Integrating Technology

On a TI-83/TI-83 Plus/TI-84 Plus graphing calculator, the logistic growth function is given in the form

$$y = \frac{c}{1 + ae^{-bx}}$$

Think of the variable x as the time t and the variable y as $P(t)$.

2. ***Find the equation of the logistic growth function*** On a TI-83/TI-83 Plus/TI-84 Plus graphing calculator, select B: Logistic, which is in the STAT CALC menu. The logistic growth function for the data is

$$y \approx \frac{3965.3}{1 + 11.445e^{-0.31152x}}$$

See Figure 4.56.

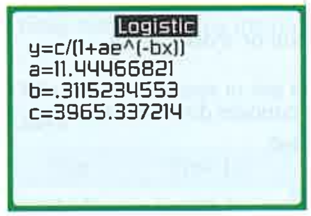


Figure 4.56

3. ***Examine the fit*** A TI-83/TI-83 Plus/TI-84 Plus calculator does not compute the coefficient of determination or the correlation coefficient for a logistic function. However, Figure 4.57 shows that the logistic function provides a good fit to the data. To use the function to predict the deer population in 2018 (the year 2018 is represented by an x value of 20), find the y value of the logistic function for $x = 20$. The VALUE command in the CALCULATE menu shows that the logistic function predicts a deer population of about 3878 in 2018. See Figure 4.58.

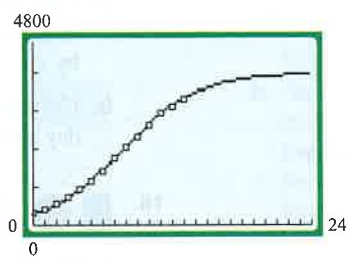


Figure 4.57

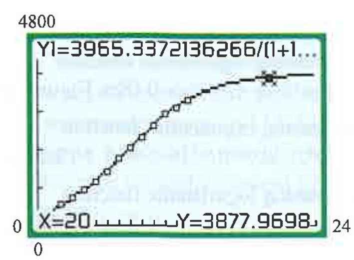


Figure 4.58

▸ **Try Exercise 24, page 416**

Answer graphs for Exercises 5–10 and answers for Exercises 11–14, 17–23, 25–27, 29, 30, and 34b are on pages AA19–AA20.

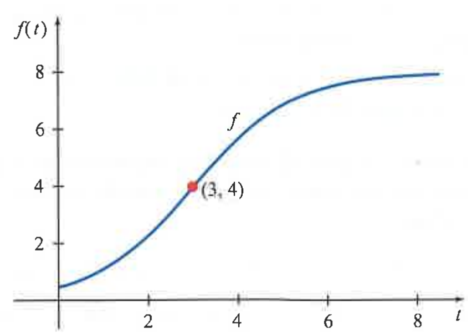
EXERCISE SET 4.7

Concept Check

1. The smooth continuous graph of f lies above all of its tangents on the interval $[x_1, x_2]$. Is f concave upward or concave downward on $[x_1, x_2]$? Concave upward

2. The following figure shows the graph of the logistic function

$$f(t) = \frac{8}{1 + 15e^{-0.9026834t}}, \quad t > 0$$



Indicates Try It Exercises

The point (3, 4) on the graph of f is called an *inflection point*, because the graph changes its concavity at that point. Examine the graph to determine whether f is concave upward or concave downward on the following intervals.

a. $[0, 3]$ Concave upward

b. $[3, \infty)$ Concave downward

3. Let $f(x) = a + b \ln x, b < 0$.

a. What is the domain of f? All positive real numbers

b. Is f an increasing function or a decreasing function? Decreasing

c. Is f concave upward or concave downward on its domain? Concave upward

4. Let $f(x) = a + b \ln x, b > 0$.

a. What is the domain of f? All positive real numbers

b. Is f an increasing function or a decreasing function? Increasing

c. Is f concave upward or concave downward on its domain? Concave downward

In Exercises 5 to 10, use a scatter plot of the given data to determine which of the following types of functions might provide a suitable model of the data.

- **An increasing exponential function**
  $y = ab^x, \quad a > 0, \quad b > 1$ (See Figure 4.43a, page 407.)
- **An increasing logarithmic function**
  $y = a + b \ln x, \quad b > 0$ (See Figure 4.43c.)
- **A decreasing exponential function**
  $y = ab^x, \quad a > 0, \quad 0 < b < 1$ (See Figure 4.43b.)
- **A decreasing logarithmic function**
  $y = a + b \ln x, \quad b < 0$ (See Figure 4.43d.)

(*Note:* Some data sets can be closely modeled by more than one type of function.)

5. $\{(1,3),(1.5,4),(2,6),(3,13),(3.5,19),(4,27)\}$
Increasing exponential function

6. $\{(1.0,1.12),(2.1,0.87),(3.2,0.68),(3.5,0.63),(4.4,0.52)\}$
Decreasing exponential function, decreasing logarithmic function

7. $\{(1,2.4),(2,1.1),(3,0.5),(4,0.2),(5,0.1)\}$
Decreasing exponential function, decreasing logarithmic function

8. $\{(5,2.3),(7,3.9),(9,4.5),(12,5.0),(16,5.4),(21,5.8),(26,6.1)\}$
Increasing logarithmic function

9. $\{(1,2.5),(1.5,1.7),(2,0.7),(3,-0.5),(3.5,-1.3),(4,-1.5)\}$
Decreasing logarithmic function

10. $\{(1,3),(1.5,3.8),(2,4.4),(3,5.2),(4,5.8),(6,6.6)\}$
Increasing logarithmic function

In Exercises 11 and 12, find the exponential regression function for the data. State the coefficient of determination.

11. $\{(10,6.8),(12,6.9),(14,15.0),(16,16.1),(18,50.0),(19,68.5)\}$

12. $\{(2.6,16.2),(3.8,48.8),(5.1,160.1),(6.5,590.2),(7,911.2)\}$

In Exercises 13 and 14, find the logarithmic regression function for the data. State the coefficient of determination.

13. $\{(5,2.7),(6,2.5),(7.2,2.2),(9.3,1.9),(11.4,1.6),(14.2,1.3)\}$

14. $\{(11,15.75),(14,15.52),(17,15.34),(20,15.18),(23,15.05)\}$

In Exercises 15 and 16, find the logistic regression function for the data.

15. $\{(0,81),(2,87),(6,98),(10,110),(15,125)\}$
$y \approx \dfrac{235.58598}{1 + 1.90188e^{-0.05101x}}$

16. $\{(0,175),(5,195),(10,217),(20,264),(35,341)\}$
$y \approx \dfrac{710.56899}{1 + 3.06263e^{-0.02964x}}$

17. **Lift Ticket Prices** The following table shows the price of an all-day lift ticket at a ski resort for selected years from 2001 to 2013.

All-Day Lift Ticket Prices, 2001–2013

Year	Price	Year	Price
2001	\$38	2009	\$75
2003	\$42	2011	\$87
2005	\$54	2013	\$101
2007	\$61		

a. Find an exponential regression function for the data. Represent 2001 by $x = 1$, 2003 by $x = 3, \ldots,$ and 2013 by $x = 13$.

b. Use the exponential function to predict the price of an all-day lift ticket for 2016 ($x = 16$). Round to the nearest dollar.

18. **Recycling Rates** U.S. recycling rates have been increasing over the last few decades. The following table shows the percent of municipal solid waste (MSW) that was recycled for selected years from 1960 to 2009.

MSW Recycling Rates, 1960–2009

Year	Recycling Rate	Year	Recycling Rate
1960	6.4%	1995	26.0%
1970	6.6%	2000	29.1%
1980	9.6%	2006	32.5%
1990	16.2%	2009	33.8%

Source: U.S. Environmental Protection Agency.

a. Find an exponential regression function and a logarithmic regression function that model the recycling rate R for the year t. Use $t = 60$ to represent 1960, $t = 70$ to represent 1970, . . . , and $t = 109$ to represent 2009.

b. Examine the coefficient of determination of the two regression functions to determine which provides a better fit for the data.

c. Use each of the regression functions to predict the recycling rate for 2013. Round each rate to the nearest tenth of a percent.

b. Use the exponential function to predict the median price of existing homes in the United States in 2014. Round to the nearest thousand dollars.

c. Use the logarithmic function to predict the median price of existing homes in the United States in 2014. Round to the nearest thousand dollars.

24. **Population of Hawaii** The following table shows the population of the state of Hawaii for selected years from 1950 to 2010.

Population of Hawaii, 1950–2010

Year	Population, P
1950	499,000
1960	642,000
1970	762,920
1980	967,710
1990	1,113,491
2000	1,211,566
2010	1,360,301

Source: Economagic.com.

a. Find the logistic regression function that approximates the population of Hawaii as a function of the year, t. Use $t = 0$ to represent 1950, $t = 10$ to represent 1960, . . . , and $t = 60$ to represent 2010. $P(t) = \dfrac{1{,}706{,}969.905}{1 + 2.440345097e^{-0.0371324816t}}$

b. Use the logistic regression function to predict the population of Hawaii in 2015 (represented by $t = 65$). Round to the nearest thousand. 1,401,000 people

c. What is the carrying capacity of the logistic model? Round to the nearest thousand. 1,707,000 people

25. **Optometry** The *near point p* of a person is the closest distance at which the person can see an object distinctly. As one grows older, one's near point increases. The table below shows data for the average near point of various people with normal eyesight.

Age y (years)	Near Point p (cm)
15	11
20	13
25	15
30	17
35	20
40	23
50	26

a. Find an exponential regression function for these data.

b. What near point does this function predict for a person 60 years old? Round to the nearest centimeter.

26. **Chemistry** The amount of oxygen x, in milliliters per liter, that can be absorbed by water at a certain temperature T, in degrees Fahrenheit, is given in the following table.

Temperature (°F)	Oxygen Absorbed (ml/L)
32	10.5
38	8.4
46	7.6
52	7.1
58	6.8
64	6.5

a. Find a logarithmic regression function for these data.

b. Using your function, how much oxygen, to the nearest tenth of a milliliter per liter, can be absorbed in water that is 50°F?

27. **The Henderson–Hasselbach Function** The scientists Henderson and Hasselbach determined that the pH of blood is a function of the ratio q of the amounts of bicarbonate and carbonic acid in the blood.

a. Determine a logarithmic regression function for the data. Use q as the independent variable (domain) and pH as the dependent variable (range).

q	7.9	12.6	31.6	50.1	79.4
pH	7.0	7.2	7.6	7.8	8.0

b. Use the logarithmic regression function to find the q value associated with a pH of 8.2. Round to the nearest tenth.

28. **World Population** The following table lists the years in which the world's population first reached 3 billion, 4 billion, 5 billion, 6 billion, and 7 billion.

World Population Milestones

Year	Population, P
1960	3 billion
1974	4 billion
1987	5 billion
1999	6 billion
2012	7 billion

Source: Geohive.com.

a. $P(t) = \dfrac{10.86679441}{1 + 2.620127542e^{-0.0299591869t}}$

a. Find a logistic regression function P for the data. Let t represent the number of years after 1960.

b. Use the logistic function to predict the year in which the world's population will first reach 8 billion. 2026

c. According to the logistic function, what will the world's population approach as $t \to \infty$? Round to the nearest billion. 11 billion people

29. **Average Ticket Prices** The following table shows the average movie theater ticket price, in the United States, for the years from 2004 to 2011.

Average U.S. Movie Theater Ticket Prices

Year	Average Ticket Price (dollars)
2004	6.21
2005	6.41
2006	6.55
2007	6.88
2008	7.18
2009	7.50
2010	7.89
2011	7.93

Source: The World Almanac and Book of Facts 2013.

a. Find an exponential regression function and a logarithmic regression function that model the average ticket price P, as a function of the year t. Use $t = 4$ to represent 2004, $t = 5$ to represent 2005, . . . , and $t = 11$ to represent 2011.

b. Which function provides a better fit to the data?

c. Use the regression function you selected in **b** to predict the average movie theater ticket price in 2015 ($t = 15$).

30. Temperature of Coffee A cup of coffee is placed in a room that maintains a constant temperature of 70°F. The following table shows both the coffee temperature T after t minutes and the difference between the coffee temperature and the room temperature after t minutes.

Time t (minutes)	0	5	10	15	20	25
Coffee Temp. T (°F)	165°	140°	121°	107°	97°	89°
$T - 70°$	95°	70°	51°	37°	27°	19°

a. Find an exponential regression function for the difference $T - 70$ as a function of t.

b. Use the regression function in **a** to predict how long it will take (to the nearest minute) for the coffee to cool to 80°F.

31. A Cat's Age in Human Years The following table lists the approximate age, in human years, of a cat as it ages from 2 months to 24 months.

A Cat's Age in Human Years

Calendar Age, x (in months)	Approximate Age (in human years)
2	3
4	6
6	9
8	11
10	13
12	15
18	20
24	24

Source: The Cat Owner's Manual, by Quirk Books.

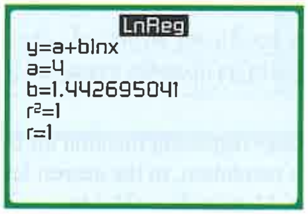
a. Find a logistic regression function A that models the data. $A(x) = \dfrac{24.64946104}{1 + 6.879954697e^{-0.2019509749x}}$

b. Use the logistic function to estimate the age, in human years, of a 5-month-old cat. Round to the nearest tenth of a year. Do you think this is a realistic estimate? 7.0 years; answers will vary.

c. Use the logistic function to estimate the age, in human years, of a 72-month-old cat. Round to the nearest tenth of a year. Do you think this is a realistic estimate? 24.6 years; answers will vary.

d. The table in this exercise is the same as the table in Exercise 73, page 285, where you used a cubic regression function to model the data and to make estimations. In this exercise, you used a logistic regression function to model the data and make estimations. Write a few sentences that compare the strengths and the weaknesses of each function regarding its ability to model the data and produce reasonable estimations. Answers will vary.

32. A Correlation Coefficient of 1 A scientist uses a graphing calculator to model the data set $\{(2,5),(4,6)\}$ with a logarithmic function. The following display shows the results.



What is the significance of the fact that the correlation coefficient for the regression function is $r = 1$? The graph of the logarithmic regression function passes through both data points.

33. Duplicate Data Points An engineer needs to model the data in set A with an exponential function.

$$A = \{(2,5),(3,10),(4,17),(4,17),(5,28)\}$$

Because the ordered pair $(4,17)$ is listed twice, the engineer decides to eliminate one of these ordered pairs and model the data in set B.

$$B = \{(2,5),(3,10),(4,17),(5,28)\}$$

Determine whether A and B both have the same exponential regression function.
A and *B* have different exponential regression functions.

34. Domain Error A scientist needs to model the data in set A.

$$A = \{(0,1.2),(1,2.3),(2,2.8),(3,3.1),(4,3.3),(5,3.4)\}$$

The scientist views a scatter plot of the data and decides to model the data with a logarithmic function of the form $y = a + b\ln x$.

a. When the scientist attempts to use a graphing calculator to determine the logarithmic regression function, the calculator displays the message

"ERR:DOMAIN"

Explain why the calculator was unable to determine the logarithmic regression function for the data. The x-coordinate of the first ordered pair is 0, and 0 is not in the domain of $y = \ln x$.

b. Explain what the scientist could do so that the data in set A could be modeled by a logarithmic function of the form $y = a + b \ln x$.

Enrichment Exercises

35. Power Functions A function that can be written in the form $y = ax^b$ is said to be a **power function.** Some data sets can best be modeled by a power function. On a TI-83/84 calculator, the PwrReg instruction is used to produce a power regression function for a set of data. Find the power regression function for the following data.

$y \approx 2.09385(x)^{1.40246}$

x	1	2	3	4	5	6
y	2.1	5.5	9.8	14.6	20.1	25.8

36. Period of a Pendulum The following table shows the time t (in seconds) of the period of a pendulum of length l (in feet). (*Note*: The period of a pendulum is the time it takes the pendulum to complete a swing from the right to the left and back.)

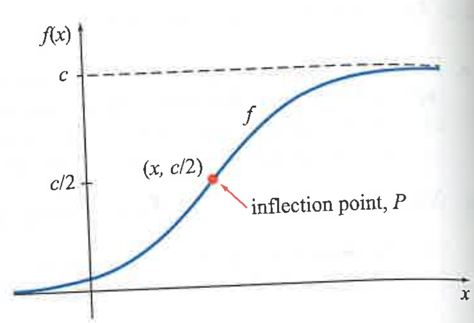
Length l (*ft*)	1	2	3	4	6	8
Time t (*s*)	1.11	1.57	1.92	2.25	2.72	3.14

Use the power regression function for the data to estimate the length of a pendulum, to the nearest tenth of a foot, that has a period of 12 seconds. 115.4 ft

37. Inflection Point of a Logistic Function An important feature of the graph of a logistic function,

$$f(x) = \frac{c}{1 + ae^{-bx}} \qquad (1)$$

with domain the set of real numbers, is its shape. The graph of every logistic function has an S-shape and a single inflection point, which separates the graph into two equal regions of opposite concavity. For instance, in the following graph f is concave upward to the left of its inflection point P and it is concave downward to the right of P.



It is easy to identify the y-coordinate of the inflection point P, because the graph of f is symmetrical about its inflection point. Thus the inflection point must occur halfway up the graph at a height of $y = c/2$.

Determine the x-coordinate of the inflection point of f. (*Hint:* Replace $f(x)$ with $c/2$ in Equation (1) and solve for x.)

$x = \dfrac{\ln a}{b}$

Scan the following QR code to access WolframAlpha on a mobile device.

www.wolframalpha.com

Exploring Concepts with Technology

Use WolframAlpha to Find Exponential and Logarithmic Regression Functions

Exponential Regression

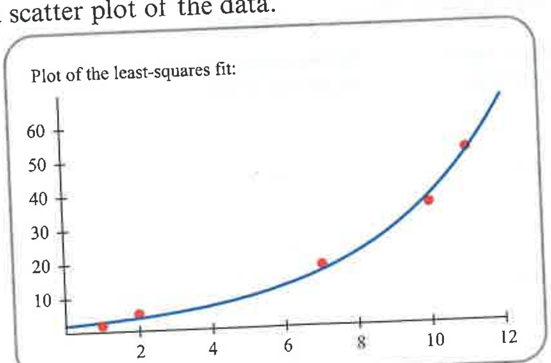
Enter the following text in the input field to find an exponential regression function for {(1, 2), (2, 5), (7, 18), (10, 35), (11, 51)}.

exponential fit {(1,2),(2,5),(7,18),(10,35),(11,51)}

Click on the equal sign icon to display the exponential regression function

$$y = 2.38874e^{0.275633x}$$

WolframAlpha also provides a graph of the exponential regression function and a scatter plot of the data.



Enter "2.38874e^(0.275633x),x=12" to display 65.2589 as the value of the exponential regression function at $x = 12$.