Air Resistance In Exercises 51 to 54, solve the given problems related to air resistance.

51. Assuming that air resistance is proportional to velocity, the velocity v, in feet per second, of a falling object after t seconds is given by $v = 32(1 - e^{-t})$.

a. Graph this equation for $t \geq 0$.

b. Determine algebraically, to the nearest 0.01 second, when the velocity is 20 feet per second.

c. Determine the horizontal asymptote of the graph of v.

d. Write a sentence that explains the meaning of the horizontal asymptote in the context of this application.

52. Assuming that air resistance is proportional to velocity, the velocity v, in feet per second, of a falling object after t seconds is given by

$$v = 64(1 - e^{-t/2})$$

a. Graph this equation for $t \geq 0$.

b. Determine algebraically, to the nearest 0.1 second, when the velocity is 50 feet per second.

c. Determine the horizontal asymptote of the graph of v.

d. Write a sentence that explains the meaning of the horizontal asymptote in the context of this application.

53. The distance s (in feet) that the object in Exercise 51 will fall in t seconds is given by the function

$$s = 32t + 32(e^{-t} - 1)$$

a. Graph this equation for $t \geq 0$.

b. Determine, to the nearest 0.1 second, the time it takes the object to fall 50 feet.

c. Calculate the slope of the secant line through $(1, s(1))$ and $(2, s(2))$.

d. Write a sentence that explains the meaning of the slope of the secant line you calculated in **c.**

54. The distance s (in feet) that the object in Exercise 52 will fall in t seconds is given by the function

$$s = 64t + 128(e^{-t/2} - 1)$$

a. Graph this equation for $t \geq 0$.

b. Determine, to the nearest 0.1 second, the time it takes the object to fall 50 feet.

c. Calculate the slope of the secant line through $(1, s(1))$ and $(2, s(2))$.

d. Write a sentence that explains the meaning of the slope of the secant line you calculated in **c.**

55. Learning Theory The logistic model is also used in learning theory. Suppose that historical records from employee training at a company show that the percent score on a product information test is given by

$$P = \frac{100}{1 + 25e^{-0.095t}}$$

where t is the number of hours of training. What is the number of hours (to the nearest hour) of training needed before a new employee will answer 75% of the questions correctly?

56. Learning Theory A company provides training in the assembly of a computer circuit to new employees. Past experience has shown that the number of correctly assembled circuits per week can be modeled by

$$N = \frac{250}{1 + 249e^{-0.503t}}$$

where t is the number of weeks of training. What is the number of weeks (to the nearest week) of training needed before a new employee will correctly make 140 circuits per week?

57. Medication Level A patient is given three doses of aspirin. Each dose contains 1 gram of aspirin. The second and third doses are each taken 3 hours after the previous dose is administered. The half-life of the aspirin is 2 hours. The amount of aspirin A in the patient's body t hours after the first dose is administered is

$$A(t) = \begin{cases} 0.5^{t/2} & 0 \leq t < 3 \\ 0.5^{t/2} + 0.5^{(t-3)/2} & 3 \leq t < 6 \\ 0.5^{t/2} + 0.5^{(t-3)/2} + 0.5^{(t-6)/2} & t \geq 6 \end{cases}$$

Find, to the nearest hundredth of a gram, the amount of aspirin in the patient's body when

a. $t = 1$ **b.** $t = 4$ **c.** $t = 9$

58. Medication Level Use the dosage formula in Exercise 57 to determine when, to the nearest tenth of an hour, the amount of aspirin in the patient's body first reaches 0.25 gram.

59. Annual Growth Rate The exponential growth function for the population of a city is $N(t) = 78{,}245e^{0.0245t}$, where t is in years. Because

$$e^{0.0245t} = (e^{0.0245})^t \approx (1.0248)^t$$

we can write the growth function as

$$N(t) = 78{,}245(1.0248)^t \approx 78{,}245\left(1 + \frac{0.0248}{1}\right)^{1 \cdot t}$$

In this form we can see that the city's population is growing by 2.48% per year.

The population of the city of Lake Tahoe, Nevada, can be modeled by the exponential growth function $N(t) = 22{,}755e^{0.0287t}$. Find the annual growth rate, expressed as a percent, of Lake Tahoe. Round to the nearest hundredth of a percent.

60. Oil Spills Crude oil leaks from a tank at a rate that depends on the amount of oil that remains in the tank. Because $\frac{1}{8}$ of the oil in the tank leaks out every 2 hours, the volume $V(t)$ of oil in the tank after t hours is given by $V(t) = V_0(0.875)^{t/2}$, where $V_0 = 350{,}000$ gallons, the number of gallons in the tank at the time the tank started to leak ($t = 0$).

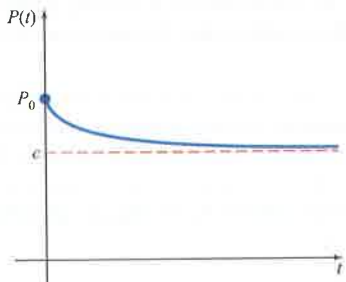
a. How many gallons does the tank hold after 3 hours?

b. How many gallons does the tank hold after 5 hours?

c. How long, to the nearest hour, will it take until 90% of the oil has leaked from the tank?

Enrichment Exercises

If $P_0 > c$ (which implies that $-1 < a < 0$), then the logistic function $P(t) = \dfrac{c}{1 + ae^{-bt}}$ decreases as t increases. Biologists often use this type of logistic function to model populations that decrease over time. See the following figure. Apply this information to Exercises 61 to 63.



$$P(t) = \frac{c}{1 + ae^{-bt}}, \quad P_0 > c$$

61. **A Declining Fish Population** A biologist finds that the fish population in a small lake can be closely modeled by the logistic function

$$P(t) = \frac{1000}{1 + (-0.3333)e^{-0.05t}}$$

where t is the time, in years, since the lake was first stocked with fish.

a. What was the fish population when the lake was first stocked with fish?

b. According to the logistic model, what will the fish population approach in the long-term future?

62. **A Declining Deer Population** The deer population in a reserve is given by the logistic function

$$P(t) = \frac{1800}{1 + (-0.25)e^{-0.07t}}$$

where t is the time, in years, since July 1, 2010.

a. What was the deer population on July 1, 2010? What was the deer population on July 1, 2012?

b. According to the logistic model, what will the deer population approach in the long-term future?

63. **Modeling World Record Times in the Men's Mile Race** In the early 1950s, many people speculated that no runner would ever run a mile race in under 4 minutes. During the period from 1913 to 1945, the world record in the mile event had been reduced from 4.14.4 (4 minutes, 14.4 seconds) to 4.01.4, but no one seemed capable of running a sub-4-minute mile. Then, in 1954, Roger Bannister broke through the 4-minute barrier by running a mile in 3.59.6. In 1999, the current record of 3.43.13 was established. It is fun to think about future record times in the mile race. Will they ever go below 3 minutes, 30 seconds? Below 3 minutes, 20 seconds? What about a sub-3-minute mile?

A declining logistic function that closely models the world record times WR, in seconds, in the men's mile run from 1913 ($t = 0$) to 1999 ($t = 86$) is given by

$$WR(t) = \frac{199.13}{1 + (-0.21726)e^{-0.0079889t}}$$

a. Use the above logistic function to predict the world record time for the men's mile run in 2020 and 2050.

b. According to the logistic function, what time will the world record in the men's mile event approach but never break through?

SECTION 4.7

Analyzing Scatter Plots
Modeling Data
Finding a Logistic Growth Model

Modeling Data with Exponential and Logarithmic Functions

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A28.

PS1. Determine whether $N(t) = 4 - \ln t$ is an increasing or a decreasing function. [4.3]

PS2. Determine whether $P(t) = 1 - 2(1.05^t)$ is an increasing or a decreasing function. [4.2]

PS3. Evaluate $P(t) = \dfrac{108}{1 + 2e^{-0.1t}}$ for $t = 0$. [4.2]

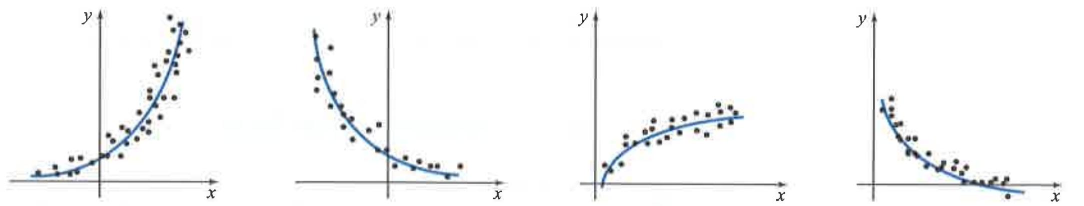
PS4. Evaluate $N(t) = 840e^{1.05t}$ for $t = 0$. [4.2]

PS5. Solve $10 = \dfrac{20}{1 + 2.2e^{-0.05t}}$ for t. Round to the nearest tenth. [4.5]

PS6. Determine the horizontal asymptote of the graph of $P(t) = \dfrac{55}{1 + 3e^{-0.08t}}$, for $t \geq 0$. [4.2]

Analyzing Scatter Plots

In Section 2.7 we used linear and quadratic functions to model several data sets. However, in some applications, data can be modeled more closely by using exponential or logarithmic functions. For instance, Figure 4.43 illustrates some scatter plots that can be modeled effectively by exponential and logarithmic functions.



a. Exponential increasing: $y = ab^x$, $a > 0$, $b > 1$

b. Exponential decreasing: $y = ab^x$, $a > 0$, $0 < b < 1$

c. Logarithmic increasing: $y = a + b \ln x$, $b > 0$

d. Logarithmic decreasing: $y = a + b \ln x$, $b < 0$

Figure 4.43

Exponential and logarithmic models

The terms *concave upward* and *concave downward* are often used to describe a graph. For instance, Figures 4.44a and 4.44b show the graphs of two increasing functions that join the points P and Q. The graphs of f and g differ in that they bend in different directions. We can distinguish between these two types of "bending" by examining the positions of *tangent lines* to the graphs. In Figures 4.44c and 4.44d, tangent lines (in red) have been drawn to the graphs of f and g. The graph of f lies above its tangent lines, and the graph of g lies below its tangent lines. The function f is said to be concave upward, and g is concave downward.

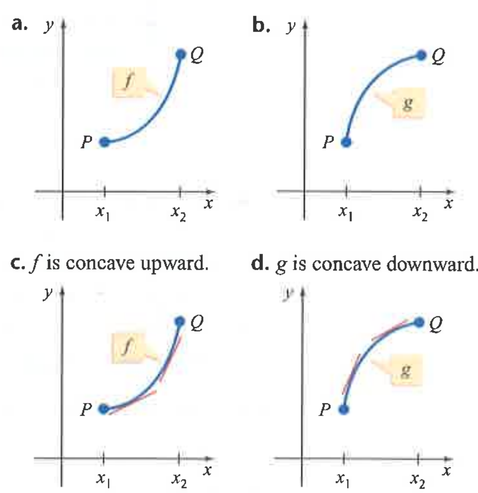


Figure 4.44

Definition of Concavity

If the graph of f lies above all of its tangents on an interval $[x_1, x_2]$, then f is **concave upward** on $[x_1, x_2]$.

If the graph of f lies below all of its tangents on an interval $[x_1, x_2]$, then f is **concave downward** on $[x_1, x_2]$.

An examination of the graphs in Figure 4.43 on page 407 shows that the graphs of all exponential functions of the form $y = ab^x, a > 0, b > 0, b \neq 1$, are concave upward. The graphs of increasing logarithmic functions of the form $y = a + b \ln x, b > 0$, are concave downward, and the graphs of decreasing logarithmic functions of the form $y = a + b \ln x, b < 0$, are concave upward.

In Example 1, we analyze scatter plots to determine whether the shape of the scatter plot can be best approximated by a function that is concave upward or concave downward.

Question • Is the graph of $y = 5 - 2 \ln x$ concave upward or concave downward?

EXAMPLE 1 Analyze Scatter Plots

For each of the following data sets, determine whether the most suitable model of the data would be an increasing exponential function or an increasing logarithmic function.

$$A = \{(1, 0.6), (2, 0.7), (2.8, 0.8), (4, 1.3), (6, 1.5), (6.5, 1.6), (8, 2.1), (11.2, 4.1), (12, 4.6), (15, 8.2)\}$$

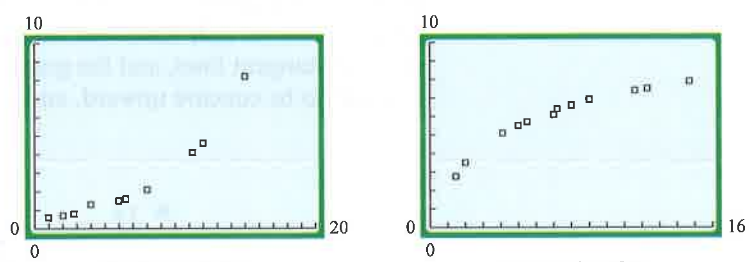
$$B = \{(1.5, 2.8), (2, 3.5), (4.1, 5.1), (5, 5.5), (5.5, 5.7), (7, 6.1), (7.2, 6.4), (8, 6.6), (9, 6.9), (11.6, 7.4), (12.3, 7.5), (14.7, 7.9)\}$$

TO REVIEW

See Section 2.7 to review the steps needed to create a scatter plot on a TI-83/TI-83 Plus/TI-84 Plus calculator.

Solution

For each set, construct a scatter plot of the data. See Figure 4.45.



Scatter plot of A Scatter plot of B

Figure 4.45

The scatter plot of A suggests that A is an increasing function that is concave upward. Thus an increasing exponential function would be the most suitable model for data set A.

The scatter plot of B suggests that B is an increasing function that is concave downward. Thus an increasing logarithmic function would be the most suitable model for data set B.

▶ Try Exercise 8, page 414

Answer • The graph of $y = 5 - 2 \ln x$ is concave upward because the b value, -2, is less than 0. See Figure 4.43d on page 407.

Modeling Data

The methods of modeling data using exponential or logarithmic functions are similar to the methods used in Section 2.7 to model data using linear or quadratic functions. Here is a summary of the modeling process.

Modeling Process

Use a graphing utility to perform the following steps.

1. **Construct a *scatter plot* of the data** to determine which type of function will effectively model the data.
2. **Find the *equation* of the modeling function** and the correlation coefficient or the coefficient of determination for the equation.
3. **Examine the *correlation coefficient* or the *coefficient of determination* and view a *graph*** that displays both the modeling function and the scatter plot to determine how well your function fits the data.

In the following example, we apply the modeling process to find a function that models the number of smartphones sold in the United States during recent years.

Integrating Technology

Most graphing utilities have built-in routines that can be used to determine the exponential or logarithmic regression function that models a set of data. On a TI-83/TI-83 Plus/TI-84 Plus calculator, the **ExpReg** instruction is used to find the exponential regression function and the **LnReg** instruction is used to find the logarithmic regression function. The TI-83/TI-83 Plus/TI-84 Plus calculator does not show the value of the regression coefficient r or the coefficient of determination unless the **DiagnosticOn** command has been entered. The **DiagnosticOn** command is in the CATALOG menu.

EXAMPLE 2 Model an Application

The following table lists the number of smartphones sold in the United States for selected years from 2004 to 2011.

a. Use a graphing utility to construct a scatter plot of the data.

b. Examine the scatter plot and determine a regression function that closely models the data.

Table 4.13 U.S. Sales of Smartphones

Calendar Year	Units Sold (millions)
2004	4.5
2006	14.0
2008	28.6
2010	54.1
2011	78.6

Source: The World Almanac and Book of Facts 2012.

Solution

a. Construct a scatter plot of the data.
Use the STAT EDIT menu to enter the data and view a scatter plot of the data. We have used $x = 1$ to represent 2004, $x = 3$ to represent 2006, $x = 5$ to represent 2008, . . . , and $x = 8$ to represent 2011. See Figure 4.46.

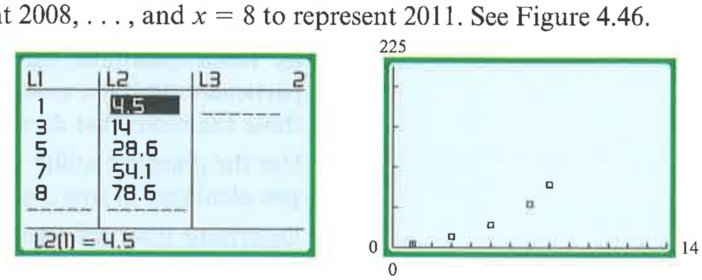
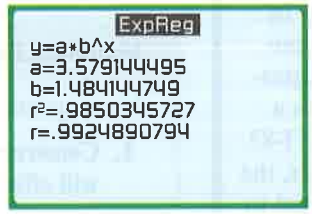


Figure 4.46

(continued)

b. Examine the scatter plot and find a regression function that models the data. From the scatter plot, we see that a good model must be increasing and concave upward. Thus it appears that the data can be closely modeled by an increasing exponential function.



ExpReg display (DiagnosticOn)

Figure 4.47

The calculator display in Figure 4.47 shows that the exponential regression function is $y \approx 3.579144(1.484145^x)$, where y is the number of smartphones sold in year x. The correlation coefficient $r \approx 0.992489$ is very close to 1. Thus we know that the exponential regression function provides a good fit for the data, as shown by the graph in Figure 4.48.

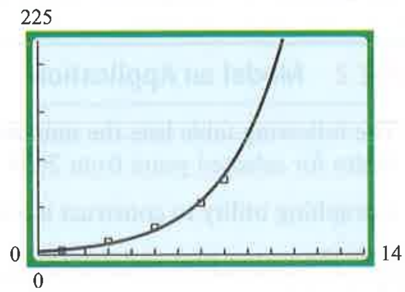


Figure 4.48

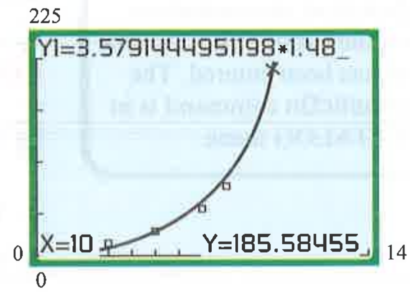


Figure 4.49

To estimate the number of smartphones that will be sold in 2013, use the VALUE command in the CALCULATE menu to evaluate the exponential regression function at $x = 10$. See Figure 4.49. According to the exponential regression function, about 185.6 million smartphones will be sold in 2013.

▸ **Try Exercise 20, page 415**

When you are selecting a function to model a given set of data, try to find a function that provides a good fit to the data *and* is likely to produce realistic predictions. The following guidelines may facilitate the selection process.

Guidelines for Selecting a Modeling Function for an Application

1. Use a graphing utility to construct a scatter plot of the data.
2. Compare the graphical features of the scatter plot with the graphical features of the basic modeling functions available on the graphing utility: linear, quadratic, cubic, exponential, logarithmic, or logistic. Pay particular attention to the concave nature of each function. Eliminate those functions that do not display the desired concavity.
3. Use the graphing utility to find the equation of each type of function you identified in step 2 as a possible model.
4. Determine how well each function fits the given data, and compare the graphs of the functions to determine which function is most likely to produce realistic predictions for the given application.

EXAMPLE 3 Select a Modeling Function and Make a Prediction

Table 4.14 shows the winning times in the women's Olympic 100-meter freestyle event for the years 1968 to 2012.

AP Photo/Matt Slocum

Ranomi Kromowidjojo wins the 100-meter freestyle in an Olympic-record time of 53.00 seconds.

Table 4.14 **Women's Olympic 100-Meter Freestyle Winning Times, 1968 to 2012**

Year	Time (s)	Year	Time (s)
1968	60.0	1992	54.64
1972	58.59	1996	54.50
1976	55.65	2000	53.83
1980	54.79	2004	53.84
1984	55.92	2008	53.12
1988	54.93	2012	53.00

Source: About.com: Swimming.

Find a function to model the data, and use the function to predict the winning time in the women's Olympic 100-meter freestyle event for 2016.

Solution

Construct a scatter plot of the data. See Figure 4.50. (*Note:* This scatter plot was produced using $x = 1$ to represent 1968, $x = 2$ to represent 1972, . . . , and $x = 12$ to represent 2012.) The general shape of the scatter plot suggests that we consider functions whose graphs are decreasing and concave upward. Thus we consider a decreasing exponential function and a decreasing logarithmic function as possible models. Use a graphing utility to find the exponential regression function and the logarithmic regression function for the data. See Figure 4.51 and Figure 4.52.

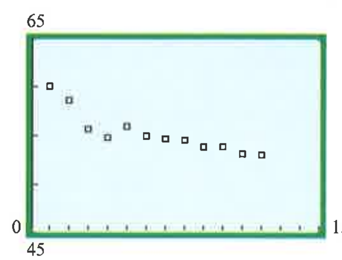


Figure 4.50

Note

See the *Exploring Concepts with Technology* feature, on page 418, for information on how to use WolframAlpha to find exponential and logarithmic regression functions for a data set.

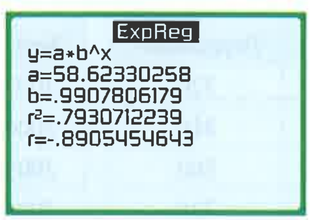


Figure 4.51

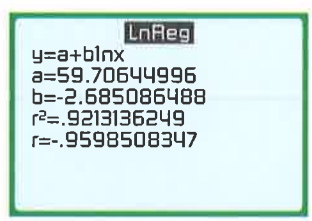


Figure 4.52

The exponential function is $y \approx 58.62330258(0.9907806179)^x$ and the logarithmic function is $y \approx 59.70644996 - 2.685086488 \ln x$. The coefficient of determination r^2 for the logarithmic regression is larger than the coefficient of determination for the exponential regression. See Figure 4.51 and Figure 4.52. Thus the logarithmic regression function provides a better fit to the data than does the exponential regression function. The correlation coefficients r can also be used to determine which function provides the better fit. For decreasing functions, the function with correlation coefficient closer to -1 provides the better fit.

(continued)

Notice that the graph of the logarithmic function has the desired behavior, as shown to the right. That is, it is a *gradually decreasing* curve, and this is the *general* behavior we would expect for future winning times in the 100-meter freestyle event. The graph of the exponential function is almost linear and is decreasing at a rapid pace, which is not what we would expect for results in an established Olympic event. See Figure 4.53. Thus we select the logarithmic function as our modeling function.

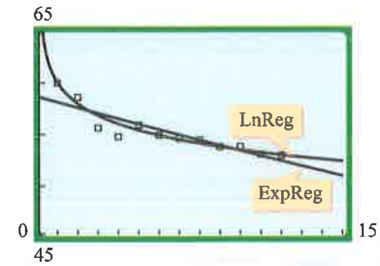


Figure 4.53

To predict the winning time for this event in 2016 (represented by $x = 13$), substitute 13 for x in the equation of the logarithmic function or use the VALUE command in the CALCULATE menu to produce the approximate time of 52.82 seconds, as shown in Figure 4.54.

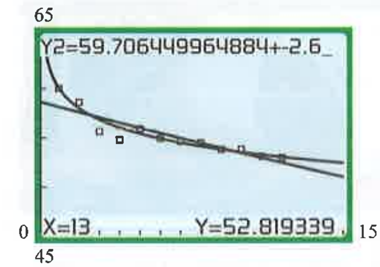


Figure 4.54

▶ Try Exercise 22, page 415

Finding a Logistic Growth Function

If a scatter plot of a set of data suggests that the data can be effectively modeled by a logistic growth function, then you can use the Logistic feature of a graphing utility to find the logistic growth function. This process is illustrated in Example 4.

EXAMPLE 4 Find a Logistic Growth Function

Table 4.15 shows the population of deer in an animal preserve for 1998 to 2012.

Table 4.15 Deer Population at an Animal Preserve

Year	Population	Year	Population	Year	Population
1998	320	2003	1150	2008	2620
1999	410	2004	1410	2009	2940
2000	560	2005	1760	2010	3100
2001	730	2006	2040	2011	3300
2002	940	2007	2310	2012	3460

Find a logistic growth function that approximates the deer population as a function of the year. Use the model to predict the deer population in 2018.

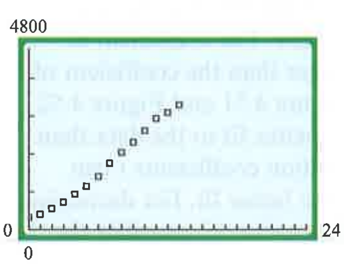


Figure 4.55

Solution

1. ***Construct a scatter plot of the data*** Enter the data into a graphing utility, and then use the utility to display a scatter plot of the data. In this example, we represent 1998 by $x = 0$, 2012 by $x = 14$, and the deer population by y.

   Figure 4.55 shows that the data can be closely approximated by a logistic growth function.

Integrating Technology

On a TI-83/TI-83 Plus/TI-84 Plus graphing calculator, the logistic growth function is given in the form

$$y = \frac{c}{1 + ae^{-bx}}$$

Think of the variable x as the time t and the variable y as $P(t)$.

2. ***Find the equation of the logistic growth function*** On a TI-83/TI-83 Plus/TI-84 Plus graphing calculator, select B: Logistic, which is in the STAT CALC menu. The logistic growth function for the data is

$$y \approx \frac{3965.3}{1 + 11.445e^{-0.31152x}}$$

See Figure 4.56.

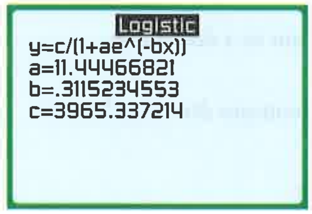


Figure 4.56

3. ***Examine the fit*** A TI-83/TI-83 Plus/TI-84 Plus calculator does not compute the coefficient of determination or the correlation coefficient for a logistic function. However, Figure 4.57 shows that the logistic function provides a good fit to the data. To use the function to predict the deer population in 2018 (the year 2018 is represented by an x value of 20), find the y value of the logistic function for $x = 20$. The VALUE command in the CALCULATE menu shows that the logistic function predicts a deer population of about 3878 in 2018. See Figure 4.58.

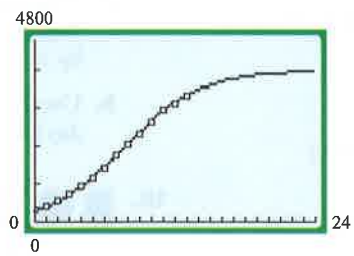


Figure 4.57

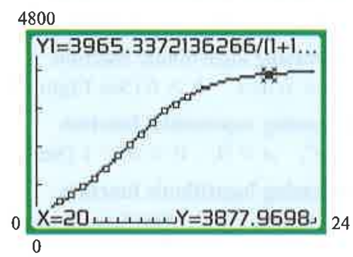


Figure 4.58

▶ Try Exercise 24, page 416

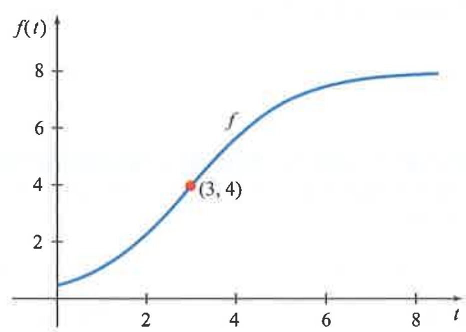
EXERCISE SET 4.7

Concept Check

1. The smooth continuous graph of f lies above all of its tangents on the interval $[x_1, x_2]$. Is f concave upward or concave downward on $[x_1, x_2]$?

2. The following figure shows the graph of the logistic function

$$f(t) = \frac{8}{1 + 15e^{-0.9026834t}}, \; t > 0$$



Indicates Try It Exercises

The point (3, 4) on the graph of f is called an *inflection point,* because the graph changes its concavity at that point. Examine the graph to determine whether f is concave upward or concave downward on the following intervals.

a. [0, 3]

b. [3, ∞)

3. Let $f(x) = a + b \ln x, b < 0$.

a. What is the domain of f?

b. Is f an increasing function or a decreasing function?

c. Is f concave upward or concave downward on its domain?

4. Let $f(x) = a + b \ln x, b > 0$.

a. What is the domain of f?

b. Is f an increasing function or a decreasing function?

c. Is f concave upward or concave downward on its domain?

In Exercises 5 to 10, use a scatter plot of the given data to determine which of the following types of functions might provide a suitable model of the data.

- **An increasing exponential function**
  $y = ab^x, \quad a > 0, \quad b > 1$ (See Figure 4.43a, page 407.)
- **An increasing logarithmic function**
  $y = a + b \ln x, \quad b > 0$ (See Figure 4.43c.)
- **A decreasing exponential function**
  $y = ab^x, \quad a > 0, \quad 0 < b < 1$ (See Figure 4.43b.)
- **A decreasing logarithmic function**
  $y = a + b \ln x, \quad b < 0$ (See Figure 4.43d.)

(*Note:* Some data sets can be closely modeled by more than one type of function.)

5. $\{(1,3),(1.5,4),(2,6),(3,13),(3.5,19),(4,27)\}$

6. $\{(1.0,1.12),(2.1,0.87),(3.2,0.68),(3.5,0.63),(4.4,0.52)\}$

7. $\{(1,2.4),(2,1.1),(3,0.5),(4,0.2),(5,0.1)\}$

8. $\{(5,2.3),(7,3.9),(9,4.5),(12,5.0),(16,5.4),(21,5.8),(26,6.1)\}$

9. $\{(1,2.5),(1.5,1.7),(2,0.7),(3,-0.5),(3.5,-1.3),(4,-1.5)\}$

10. $\{(1,3),(1.5,3.8),(2,4.4),(3,5.2),(4,5.8),(6,6.6)\}$

In Exercises 11 and 12, find the exponential regression function for the data. State the coefficient of determination.

11. $\{(10,6.8),(12,6.9),(14,15.0),(16,16.1),(18,50.0),(19,68.5)\}$

12. $\{(2.6,16.2),(3.8,48.8),(5.1,160.1),(6.5,590.2),(7,911.2)\}$

In Exercises 13 and 14, find the logarithmic regression function for the data. State the coefficient of determination.

13. $\{(5,2.7),(6,2.5),(7.2,2.2),(9.3,1.9),(11.4,1.6),(14.2,1.3)\}$

14. $\{(11,15.75),(14,15.52),(17,15.34),(20,15.18),(23,15.05)\}$

In Exercises 15 and 16, find the logistic regression function for the data.

15. $\{(0,81),(2,87),(6,98),(10,110),(15,125)\}$

16. $\{(0,175),(5,195),(10,217),(20,264),(35,341)\}$

17. **Lift Ticket Prices** The following table shows the price of an all-day lift ticket at a ski resort for selected years from 2001 to 2013.

All-Day Lift Ticket Prices, 2001–2013

Year	Price	Year	Price
2001	\$38	2009	\$75
2003	\$42	2011	\$87
2005	\$54	2013	\$101
2007	\$61		

a. Find an exponential regression function for the data. Represent 2001 by $x = 1$, 2003 by $x = 3, \ldots,$ and 2013 by $x = 13$.

b. Use the exponential function to predict the price of an all-day lift ticket for 2016 ($x = 16$). Round to the nearest dollar.

18. **Recycling Rates** U.S. recycling rates have been increasing over the last few decades. The following table shows the percent of municipal solid waste (MSW) that was recycled for selected years from 1960 to 2009.

MSW Recycling Rates, 1960–2009

Year	Recycling Rate	Year	Recycling Rate
1960	6.4%	1995	26.0%
1970	6.6%	2000	29.1%
1980	9.6%	2006	32.5%
1990	16.2%	2009	33.8%

Source: U.S. Environmental Protection Agency.

a. Find an exponential regression function and a logarithmic regression function that model the recycling rate R, for the year t. Use $t = 60$ to represent 1960, $t = 70$ to represent 1970, . . . , and $t = 109$ to represent 2009.

b. Examine the coefficient of determination of the two regression functions to determine which provides a better fit for the data.

c. Use each of the regression functions to predict the recycling rate for 2013. Round each rate to the nearest tenth of a percent.

d. In your opinion, which of the recycling rate predictions you determined in **c** is the more realistic prediction? Explain.

19. **Hypothermia** The following table shows the time T, in hours, before a scuba diver wearing a 3-millimeter-thick wet suit reaches hypothermia (95°F) for various water temperatures F, in degrees Fahrenheit.

Water Temperature F (°F)	Time T (h)
41	1.1
46	1.4
50	1.8
59	3.7

a. Find an exponential regression function for the data.

b. Use the model from **a** to estimate the time it takes for the diver to reach hypothermia in water that has a temperature of 65°F. Round to the nearest tenth of an hour.

20. **Atmospheric Pressure** The following table shows the Earth's atmospheric pressure y (in newtons per square centimeter) at an altitude of x kilometers. Find an exponential regression function that models the atmospheric pressure as a function of the altitude. Use the function to estimate the atmospheric pressure at an altitude of 24 kilometers. Round to the nearest tenth of a newton per square centimeter.

Altitude x (km)	Pressure y (N/cm^2)
0	10.3
2	8.0
4	6.4
6	5.1
8	4.0
10	3.2
12	2.5
14	2.0
16	1.6
18	1.3

21. **Hypothermia** The following table shows the time T, in hours, before a scuba diver wearing a 4-millimeter-thick wet suit reaches hypothermia (95°F) for various water temperatures F, in degrees Fahrenheit.

Water Temperature F (°F)	Time T (h)
41	1.5
46	1.9
50	2.4
59	5.2

a. Find an exponential regression function for the data.

b. Use the function from **a** to estimate the time it takes for the diver to reach hypothermia in water that has a temperature of 65°F. Round to the nearest tenth of an hour. How much greater is this result compared with the answer to Exercise 19**b**?

22. **400-Meter Race** The following table lists the progression of world record times in the men's 400-meter race from 1948 to 2011. (*Note*: No new world record times were set during the time period from 2000 to 2013.)

World Record Times in the Men's 400-Meter Race, 1948 to 2013

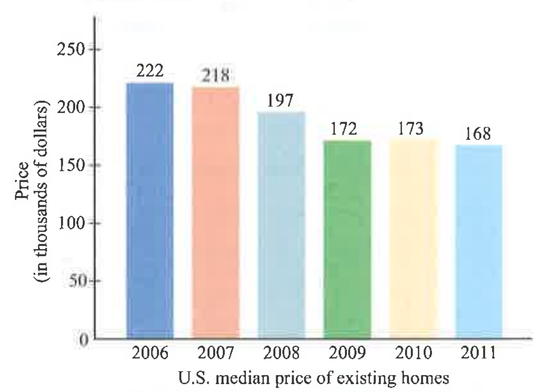
Year	Time (s)	Year	Time (s)
1948	45.9	1964	44.9
1950	45.8	1967	44.5
1955	45.4	1968	44.1
1956	45.2	1968	43.86
1960	44.9	1988	43.29
1963	44.9	1999	43.18

Source: Track and Field Statistics, http://trackfield.brinkster.net.

a. Determine whether the data can better be modeled by an exponential function or a logarithmic function. Let $x = 48$ represent 1948, $x = 50$ represent 1950, and so forth.

b. Assume that a new world record time will be established in 2015, which is represented by $x = 115$. Use the function you chose in **a** to predict the world record time in the men's 400-meter race for 2015. Round to the nearest hundredth of a second.

23. **Median Price of Homes** The following bar graph shows the median price, P, of existing homes in the United States for the years from 2006 to 2011.



Source: The World Almanac and Book of Facts 2012.

a. Find an exponential regression function and a logarithmic regression function for the data. Use $t = 6$ to represent 2006, $t = 7$ to represent 2007, . . . , and $t = 11$ to represent 2011.

b. Use the exponential function to predict the median price of existing homes in the United States in 2014. Round to the nearest thousand dollars.

c. Use the logarithmic function to predict the median price of existing homes in the United States in 2014. Round to the nearest thousand dollars.

24. **Population of Hawaii** The following table shows the population of the state of Hawaii for selected years from 1950 to 2010.

Population of Hawaii, 1950–2010

Year	Population, P	Year	Population, P
1950	499,000	1990	1,113,491
1960	642,000	2000	1,211,566
1970	762,920	2010	1,360,301
1980	967,710		

Source: Economagic.com.

a. Find the logistic regression function that approximates the population of Hawaii as a function of the year, t. Use $t = 0$ to represent 1950, $t = 10$ to represent 1960, . . . , and $t = 60$ to represent 2010.

b. Use the logistic regression function to predict the population of Hawaii in 2015 (represented by $t = 65$). Round to the nearest thousand.

c. What is the carrying capacity of the logistic model? Round to the nearest thousand.

25. **Optometry** The *near point* p of a person is the closest distance at which the person can see an object distinctly. As one grows older, one's near point increases. The table below shows data for the average near point of various people with normal eyesight.

Age y (years)	Near Point p (cm)
15	11
20	13
25	15
30	17
35	20
40	23
50	26

a. Find an exponential regression function for these data.

b. What near point does this function predict for a person 60 years old? Round to the nearest centimeter.

26. **Chemistry** The amount of oxygen x, in milliliters per liter, that can be absorbed by water at a certain temperature T, in degrees Fahrenheit, is given in the following table.

Temperature (°F)	Oxygen Absorbed (ml/L)
32	10.5
38	8.4
46	7.6
52	7.1
58	6.8
64	6.5

a. Find a logarithmic regression function for these data.

b. Using your function, how much oxygen, to the nearest tenth of a milliliter per liter, can be absorbed in water that is 50°F?

27. **The Henderson-Hasselbach Function** The scientists Henderson and Hasselbach determined that the pH of blood is a function of the ratio q of the amounts of bicarbonate and carbonic acid in the blood.

a. Determine a logarithmic regression function for the data. Use q as the independent variable (domain) and pH as the dependent variable (range).

q	7.9	12.6	31.6	50.1	79.4
pH	7.0	7.2	7.6	7.8	8.0

b. Use the logarithmic regression function to find the q value associated with a pH of 8.2. Round to the nearest tenth.

28. **World Population** The following table lists the years in which the world's population first reached 3 billion, 4 billion, 5 billion, 6 billion, and 7 billion.

World Population Milestones

Year	Population, P
1960	3 billion
1974	4 billion
1987	5 billion
1999	6 billion
2012	7 billion

Source: Geohive.com.

a. Find a logistic regression function P for the data. Let t represent the number of years after 1960.

b. Use the logistic function to predict the year in which the world's population will first reach 8 billion.

c. According to the logistic function, what will the world's population approach as $t \to \infty$? Round to the nearest billion.

29. **Average Ticket Prices** The following table shows the average movie theater ticket price, in the United States, for the years from 2004 to 2011.

Average U.S. Movie Theater Ticket Prices

Year	Average Ticket Price (dollars)	Year	Average Ticket Price (dollars)
2004	6.21	2008	7.18
2005	6.41	2009	7.50
2006	6.55	2010	7.89
2007	6.88	2011	7.93

Source: The World Almanac and Book of Facts 2013.

a. Find an exponential regression function and a logarithmic regression function that model the average ticket price P, as a function of the year t. Use $t = 4$ to represent 2004, $t = 5$ to represent 2005, . . . , and $t = 11$ to represent 2011.

b. Which function provides a better fit to the data?

c. Use the regression function you selected in **b** to predict the average movie theater ticket price in 2015 ($t = 15$).

30. **Temperature of Coffee** A cup of coffee is placed in a room that maintains a constant temperature of 70°F. The following table shows both the coffee temperature T after t minutes and the difference between the coffee temperature and the room temperature after t minutes.

Time t (minutes)	0	5	10	15	20	25
Coffee Temp. T (°F)	165°	140°	121°	107°	97°	89°
$T - 70°$	95°	70°	51°	37°	27°	19°

a. Find an exponential regression function for the difference $T - 70$ as a function of t.

b. Use the regression function in **a** to predict how long it will take (to the nearest minute) for the coffee to cool to 80°F.

31. **A Cat's Age in Human Years** The following table lists the approximate age, in human years, of a cat as it ages from 2 months to 24 months.

A Cat's Age in Human Years

Calendar Age, x (in months)	Approximate Age (in human years)
2	3
4	6
6	9
8	11
10	13
12	15
18	20
24	24

Source: The Cat Owner's Manual, by Quirk Books.

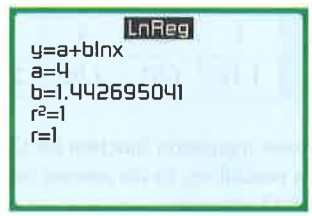
a. Find a logistic regression function A that models the data.

b. Use the logistic function to estimate the age, in human years, of a 5-month-old cat. Round to the nearest tenth of a year. Do you think this is a realistic estimate?

c. Use the logistic function to estimate the age, in human years, of a 72-month-old cat. Round to the nearest tenth of a year. Do you think this is a realistic estimate?

d. The table in this exercise is the same as the table in Exercise 73, page 285, where you used a cubic regression function to model the data and to make estimations. In this exercise, you used a logistic regression function to model the data and make estimations. Write a few sentences that compare the strengths and the weaknesses of each function regarding its ability to model the data and produce reasonable estimations.

32. **A Correlation Coefficient of 1** A scientist uses a graphing calculator to model the data set $\{(2, 5), (4, 6)\}$ with a logarithmic function. The following display shows the results.



What is the significance of the fact that the correlation coefficient for the regression function is $r = 1$?

33. **Duplicate Data Points** An engineer needs to model the data in set A with an exponential function.

$$A = \{(2, 5), (3, 10), (4, 17), (4, 17), (5, 28)\}$$

Because the ordered pair $(4, 17)$ is listed twice, the engineer decides to eliminate one of these ordered pairs and model the data in set B.

$$B = \{(2, 5), (3, 10), (4, 17), (5, 28)\}$$

Determine whether A and B both have the same exponential regression function.

34. **Domain Error** A scientist needs to model the data in set A.

$$A = \{(0, 1.2), (1, 2.3), (2, 2.8), (3, 3.1), (4, 3.3), (5, 3.4)\}$$

The scientist views a scatter plot of the data and decides to model the data with a logarithmic function of the form $y = a + b \ln x$.

a. When the scientist attempts to use a graphing calculator to determine the logarithmic regression function, the calculator displays the message

"ERR:DOMAIN"

Explain why the calculator was unable to determine the logarithmic regression function for the data.

b. Explain what the scientist could do so that the data in set A could be modeled by a logarithmic function of the form $y = a + b \ln x$.

Enrichment Exercises

35. **Power Functions** A function that can be written in the form $y = ax^b$ is said to be a **power function.** Some data sets can best be modeled by a power function. On a TI-83/84 calculator, the PwrReg instruction is used to produce a power regression function for a set of data. Find the power regression function for the following data.

x	1	2	3	4	5	6
y	2.1	5.5	9.8	14.6	20.1	25.8

36. **Period of a Pendulum** The following table shows the time t (in seconds) of the period of a pendulum of length l (in feet). (*Note*: The period of a pendulum is the time it takes the pendulum to complete a swing from the right to the left and back.)

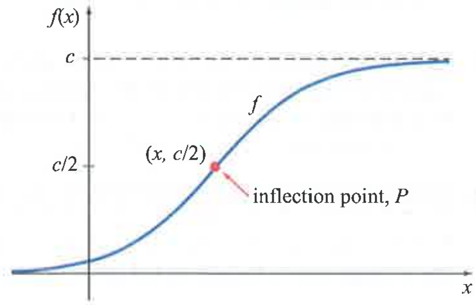
Length l (*ft*)	1	2	3	4	6	8
Time t (*s*)	1.11	1.57	1.92	2.25	2.72	3.14

Use the power regression function for the data to estimate the length of a pendulum, to the nearest tenth of a foot, that has a period of 12 seconds.

37. **Inflection Point of a Logistic Function** An important feature of the graph of a logistic function,

$$f(x) = \frac{c}{1 + ae^{-bx}} \qquad (1)$$

with domain the set of real numbers, is its shape. The graph of every logistic function has an S-shape and a single inflection point, which separates the graph into two equal regions of opposite concavity. For instance, in the following graph f is concave upward to the left of its inflection point P and it is concave downward to the right of P.



It is easy to identify the y-coordinate of the inflection point P, because the graph of f is symmetrical about its inflection point. Thus the inflection point must occur halfway up the graph at a height of $y = c/2$.

Determine the x-coordinate of the inflection point of f. (*Hint:* Replace $f(x)$ with $c/2$ in Equation (1) and solve for x.)

Exploring Concepts with Technology

Use WolframAlpha to Find Exponential and Logarithmic Regression Functions

Scan the following QR code to access WolframAlpha on a mobile device.

www.wolframalpha.com

Exponential Regression

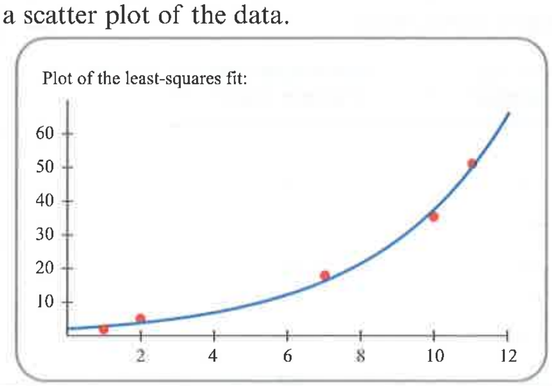
Enter the following text in the input field to find an exponential regression function for {(1, 2), (2, 5), (7, 18), (10, 35), (11, 51)}.

exponential fit {(1,2),(2,5),(7,18),(10,35),(11,51)}

Click on the equal sign icon to display the exponential regression function

$$y = 2.38874e^{0.275633x}$$

WolframAlpha also provides a graph of the exponential regression function and a scatter plot of the data.



Enter "2.38874e^(0.275633x),x=12" to display 65.2589 as the value of the exponential regression function at $x = 12$.

Note

WolframAlpha does not provide a convenient method for finding the logistic regression function for a given data set.

Logarithmic Regression

Enter the following text in the input field to find a logarithmic regression function for {(1, 3), (3, 4), (5, 4.5), (7, 5.2)}.

```
logarithmic fit {(1,3),(3,4),(5,4.5),(7,5.2)}
```

WolframAlpha displays the logarithmic regression function

$$y = 1.07211 \log(15.3438\, x)$$

Recall that WolframAlpha uses "log" to represent the natural logarithmic function. Thus this logarithmic regression function can be written as

$$y = 1.07211 \ln(15.3438\, x)$$

Caution: The TI-83/84 calculators and WolframAlpha use different algorithms to find exponential and logarithmic regression functions. Thus, for a given data set, the exponential and logarithmic regression functions produced using the TI-83/84 calculators and the exponential and logarithmic regression functions produced using WolframAlpha are not identical.

In the *Answers to Selected Exercises* appendix, the exponential and logarithmic regression functions produced using the TI calculators as well as the exponential and logarithmic regression functions produced using WolframAlpha are provided to accommodate users of each of these technologies.

CHAPTER 4 TEST PREP

The following test prep table summarizes essential concepts in this chapter. The references given in the right-hand column list Examples and Exercises that can be used to test your understanding of a concept.

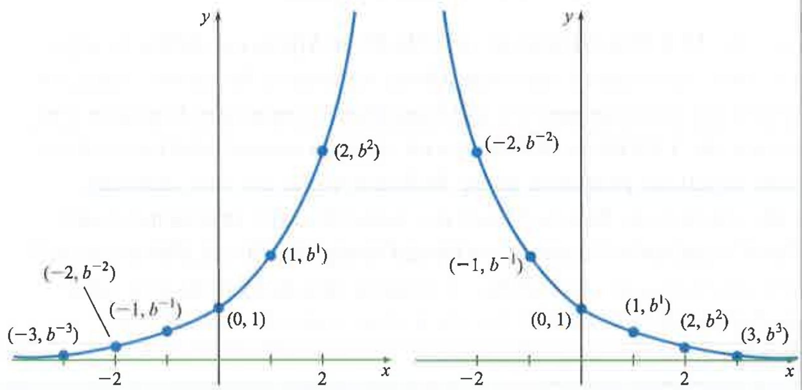
4.1 Inverse Functions

Graph the Inverse of a Function A function f has an inverse function if and only if it is a one-to-one function. The graph of f and the graph of its inverse f^{-1} are symmetric with respect to the line given by $y = x$.	See Example 1, page 338, and then try Exercises 1 and 2, page 423.
Composition of Inverse Functions Property If f is a one-to-one function, then f^{-1} is the inverse function of f if and only if $(f \circ f^{-1})(x) = f[f^{-1}(x)] = x$ for all x in the domain of f^{-1} and $(f^{-1} \circ f)(x) = f^{-1}[f(x)] = x$ for all x in the domain of f.	See Example 2, page 339, and then try Exercises 3 and 6, page 423.
Find the Inverse of a Function If a one-to-one function f is defined by an equation, then you can often use the following procedure to find the equation of f^{-1}. **1.** Substitute y for $f(x)$. **2.** Interchange x and y. **3.** Solve, if possible, for y in terms of x. **4.** Substitute $f^{-1}(x)$ for y.	See Examples 4 and 5, pages 341 and 342, and then try Exercises 9 and 11, page 423.

4.2 Exponential Functions and Their Applications

Properties of $f(x) = b^x$ For positive real numbers b, $b \neq 1$, the exponential function defined by $f(x) = b^x$ has the following properties.

- The function f is a one-to-one function. It has the set of real numbers as its domain and the set of positive real numbers as its range.
- The graph of f is a smooth, continuous curve with a y-intercept of $(0, 1)$, and the graph passes through $(1, b)$.
- If $b > 1$, f is an increasing function and its graph is asymptotic to the negative x-axis. See Figure a.
- If $0 < b < 1$, f is a decreasing function and its graph is asymptotic to the positive x-axis. See Figure b.



a. $f(x) = b^x, b > 1$ **b.** $f(x) = b^x, 0 < b < 1$

See Example 2, page 351, and then try Exercises 25 and 26, page 423.

Graphing Techniques The graphs of some functions can be constructed by translating, stretching, compressing, or reflecting another graph or by combining these techniques.

See Examples 3 and 4, pages 352 and 353, and then try Exercises 29 and 30, page 423.

Natural Exponential Function The number e is defined as the number that

$$\left(1 + \frac{1}{n}\right)^n$$

approaches as n increases without bound. The value of e accurate to 8 decimal places is 2.71828183. The function $f(x) = e^x$, where x is a real number, is called the natural exponential function. Many applications can be modeled by functions that involve e^{kx}, where k is a constant.

See Example 5, page 355, and then try Exercise 84, page 424.

4.3 Logarithmic Functions and Their Applications

Exponential and Logarithmic Form
The exponential form of $y = \log_b x$ is $b^y = x$.
The logarithmic form of $b^y = x$ is $y = \log_b x$.

See Examples 1 and 2, page 362, and then try Exercises 39 and 43, page 423.

Basic Logarithmic Properties
$\log_b b = 1$ $\quad \log_b 1 = 0$ $\quad \log_b (b^x) = x$ $\quad b^{\log_b x} = x$

See Example 3, page 363, and then try Exercise 16, page 423.

Properties of $f(x) = \log_b x$ For positive real numbers b, $b \neq 1$, the logarithmic function $f(x) = \log_b x$ has the following properties.

- The domain of f is the set of positive real numbers, and its range is the set of all real numbers.

See Example 4, page 364, and then try Exercises 31 and 32, page 423.