4.7 Transformations of Polynomial Functions

GO DIGITAL

Learning Target Describe and graph transformations of polynomial functions.

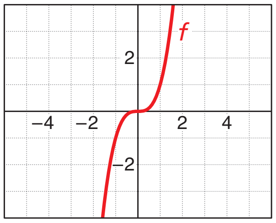
Success Criteria
- I can describe transformations of polynomial functions.
- I can graph transformations of polynomial functions.
- I can write functions that represent transformations of polynomial functions.

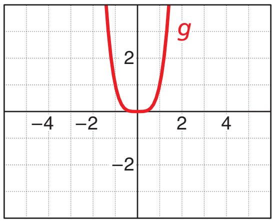
EXPLORE IT! Transforming Graphs of Cubic and Quartic Functions

Work with a partner. The graphs of the parent cubic function $f(x) = x^3$ and the parent quartic function $g(x) = x^4$ are shown.

Math Practice

Construct Arguments Why does the range of f include negative numbers, but the range of g does not?





In parts (a)–(h), use technology to explore each function for several values of k, h, and a. How does the graph change when you change the values of k, h, and a?

a. $y = f(x) + k$

b. $y = f(x - h)$

c. $y = a \cdot f(x)$

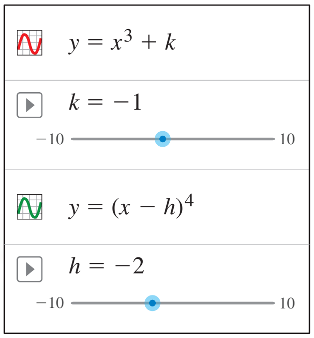
d. $y = f(ax)$

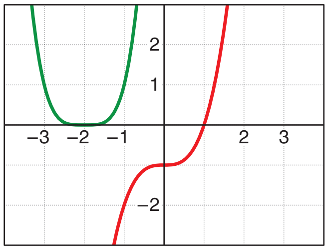
e. $y = g(x) + k$

f. $y = g(x - h)$

g. $y = a \cdot g(x)$

h. $y = g(ax)$





GO DIGITAL

Describing Transformations of Polynomial Functions

You can transform graphs of polynomial functions in the same way you transformed graphs of linear functions, absolute value functions, and quadratic functions. Examples of transformations of the graph of $f(x) = x^4$ are shown below.

KEY IDEAS

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = (x - 5)^4$ $g(x) = (x + 2)^4$	5 units right 2 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = x^4 + 1$ $g(x) = x^4 - 4$	1 unit up 4 units down
Reflection Graph flips over a line.	$f(-x)$ $-f(x)$	$g(x) = (-x)^4 = x^4$ $g(x) = -x^4$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis by a factor of $\frac{1}{a}$.	$f(ax)$	$g(x) = (2x)^4$ $g(x) = \left(\frac{1}{2}x\right)^4$	shrink by a factor of $\frac{1}{2}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis by a factor of a.	$a \cdot f(x)$	$g(x) = 8x^4$ $g(x) = \frac{1}{4}x^4$	stretch by a factor of 8 shrink by a factor of $\frac{1}{4}$

EXAMPLE 1 Translating a Polynomial Function

WATCH

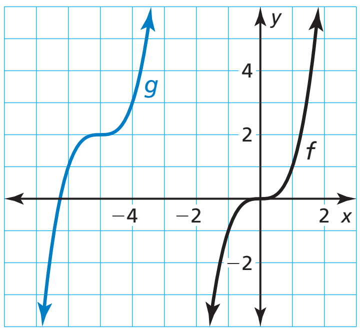
Describe the transformation of $f(x) = x^3$ represented by $g(x) = (x + 5)^3 + 2$. Then graph each function.

SOLUTION

Notice that the function is of the form $g(x) = (x - h)^3 + k$. Rewrite the function to identify h and k.

$$g(x) = (x - (-5))^3 + 2$$

h = −5, k = 2



Because $h = -5$ and $k = 2$, the graph of g is a translation 5 units left and 2 units up of the graph of f.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Describe the transformation of f represented by g. Then graph each function.

1. $f(x) = x^3$, $g(x) = x^3 - 2$

2. $f(x) = x^4$, $g(x) = (x - 3)^4 - 1$

3. **MP STRUCTURE** Describe the transformation of $f(x) = x^4$ represented by $g(x) = (x^2 - 1)(x^2 + 1)$.

GO DIGITAL

WATCH

EXAMPLE 2 Transforming Polynomial Functions

Describe the transformation of f represented by g. Then graph each function.

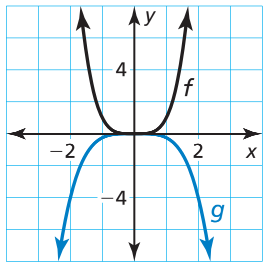
a. $f(x) = x^4$, $g(x) = -\frac{1}{4}x^4$

b. $f(x) = x^5$, $g(x) = (2x)^5 - 3$

SOLUTION

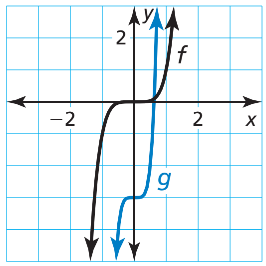
a. Notice that the function is of the form $g(x) = -ax^4$, where $a = \frac{1}{4}$.

▶ So, the graph of g is a reflection in the x-axis and a vertical shrink by a factor of $\frac{1}{4}$ of the graph of f.



b. Notice that the function is of the form $g(x) = (ax)^5 + k$, where $a = 2$ and $k = -3$.

▶ So, the graph of g is a horizontal shrink by a factor of $\frac{1}{2}$ and a translation 3 units down of the graph of f.



SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

4. Describe the transformation of $f(x) = x^3$ represented by $g(x) = 4(x + 2)^3$. Then graph each function.

5. **VOCABULARY** Describe how the vertex form of a quadratic function is similar to the form $f(x) = a(x - h)^3 + k$ for a cubic function.

Writing Transformations of Polynomial Functions

EXAMPLE 3 Writing Transformed Polynomial Functions

WATCH

Let $f(x) = x^3 + x^2 + 1$. Write a rule for g and then graph each function. Describe the graph of g as a transformation of the graph of f.

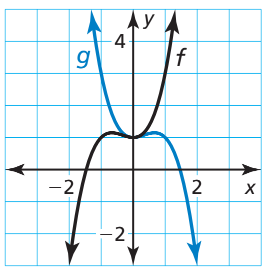
a. $g(x) = f(-x)$

b. $g(x) = 3f(x)$

SOLUTION

a. $g(x) = f(-x) = (-x)^3 + (-x)^2 + 1$

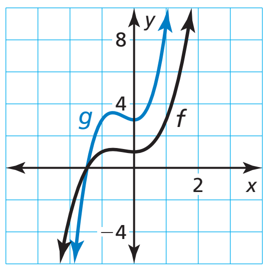
$= -x^3 + x^2 + 1$



▶ The graph of g is a reflection in the y-axis of the graph of f.

b. $g(x) = 3f(x) = 3(x^3 + x^2 + 1)$

$= 3x^3 + 3x^2 + 3$



▶ The graph of g is a vertical stretch by a factor of 3 of the graph of f.

REMEMBER

Vertical stretches and shrinks do not change the x-intercept(s) of a graph. You can observe this using the graph in Example 3(b).

EXAMPLE 4 Writing a Transformed Polynomial Function

WATCH

GO DIGITAL

Let the graph of g be a vertical stretch by a factor of 2, followed by a translation 3 units up of the graph of $f(x) = x^4 - 2x^2$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the vertical stretch of f.

$h(x) = 2 \cdot f(x)$ — Multiply the output by 2.

$= 2(x^4 - 2x^2)$ — Substitute $x^4 - 2x^2$ for $f(x)$.

$= 2x^4 - 4x^2$ — Distributive Property

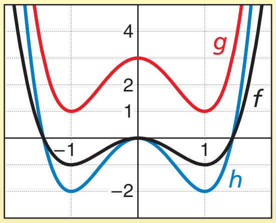
Step 2 Then write a function g that represents the translation of h.

$g(x) = h(x) + 3$ — Add 3 to the output.

$= 2x^4 - 4x^2 + 3$ — Substitute $2x^4 - 4x^2$ for $h(x)$.

▶ The transformed function is $g(x) = 2x^4 - 4x^2 + 3$.

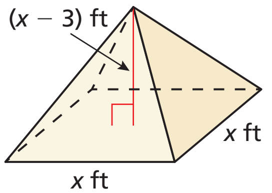
Check



EXAMPLE 5 Writing a Polynomial Model

WATCH

The function $V(x) = \frac{1}{3}x^3 - x^2$ represents the volume (in cubic feet) of the square pyramid shown. The function $W(x) = V(3x)$ represents the volume (in cubic feet) when x is measured in yards. Write a rule for W. Find and interpret $W(10)$.



SOLUTION

1. **Understand the Problem** You are given two volume functions V and W whose inputs have different units. The horizontal shrink shown by $W(x) = V(3x)$ makes sense because there are 3 feet in 1 yard. You are asked to write a rule for W and interpret the output for a given input.

2. **Make a Plan** Write the transformed function W and then find $W(10)$.

3. **Solve and Check** $W(x) = V(3x)$

$= \frac{1}{3}(3x)^3 - (3x)^2$ — Replace x with $3x$ in $V(x)$.

$= 9x^3 - 9x^2$ — Simplify.

Next, find $W(10)$.

$$W(10) = 9(10)^3 - 9(10)^2 = 9000 - 900 = 8100$$

▶ When x is 10 yards, the volume of the pyramid is 8100 cubic feet.

Check Because $W(x) = V(3x)$, you can determine that $W(10) = V(30)$. Check that your solution is correct by verifying that $V(30) = 8100$.

$V(30) = \frac{1}{3}(30)^3 - 30^2$

$= 9000 - 900$

$= 8100$ ✓

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

6. Let $f(x) = x^5 - 4x + 6$ and $g(x) = -f(x)$. Write a rule for g and then graph each function. Describe the graph of g as a transformation of the graph of f.

7. Let the graph of g be a horizontal stretch by a factor of 2, followed by a translation 3 units right of the graph of $f(x) = 8x^3 + 3$. Write a rule for g.

8. **WHAT IF?** In Example 5, the height of the pyramid is $6x$ feet, and the volume (in cubic feet) is represented by $V(x) = 2x^3$. Write a rule for W. Find and interpret $W(7)$.

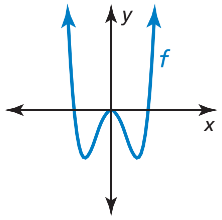
4.7 Practice WITH CalcChat® AND CalcView®

GO DIGITAL

In Exercises 1–4, describe the transformation of f represented by g. Then graph each function. *Example 1*

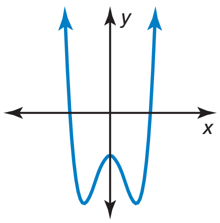
1. $f(x) = x^4$, $g(x) = x^4 + 3$
2. $f(x) = x^4$, $g(x) = (x - 5)^4$
3. $f(x) = x^5$, $g(x) = (x - 2)^5 - 1$
4. $f(x) = x^6$, $g(x) = (x + 1)^6 - 4$

ANALYZING RELATIONSHIPS **In Exercises 5–8, match the function with the correct transformation of the graph of f. Explain your reasoning.**

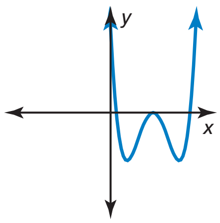


5. $y = f(x - 2)$
6. $y = f(x + 2) + 2$
7. $y = f(x - 2) + 2$
8. $y = f(x) - 2$

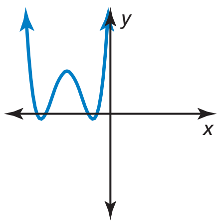
A.



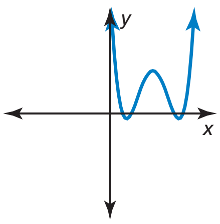
B.



C.



D.



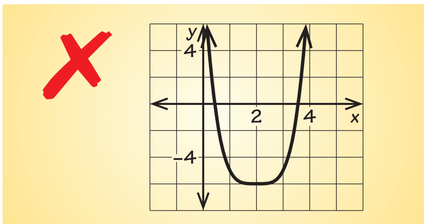
In Exercises 9–14, describe the transformation of f represented by g. Then graph each function. *Example 2*

9. $f(x) = x^4$, $g(x) = -2x^4$
10. $f(x) = x^6$, $g(x) = -\frac{1}{4}x^6$
11. $f(x) = x^3$, $g(x) = 5x^3 + 1$
12. $f(x) = x^4$, $g(x) = \frac{1}{2}x^4 + 5$
13. $f(x) = x^5$, $g(x) = \frac{3}{4}(x + 4)^5$
14. $f(x) = x^4$, $g(x) = (3x)^4 - 2$

In Exercises 15–20, write a rule for g and then graph each function. Describe the graph of g as a transformation of the graph of f. *Example 3*

15. $f(x) = x^4 + 1$, $g(x) = f(x + 2)$
16. $f(x) = x^6 - 3x^3 + 2$, $g(x) = f(x) - 3$
17. $f(x) = x^5 - 2x + 3$, $g(x) = 3f(x)$
18. $f(x) = 2x^3 - 2x^2 + 6$, $g(x) = -\frac{1}{2}f(x)$
19. $f(x) = x^4 + x^3 - 1$, $g(x) = f(-x) - 5$
20. $f(x) = \frac{1}{2}x^5 + x^3 - 4x - 4$, $g(x) = -f(2x) + 1$

21. **ERROR ANALYSIS** Describe and correct the error in graphing the transformation of $f(x) = x^4$ represented by $g(x) = (x + 2)^4 - 6$.



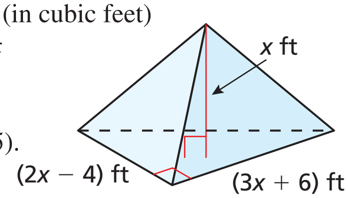
22. **ERROR ANALYSIS** Describe and correct the error in describing the transformation of $f(x) = x^5$ represented by $g(x) = (3x)^5 - 4$.

> ✗ The graph of g is a horizontal shrink by a factor of 3, followed by a translation 4 units down of the graph of f.

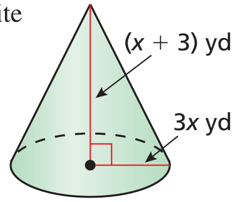
In Exercises 23–26, write a rule for g that represents the indicated transformations of the graph of f. *Example 4*

23. $f(x) = x^3 - 6$; translation 3 units left, followed by a reflection in the y-axis
24. $f(x) = x^4 + 2x + 6$; vertical stretch by a factor of 2, followed by a translation 4 units right
25. $f(x) = x^3 + 2x^2 - 9$; horizontal shrink by a factor of $\frac{1}{3}$ and a translation 2 units up, followed by a reflection in the x-axis
26. $f(x) = 2x^5 - x^3 + x^2 + 4$; reflection in the y-axis and a vertical stretch by a factor of 3, followed by a translation 1 unit down

27. **CONNECTING CONCEPTS** The function $V(x) = x^3 - 4x$ represents the volume (in cubic feet) of the pyramid. The function $W(x) = V(3x)$ represents the volume (in cubic feet) of the pyramid when x is measured in yards. Write a rule for W. Find and interpret $W(5)$. *Example 5*

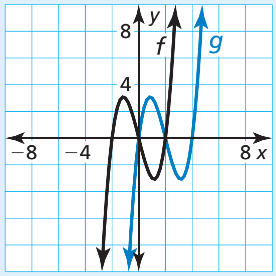


28. **CONNECTING CONCEPTS** Write a function V for the volume (in cubic yards) of the right circular cone shown. Then write a function W that represents the volume (in cubic yards) of the cone when x is measured in feet. Find and interpret $W(3)$.



29. **MAKING AN ARGUMENT** The function $V(x) = x^3$ represents the volume of a cube with edge length x. Does the volume decrease by a greater amount when you divide the volume in half or when you divide each side length in half? Justify your answer.

30. **HOW DO YOU SEE IT?** Describe the transformation of the graph of f represented by the graph of g.



31. **OPEN-ENDED** Describe two transformations of $f(x) = x^5$ where the order in which the transformations are performed is important. Then describe two transformations where the order is *not* important. Explain your reasoning.

32. **THOUGHT PROVOKING** Write a function g that has a y-intercept of -2 and is a transformation of $f(x) = -\frac{1}{4}(2x^2 - 3)(x + 2)^2$.

GO DIGITAL

REVIEW & REFRESH

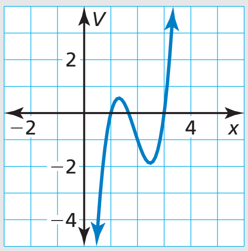
WATCH

In Exercises 33 and 34, find the minimum value or maximum value of the function. Find the domain and range of the function, and when the function is increasing and decreasing.

33. $h(x) = (x + 5)^2 - 7$ **34.** $f(x) = -2x^2 + 4x - 1$

35. Find all the real zeros of $f(x) = 2x^3 - 21x^2 + 12x + 72$.

36. **MODELING REAL LIFE** The volume (in cubic feet) of a dog kennel in the shape of a rectangular prism can be modeled by $V = 3x^3 - 17x^2 + 29x - 15$, where x is the length (in feet). Determine the values of x for which the model makes sense. Explain.



37. Write an equation in intercept form of the parabola that passes through $(-10, 10)$ and has x-intercepts -11 and -5.

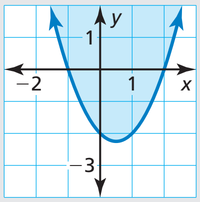
38. How many solutions does $x^4 + 8x^2 - 9 = 0$ have? Find all the solutions.

In Exercises 39–42, perform the operation. Write the answer in standard form.

39. $(12 - 4i) + (1 - i)$ **40.** $(3 + 8i) - (-6 + 2i)$

41. $7i(5 - 3i)$ **42.** $(9 - 11i)(-2 + 4i)$

43. Write an inequality that is represented by the graph.



44. Let $f(x) = -x^4 + 2x^2 - 3$ and $g(x) = 2f(x)$. Write a rule for g and then graph each function. Describe the graph of g as a transformation of the graph of f.

In Exercises 45 and 46, graph the function and its parent function. Then describe the transformation.

45. $g(x) = |x + 3|$ **46.** $h(x) = \frac{3}{2}x^2$