each function is *even*, *odd*, or *neither*.

a. $f(x) = x^3 - 7x$ **b.** $g(x) = x^4 + x^2 - 1$ **c.** $h(x) = x^3 + 2$

SOLUTION

a. Replace x with $-x$ in the equation for f, and then simplify.

$$f(-x) = (-x)^3 - 7(-x) = -x^3 + 7x = -(x^3 - 7x) = -f(x)$$

▶ Because $f(-x) = -f(x)$, the function is odd.

b. Replace x with $-x$ in the equation for g, and then simplify.

$$g(-x) = (-x)^4 + (-x)^2 - 1 = x^4 + x^2 - 1 = g(x)$$

▶ Because $g(-x) = g(x)$, the function is even.

c. Replacing x with $-x$ in the equation for h produces

$$h(-x) = (-x)^3 + 2 = -x^3 + 2.$$

▶ Because $h(x) = x^3 + 2$ and $-h(x) = -x^3 - 2$, you can conclude that $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$. So, the function is neither even nor odd.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Determine whether the function is *even*, *odd*, or *neither*.

9. $f(x) = -x^2 + 5$ **10.** $f(x) = x^4 - 5x^3$ **11.** $f(x) = 2x^5$ **12.** $f(x) = |2x^3|$

4.8 Practice WITH CalcChat® AND CalcView®

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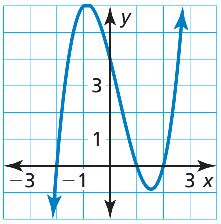
ANALYZING RELATIONSHIPS In Exercises 1–4, match the function with its graph.

1. $f(x) = (x - 1)^2(x + 2)$ **2.** $h(x) = (x + 2)^2(x + 1)$

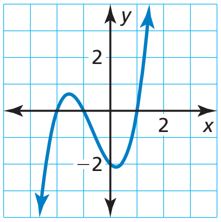
3. $f(x) = (x - 1)(x - 2)(x + 2)$

4. $g(x) = (x + 1)(x - 1)(x + 2)$

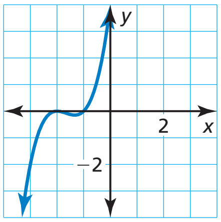
A.



B.



C.



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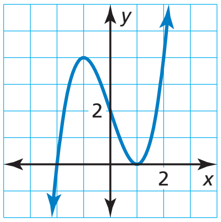


In Exercises 5–12, graph the function. *Example 1*

5. $f(x) = (x - 2)^2(x + 1)$ **6.** $f(x) = (x + 2)^2(x + 4)^2$

7. $h(x) = (x + 1)^2(x - 1)(x - 3)$

8. $g(x) = 4(x + 1)(x + 2)(x - 1)$

9. $h(x) = \frac{1}{3}(x - 5)(x + 2)(x - 3)$

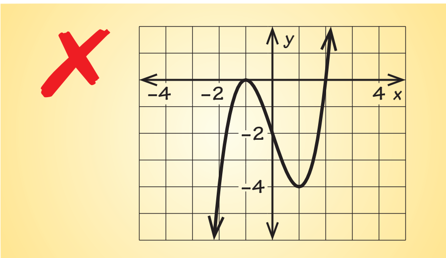
10. $g(x) = \frac{1}{12}(x + 4)(x + 8)(x - 1)$

11. $h(x) = (x - 3)(x^2 + x + 1)$

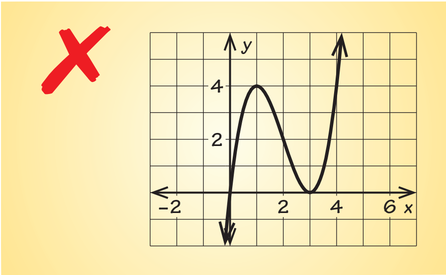
12. $f(x) = (x - 4)(2x^2 - 2x + 1)$

ERROR ANALYSIS In Exercises 13 and 14, describe and correct the error in using factors to graph f.

13. $f(x) = (x + 2)(x - 1)^2$



14. $f(x) = x^2(x - 3)^3$



In Exercises 15–22, find all the real zeros of the function. *Example 2*

15. $f(x) = x^3 - 4x^2 - x + 4$

16. $f(x) = x^3 - 3x^2 - 4x + 12$

17. $h(x) = 2x^3 + 7x^2 - 5x - 4$

18. $h(x) = 4x^3 - 2x^2 - 24x - 18$

19. $g(x) = 4x^3 + x^2 - 51x + 36$

20. $f(x) = 2x^3 - 3x^2 - 32x - 15$

21. $p(x) = 2x^4 + 5x^3 - 18x^2 - 68x - 56$

22. $m(x) = 24x^4 - 14x^3 - 37x^2 + 4x + 3$

In Exercises 23–30, graph the function. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing. *Example 3*

23. $g(x) = 2x^3 + 8x^2 - 3$

24. $g(x) = -x^4 + 3x$

25. $h(x) = x^4 - 3x^2 + x$

26. $f(x) = x^5 - 4x^3 + x^2 + 2$

27. $f(x) = 0.5x^3 - 2x + 2.5$

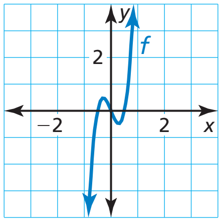
28. $f(x) = 0.7x^4 - 3x^3 + 5x$

29. $h(x) = x^5 + 2x^2 - 17x - 4$

30. $g(x) = x^4 - 5x^3 + 2x^2 + x - 3$

In Exercises 31–34, determine the least possible degree of *f*.

31.



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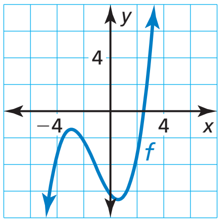

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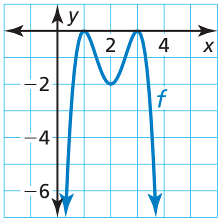

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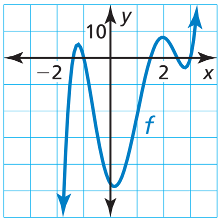

In Exercises 35 and 36, sketch a graph of a polynomial function *f* having the given characteristics.

35.
- The graph of f has x-intercepts of -4, 0, and 2.
- f has a local maximum when $x = 1$.
- f has a local minimum when $x = -2$.

36.
- The graph of f has x-intercepts of -3, 1, and 5.
- f has a local maximum when $x = 1$.
- f has a local minimum when $x = -2$ and when $x = 4$.

In Exercises 37–44, determine whether the function is *even, odd,* or *neither*. *Example 4*

37. $h(x) = 4x^7$

38. $g(x) = -2x^6 + x^2$

39. $f(x) = x^4 + 3x^2 - 2$

40. $f(x) = x^5 + 3x^3 - x$

41. $g(x) = x^2 + 5x + 1$

42. $f(x) = -x^3 + 2x - 9$

43. $f(x) = x^4 - 12x^2$

44. $h(x) = x^5 + 3x^4$

45. MODELING REAL LIFE When a swimmer does the breaststroke, the function

$$S = -241t^7 + 1060t^6 - 1870t^5 + 1650t^4 - 737t^3 + 144t^2 - 2.43t$$

models the speed S (in meters per second) of the swimmer during one complete stroke, where t is the number of seconds since the start of the stroke and $0 \le t \le 1.22$. Use technology to graph the function. At what time during the stroke is the swimmer traveling the fastest?

46. MODELING REAL LIFE The number V (in millions) of viewers of a weekly television show can be modeled by $V = 0.042x^3 - 0.45x^2 + 1.3x + 5$, where x is the number of weeks since the premiere and $0 \le x \le 7$. Use technology to graph the function. Then describe how the number of viewers changes over this period of time.

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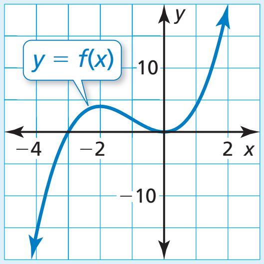
47. WRITING Why is the adjective *local*, used to describe the maximums and minimums of cubic functions, not required for quadratic functions?

48. HOW DO YOU SEE IT?
The graph of a polynomial function is shown.



a. Approximate the real zeros of the function and the points where the local maximum and local minimum occur.

b. Compare the x-intercepts of the graphs of $y = f(x)$ and $y = -f(x)$.

c. Compare the local maximums and local minimums of the functions $y = f(x)$ and $y = -f(x)$.

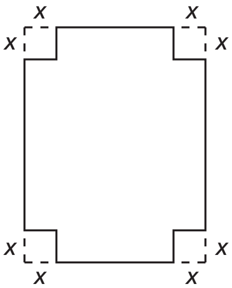
49. MP REASONING Is it possible to sketch the graph of a cubic polynomial function that has *no* turning points? Explain your reasoning.

50. MP PROBLEM SOLVING STEM Quonset huts are temporary, all-purpose structures shaped like half cylinders. You have 1100 square feet of material to build a quonset hut.

a. Write an equation that gives the surface area S of the hut. Then write an expression for ℓ in terms of r.

b. Write an equation that gives the volume V of a quonset hut as a function of r only. Then find the value of r that maximizes the volume of the hut.

51. **MP PRECISION** You can construct a rectangular box out of a sheet of paper by cutting squares with equal side lengths from the corners and folding up the sides. Choose a piece of paper and construct a box with the greatest possible volume that can be obtained. Explain how you determined the length you should make the cuts and your choice of units. State the volume of your box.

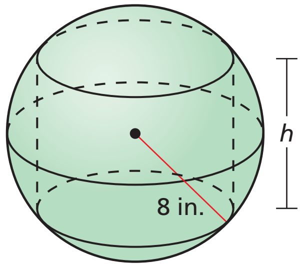


52. **THOUGHT PROVOKING**
Write and graph a polynomial function that has one real zero in each of the intervals $-2 < x < -1$, $0 < x < 1$, and $4 < x < 5$. Is there a maximum degree that such a polynomial function can have? Justify your answer.

53. **MAKING AN ARGUMENT** Is the product of two odd functions an odd function? Explain your reasoning.

GO DIGITAL

54. **DIG DEEPER** A cylinder is inscribed in a sphere of radius 8 inches. What is the maximum volume of the cylinder?



55. **PERFORMANCE TASK** Measure the dimensions of a food can of your choice, and calculate the surface area and volume. The manufacturer wants to use a can that has the same surface area and the greatest possible volume. Write a recommendation to the manufacturer describing a design for the can.

REVIEW & REFRESH

WATCH

In Exercises 56 and 57, solve the equation.

56. $n^3 + n^2 - 6n = 0$

57. $2m^4 + 10m^3 = 4m^2 + 20m$

In Exercises 58 and 59, determine whether the data are *linear*, *quadratic*, or *neither*. Explain.

58.

Month, x	0	1	2	3
Savings (dollars), y	100	150	200	250

59.

Time (seconds), x	0	1	2	3
Height (feet), y	300	284	236	156

60. **MODELING REAL LIFE** During a game of "Spud," a kickball is thrown straight up into the air. The height h (in feet) of the ball t seconds after it is thrown can be modeled by $h = -16t^2 + 24t + 5$.

a. Find the maximum height of the ball.

b. A chosen player catches the ball when it is 5 feet above the ground. How long is the ball in the air?

In Exercises 61 and 62, describe the transformation of *f* represented by *g*. Then graph each function.

61. $f(x) = x^3, g(x) = (x + 2)^3 - 5$

62. $f(x) = x^4, g(x) = -\frac{1}{2}x^4$

In Exercises 63 and 64, divide.

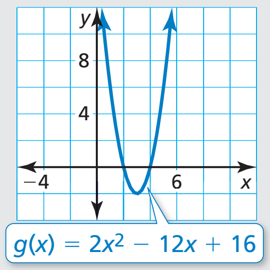
63. $(2x^2 + 9x - 5) \div (x + 3)$

64. $(7x^3 - 2x^2 + 4x + 6) \div (x^2 - x + 2)$

65. **MP REASONING** The graph of $g(x) = x - 5$ is a vertical translation 2 units down of the graph of $f(x) = x - 3$. How can you obtain the graph of g from the graph of f using a horizontal translation? Explain.

66. Find all the zeros of $f(x) = x^5 + 3x^4 + 9x^3 + 23x^2 - 36$.

67. Use the graph to solve $2x^2 + 16 = 12x$.



68. Graph $f(x) = 2x^4 - 3x^3 - 9x^2 + 6x - 4$. Identify the x-intercepts and the points where the local maximums and local minimums occur. Determine the intervals for which the function is increasing or decreasing.

In Exercises 69 and 70, find the product.

69. $(3z + 4)(3z - 4)$

70. $(6y - 10)^2$