76. **Population Growth** A yeast culture grows according to the equation

$$Y = \frac{50{,}000}{1 + 250e^{-0.305t}}$$

where Y is the number of yeast and t is time in hours.

a. Graph the equation for $t \geq 0$.

b. Use the graph to estimate (to the nearest hour) the number of hours before the yeast population reaches 35,000.

c. Use the graph to estimate the horizontal asymptote.

d. Write a sentence that explains the meaning of the horizontal asymptote in the context of this application.

77. **Consumption of Natural Resources** A model for how long our coal resources will last is given by

$$T = \frac{\ln(300r + 1)}{\ln(r + 1)}$$

where r is the percent increase in consumption from current levels of use and T is the time, in years, before the resources are depleted.

a. Graph this equation.

b. If our consumption of coal increases by 3% per year, in how many years will we deplete our coal resources?

c. What percent increase in consumption of coal will deplete the resources in 100 years? Round to the nearest tenth of a percent.

78. **Effects of Air Resistance on Velocity** If we assume that air resistance is proportional to the square of the velocity, then the time t, in seconds, required for an object to reach a velocity v in feet per second is given by

$$t = \frac{9}{24}\ln\frac{24 + v}{24 - v}, 0 \leq v < 24$$

a. Determine the velocity, to the nearest hundredth of a foot per second, of the object after 1.5 seconds.

b. Determine the vertical asymptote for the graph of this function.

c. Write a sentence that explains the meaning of the vertical asymptote in the context of this application.

79. **Terminal Velocity with Air Resistance** The velocity v, in feet per second, of an object t seconds after it has been dropped from a height above the surface of the Earth is given by the equation $v = 32t$, assuming no air resistance. If we assume that air resistance is proportional to the square of the velocity, then the velocity after t seconds is given by

$$v = 100\left(\frac{e^{0.64t} - 1}{e^{0.64t} + 1}\right)$$

a. In how many seconds will the velocity be 50 feet per second?

b. Determine the horizontal asymptote for the graph of this function.

c. Write a sentence that explains the meaning of the horizontal asymptote in the context of this application.

80. **Effects of Air Resistance on Distance** The distance s, in feet, that the object in Exercise 79 will fall in t seconds is given by

$$s = \frac{100^2}{32}\ln\left(\frac{e^{0.32t} + e^{-0.32t}}{2}\right)$$

a. Graph this equation for $t \geq 0$.

b. How long does it take for the object to fall 100 feet? Round to the nearest tenth of a second.

81. **Retirement Planning** The retirement account for a graphic designer contains $250,000 on January 1, 2013, and earns interest at a rate of 0.5% per month. On February 1, 2013, the designer withdraws $2000 and plans to continue these withdrawals as retirement income each month. The value V of the account after x months is

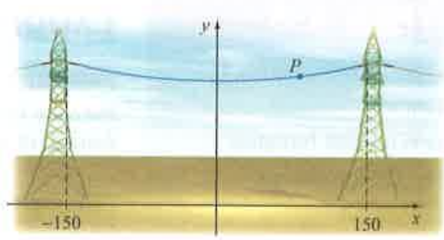
$$V = 400{,}000 - 150{,}000(1.005)^x$$

If the designer wishes to leave $100,000 to a scholarship foundation, what is the maximum number of withdrawals the designer can make from this account and still have $100,000 to donate?

82. **Transmission Cable** The height h, in feet, of any point P on the cable shown below is given by

$$h(x) = 40(e^{x/216.4} + e^{-x/216.4}), \quad -150 \leq x \leq 150$$

where $|x|$ is the horizontal distance, in feet, between P and the y-axis.



a. What is the lowest height of the cable?

b. What is the height of the cable 100 feet to the right of the y-axis? Round to the nearest tenth of a foot.

c. How far to the right of the y-axis is the cable 90 feet in height? Round to the nearest tenth of a foot.

83. The following argument seems to indicate that $0.125 > 0.25$. Find the first incorrect statement in the argument.

$$3 > 2$$
$$3(\log 0.5) > 2(\log 0.5)$$
$$\log 0.5^3 > \log 0.5^2$$
$$0.5^3 > 0.5^2$$
$$0.125 > 0.25$$

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84. The following argument seems to indicate that $4 = 6$. Find the first incorrect statement in the argument.

$$\begin{aligned}4 &= \log_2 16\\ 4 &= \log_2(8 + 8)\\ 4 &= \log_2 8 + \log_2 8\\ 4 &= 3 + 3\\ 4 &= 6\end{aligned}$$

85. A common mistake that students make is to write $\log(x + y)$ as $\log x + \log y$. If $\log(x + y) = \log x + \log y$, then what is the relationship between x and y? (*Hint:* Solve for x in terms of y.)

86. Let $f(x) = 2 \ln x$ and $g(x) = \ln x^2$. Does $f(x) = g(x)$ for all x?

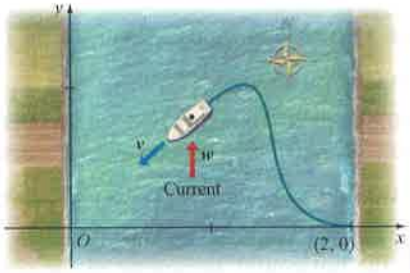
87. Explain why the functions $F(x) = 1.4^x$ and $G(x) = e^{0.336x}$ represent essentially the same function.

88. Find k such that $f(t) = 2.2^t$ and $g(t) = e^{-kt}$ represent essentially the same function.

Enrichment Exercises

89. **Navigating** The pilot of a boat is trying to cross a river to a point O, 2 miles due west of the boat's starting position, by always pointing the nose of the boat toward O. Suppose the speed of the current is w miles per hour and the speed of the boat is v miles per hour. If point O is the origin and the boat's starting position is (2, 0), then the equation of the boat's path is given by

$$y = \left(\frac{x}{2}\right)^{1-(w/v)} - \left(\frac{x}{2}\right)^{1+(w/v)}$$



a. If the speed of the current and the speed of the boat are the same, can the pilot reach point O by always having the nose of the boat pointed toward O? If not, at what point will the pilot arrive? Explain.

b. If the speed of the current is greater than the speed of the boat, can the pilot reach point O by always pointing the nose of the boat toward point O? If not, where will the pilot arrive? Explain.

c. If the speed of the current is less than the speed of the boat, can the pilot reach O by always pointing the nose of the boat toward O? If not, where will the pilot arrive? Explain.

SECTION 4.6

Exponential Growth and Decay

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A27.

PS1. Evaluate $A = 1000\left(1 + \frac{0.1}{12}\right)^{12t}$ for $t = 2$. Round to the nearest hundredth. [4.2]

PS2. Evaluate $A = 600\left(1 + \frac{0.04}{4}\right)^{4t}$ for $t = 8$. Round to the nearest hundredth. [4.2]

PS3. Solve $0.5 = e^{14k}$ for k. Round to the nearest ten-thousandth. [4.5]

PS4. Solve $0.85 = 0.5^{t/5730}$ for t. Round to the nearest ten. [4.5]

PS5. Solve $6 = \dfrac{70}{5 + 9e^{-k\cdot 12}}$ for k. Round to the nearest thousandth. [4.5]

PS6. Solve $2{,}000{,}000 = \dfrac{3^{n+1} - 3}{2}$ for n. Round to the nearest tenth. [4.5]

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Exponential Growth and Decay

In many applications, a quantity changes at a rate proportional to the amount present. In these applications, the amount present at time t is given by a special function called an *exponential growth function* or an *exponential decay function.*

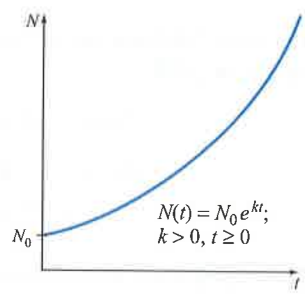
Definition of Exponential Growth and Decay Functions

If a quantity N increases or decreases at a rate proportional to the amount present at time t, then the quantity can be modeled by

$$N(t) = N_0 e^{kt}$$

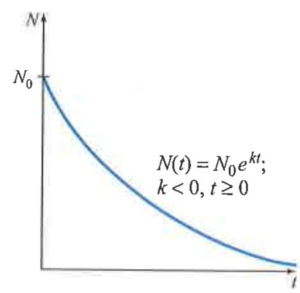
where N_0 is the value of N at time $t = 0$ and k is a constant called the **growth rate constant.**

- If k is positive, N increases as t increases and $N(t) = N_0 e^{kt}$ is called an **exponential growth function.** See Figure 4.39.
- If k is negative, N decreases as t increases and $N(t) = N_0 e^{kt}$ is called an **exponential decay function.** See Figure 4.40.



Exponential growth function

Figure 4.39



Exponential decay function

Figure 4.40

Question • Is $N(t) = 1450e^{0.05t}$ an exponential growth function or an exponential decay function?

In Example 1, we find an exponential growth function that models the population growth of a city.

EXAMPLE 1 Find the Exponential Growth Function That Models Population Growth

a. The population of a city is growing exponentially. The population of the city was 16,400 in 2003 and 20,200 in 2013. Find the exponential growth function that models the population growth of the city.

b. Use the function from **a** to predict, to the nearest 100, the population of the city in 2018.

(continued)

Answer • Because the growth rate constant $k = 0.05$ is positive, the function is an exponential growth function.

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Solution

a. We need to determine N_0 and k in $N(t) = N_0 e^{kt}$. If we represent 2003 by $t = 0$, then our given data are $N(0) = 16{,}400$ and $N(10) = 20{,}200$. Because N_0 is defined to be $N(0)$, we know that $N_0 = 16{,}400$. To determine k, substitute $t = 10$ and $N_0 = 16{,}400$ into $N(t) = N_0 e^{kt}$ to produce

$$N(10) = 16{,}400e^{k \cdot 10}$$

$$20{,}200 = 16{,}400e^{10k}$$ • Substitute 20,200 for $N(10)$.

$$\frac{20{,}200}{16{,}400} = e^{10k}$$ • Solve for e^{10k}.

$$\ln\frac{20{,}200}{16{,}400} = 10k$$ • Write in logarithmic form.

$$\frac{1}{10}\ln\frac{20{,}200}{16{,}400} = k$$ • Solve for k.

$$0.0208 \approx k$$

The exponential growth function is $N(t) \approx 16{,}400e^{0.0208t}$.

b. The year 2003 was represented by $t = 0$, so we will use $t = 15$ to represent 2018.

$$N(t) \approx 16{,}400e^{0.0208t}$$

$$N(15) \approx 16{,}400e^{0.0208 \cdot 15}$$

$$\approx 22{,}400$$ • Round to the nearest 100.

The exponential growth function yields 22,400 as the approximate population of the city in 2018.

▶ Try Exercise 10, page 402

Many radioactive materials *decrease* in mass exponentially over time. This decrease, called radioactive decay, is measured in terms of **half-life,** which is defined as the time required for the disintegration of half the atoms in a sample of a radioactive substance. Table 4.11 shows the half-lives of selected radioactive isotopes.

Table 4.11

Isotope	Half-Life
Carbon (^{14}C)	5730 years
Radium (^{226}Ra)	1660 years
Polonium (^{210}Po)	138 days
Phosphorus (^{32}P)	14 days
Polonium (^{214}Po)	1/10,000 of a second

EXAMPLE 2 Find an Exponential Decay Function

Find the exponential decay function for the amount of phosphorus (^{32}P) that remains in a sample after t days.

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Solution

When $t = 0$, $N(0) = N_0e^{k(0)} = N_0$. Thus $N(0) = N_0$. Also, because the phosphorus has a half-life of 14 days (from Table 4.11), $N(14) = 0.5N_0$. To find k, substitute $t = 14$ into $N(t) = N_0e^{kt}$ and solve for k.

$$N(14) = N_0 \cdot e^{k \cdot 14}$$

$$0.5N_0 = N_0e^{14k} \quad \text{• Substitute } 0.5N_0 \text{ for } N(14).$$

$$0.5 = e^{14k} \quad \text{• Divide each side by } N_0.$$

$$\ln 0.5 = 14k \quad \text{• Write in logarithmic form.}$$

$$\frac{1}{14}\ln 0.5 = k \quad \text{• Solve for } k.$$

$$-0.0495 \approx k$$

The exponential decay function is $N(t) \approx N_0e^{-0.0495t}$.

▶ Try Exercise 12, page 402

Study tip

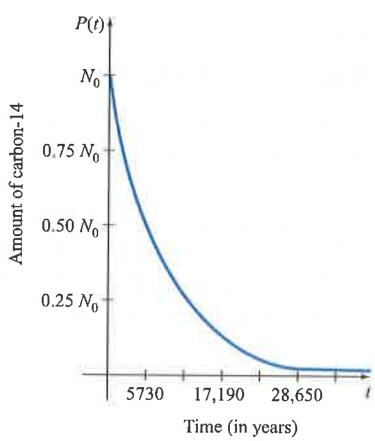
Because $e^{-0.0495} \approx (0.5)^{1/14}$, the decay function $N(t) = N_0e^{-0.0495t}$ can also be written as $N(t) = N_0(0.5)^{t/14}$. In this form, it is easy to see that if t is increased by 14 then N will decrease by a factor of 0.5.

Carbon Dating

The bone tissue in all living animals contains both carbon-12, which is nonradioactive, and carbon-14, which is radioactive and has a half-life of approximately 5730 years. See Figure 4.41. As long as the animal is alive, the ratio of carbon-14 to carbon-12 remains constant. When the animal dies ($t = 0$), the carbon-14 begins to decay. Thus a bone that has a smaller ratio of carbon-14 to carbon-12 is older than a bone that has a larger ratio. The percent of carbon-14 present at time t, in years, is

$$P(t) = 0.5^{t/5730}$$

The process of using the percent of carbon-14 present at a given time to estimate the age of a bone is called **carbon dating.**



$P(t) = 0.5^{t/5730}$

Figure 4.41

EXAMPLE 3 A Carbon Dating Application

Estimate the age of a bone if it now has 85% of the carbon-14 it had at time $t = 0$.

Solution

Let t be the time, in years, at which $P(t) = 0.85$.

$$0.85 = 0.5^{t/5730}$$

$$\ln 0.85 = \ln 0.5^{t/5730} \quad \text{• Take the natural logarithm of each side.}$$

$$\ln 0.85 = \frac{t}{5730}\ln 0.5 \quad \text{• Apply the power property.}$$

$$5730\left(\frac{\ln 0.85}{\ln 0.5}\right) = t \quad \text{• Solve for } t.$$

$$1340 \approx t$$

The bone is approximately 1340 years old.

▶ Try Exercise 16, page 403

Math Matters

The chemist Willard Frank Libby developed the carbon dating process in 1947. In 1960 he was awarded the Nobel Prize in chemistry for this achievement.

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Compound Interest Formulas

Interest is money paid for the use of money. The interest I is called **simple interest** if it is a fixed percent r, per time period t, of the amount of money invested. The amount of money invested is called the **principal** P. Simple interest is computed using the formula $I = Prt$. For example, if \$1000 is invested at 12% for 3 years, the simple interest is

$$I = Prt = \$1000(0.12)(3) = \$360$$

The balance after t years is $A = P + I = P + Prt$. In the preceding example, the \$1000 invested for 3 years produced \$360 interest. Thus the balance after 3 years is \$1000 + \$360 = \$1360.

In many financial transactions, interest is added to the principal at regular intervals so that interest is paid on interest, as well as on the principal. Interest earned in this manner is called **compound interest.** For example, if \$1000 is invested at 12% annual interest compounded annually for 3 years, then the total interest after 3 years is

First-year interest	\$1000(0.12) = \$120.00	
Second-year interest	\$1120(0.12) = \$134.40	
Third-year interest	\$1254.40(0.12) ≈ \$150.53	
	\$404.93	• Total interest

This method of computing the balance can be tedious and time-consuming. A *compound interest formula* can be used to determine the balance due after t years of compounding.

Note that if P dollars is invested at an interest rate of r per year, then the balance after 1 year is $A_1 = P + Pr = P(1 + r)$, where Pr represents the interest earned for the year. Observe that A_1 is the product of the original principal P and $(1 + r)$. If the amount A_1 is reinvested for another year, then the balance after the second year is

$$A_2 = (A_1)(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$$

Table 4.12

Number of Years	Balance
3	$A_3 = P(1 + r)^3$
4	$A_4 = P(1 + r)^4$
⋮	⋮
t	$A_t = P(1 + r)^t$

Successive reinvestments lead to the results shown in Table 4.12. The equation $A_t = P(1 + r)^t$ is valid if r is the annual interest rate paid during each of the t years.

If r is an annual interest rate and n is the number of compounding periods per year, then the interest rate each period is r/n, and the number of compounding periods after t years is nt. Thus the compound interest formula is as follows.

Compound Interest Formula

A principal P invested at an annual interest rate r, expressed as a decimal and compounded n times per year for t years, produces the balance

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

EXAMPLE 4 Solve a Compound Interest Application

Find the balance if \$1000 is invested at an annual interest rate of 10% for 2 years compounded on the following basis.

a. Monthly

b. Daily

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Solution

a. Because there are 12 months in a year, use $n = 12$.

$$A = \$1000\left(1 + \frac{0.1}{12}\right)^{12\cdot 2} \approx \$1000(1.008333333)^{24} \approx \$1220.39$$

b. Because there are 365 days in a year, use $n = 365$.

$$A = \$1000\left(1 + \frac{0.1}{365}\right)^{365\cdot 2} \approx \$1000(1.000273973)^{730} \approx \$1221.37$$

▶ **Try Exercise 20, page 403**

To **compound continuously** means to increase the number of compounding periods per year, n, without bound.

To derive a continuous compounding interest formula, substitute $\frac{1}{m}$ for $\frac{r}{n}$ in the compound interest formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \tag{1}$$

to produce

$$A = P\left(1 + \frac{1}{m}\right)^{nt} \tag{2}$$

This substitution is motivated by the desire to express $\left(1 + \frac{r}{n}\right)^{n}$ as $\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{r}$, which approaches e^{r} as m gets larger without bound.

Solving the equation $\frac{1}{m} = \frac{r}{n}$ for n yields $n = mr$, so the exponent nt can be written as mrt. Therefore, Equation (2) can be expressed as

$$A = P\left(1 + \frac{1}{m}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} \tag{3}$$

By the definition of e, we know that as m increases without bound,

$$\left(1 + \frac{1}{m}\right)^{m} \quad \text{approaches} \quad e$$

Thus, using continuous compounding, Equation (3) simplifies to $A = Pe^{rt}$.

Continuous Compounding Interest Formula

If an account with principal P and annual interest rate r is compounded continuously for t years, then the balance is $A = Pe^{rt}$.

EXAMPLE 5 Solve a Continuous Compound Interest Application

Find the balance after 4 years on \$800 invested at an annual rate of 6% compounded continuously.

Algebraic Solution

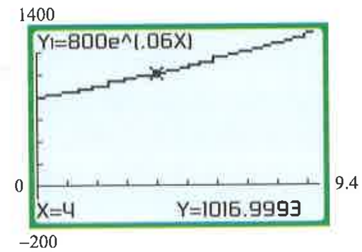
Use the continuous compounding formula with $P = 800$, $r = 0.06$, and $t = 4$.

$$\begin{aligned} A &= Pe^{rt} \\ &= 800e^{0.06(4)} && \text{• Substitute given values.} \\ &= 800e^{0.24} && \text{• Simplify.} \\ &\approx 800(1.27124915) \\ &\approx 1017.00 && \text{• Round to the nearest cent.} \end{aligned}$$

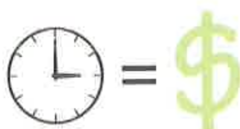
The balance after 4 years will be \$1017.00.

Visualize the Solution

The following graph of $A = 800e^{0.06t}$ shows that the balance is about \$1017.00 when $t = 4$.



▶ Try Exercise 22, page 403



You have probably heard it said that time is money. In fact, many investors ask the question "How long will it take to double my money?" The following example answers this question for two different investments.

EXAMPLE 6 Double Your Money

Find the time required for money invested at an annual rate of 6% to double in value if the investment is compounded on the following basis.

a. Semiannually **b.** Continuously

Solution

a. Use $A = P\left(1 + \frac{r}{n}\right)^{nt}$ with $r = 0.06$, $n = 2$, and the balance A equal to twice the principal ($A = 2P$).

$$\begin{aligned} 2P &= P\left(1 + \frac{0.06}{2}\right)^{2t} \\ 2 &= \left(1 + \frac{0.06}{2}\right)^{2t} && \text{• Divide each side by } P. \\ \ln 2 &= \ln\left(1 + \frac{0.06}{2}\right)^{2t} && \text{• Take the natural logarithm of each side.} \\ \ln 2 &= 2t\ln\left(1 + \frac{0.06}{2}\right) && \text{• Apply the power property.} \\ 2t &= \frac{\ln 2}{\ln\left(1 + \frac{0.06}{2}\right)} && \text{• Solve for } t. \end{aligned}$$

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$$t = \frac{1}{2} \cdot \frac{\ln 2}{\ln\left(1 + \frac{0.06}{2}\right)}$$

$$t \approx 11.72$$

If the investment is compounded semiannually, it will double in value in about 11.72 years.

b. Use $A = Pe^{rt}$ with $r = 0.06$ and $A = 2P$.

$$2P = Pe^{0.06t}$$

$$2 = e^{0.06t}$$ • Divide each side by P.

$$\ln 2 = 0.06t$$ • Write in logarithmic form.

$$t = \frac{\ln 2}{0.06}$$ • Solve for t.

$$t \approx 11.55$$

If the investment is compounded continuously, it will double in value in about 11.55 years.

Try Exercise 26, page 403

Restricted Growth Models

The exponential growth function $N(t) = N_0e^{kt}$ is an *unrestricted growth model* that does not consider any limited resources that eventually will curb population growth.

The **logistic model** is a *restricted growth model* that takes into consideration the effects of limited resources. The logistic model was developed by Pierre Verhulst in 1836.

Definition of the Logistic Model (Restricted Growth Model)

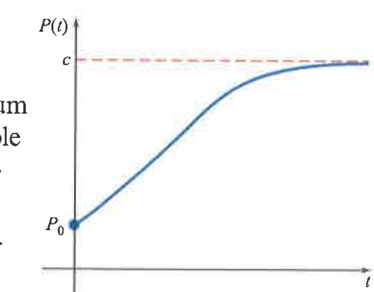
The magnitude of a population at time $t \geq 0$ is given by

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

where c is the **carrying capacity** (the maximum population that can be supported by available resources as $t \to \infty$) and b is a positive constant called the **growth rate constant.**

The **initial population** is $P_0 = P(0)$. The constant a is related to the initial population P_0 and the carrying capacity c by the formula

$$a = \frac{c - P_0}{P_0}$$



$P(t) = \frac{c}{1 + ae^{-bt}}, 0 < P_0 < c$

In the following example, we determine a logistic growth model for a coyote population.

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EXAMPLE 7 Find and Use a Logistic Model

At the beginning of 2011, the coyote population in a wilderness area was estimated at 200. By the beginning of 2013, the coyote population had increased to 250. A park ranger estimates that the carrying capacity of the wilderness area is 500 coyotes.

a. Use the given data to determine the growth rate constant for the logistic model of this coyote population.

b. Use the logistic model determined in **a** to predict the year in which the coyote population will first reach 400.

Solution

a. If we represent the beginning of 2011 by $t = 0$, then the beginning of 2013 will be represented by $t = 2$. In the logistic model, make the following substitutions:

$$P(2) = 250,\ c = 500, \text{ and } a = \frac{c - P_0}{P_0} = \frac{500 - 200}{200} = 1.5.$$

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

$$P(2) = \frac{500}{1 + 1.5e^{-b\cdot 2}}$$ • Substitute the given values for t, c, and a.

$$250 = \frac{500}{1 + 1.5e^{-b\cdot 2}}$$ • $P(2) = 250$

$$250(1 + 1.5e^{-b\cdot 2}) = 500$$ • Solve for the growth rate constant b.

$$1 + 1.5e^{-b\cdot 2} = \frac{500}{250}$$

$$1.5e^{-b\cdot 2} = 2 - 1$$

$$e^{-b\cdot 2} = \frac{1}{1.5}$$

$$-2b = \ln\left(\frac{1}{1.5}\right)$$ • Write in logarithmic form.

$$b = -\frac{1}{2}\ln\left(\frac{1}{1.5}\right)$$

$$b \approx 0.20273255$$

Using $a = 1.5$, $b = 0.20273255$, and $c = 500$ gives us the following logistic model.

$$P(t) = \frac{500}{1 + 1.5e^{-0.20273255t}}$$

b. To determine in which year the logistic model predicts that the coyote population will first reach 400, replace $P(t)$ with 400 and solve for t.

$$400 = \frac{500}{1 + 1.5e^{-0.20273255t}}$$

$$400(1 + 1.5e^{-0.20273255t}) = 500$$

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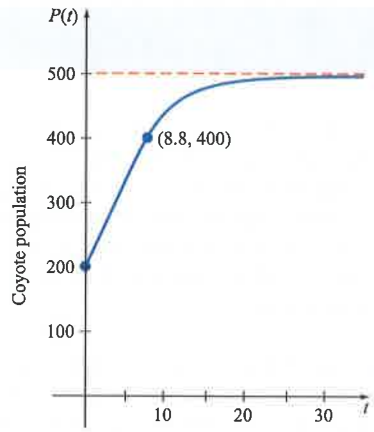


Year ($t = 0$ represents the beginning of 2011)

$$P(t) = \frac{500}{1 + 1.5e^{-0.20273255t}}$$

Figure 4.42

$$1 + 1.5e^{-0.20273255t} = \frac{500}{400}$$

$$1.5e^{-0.20273255t} = 1.25 - 1$$

$$e^{-0.20273255t} = \frac{0.25}{1.5}$$

$$-0.20273255t = \ln\left(\frac{0.25}{1.5}\right)$$ • Write in logarithmic form.

$$t = \frac{1}{-0.20273255}\ln\left(\frac{0.25}{1.5}\right)$$ • Solve for t.

$$\approx 8.8$$

According to the logistic model, the coyote population will reach 400 about 8.8 years after the beginning of 2011, which is during 2019. The graph of the logistic model is shown in Figure 4.42. Note that $P(8.8) \approx 400$ and that as $t \to \infty$, $P(t) \to 500$.

▶ Try Exercise 42, page 404

In Example 8, we use a function of the form $v = a(1 - e^{-kt})$ to model the velocity of an object that has been dropped from a high elevation.

EXAMPLE 8 Application to Air Resistance

Assuming that air resistance is proportional to the velocity of a falling object, the velocity (in feet per second) of the object t seconds after it has been dropped is given by $v = 82(1 - e^{-0.39t})$.

a. Determine when the velocity will be 70 feet per second.

b. The graph of v has $v = 82$ as a horizontal asymptote. Explain the meaning of this asymptote in the context of this example.

Algebraic Solution

a.

$$v = 82(1 - e^{-0.39t})$$

$$70 = 82(1 - e^{-0.39t})$$ • Substitute 70 for v.

$$\frac{70}{82} = 1 - e^{-0.39t}$$ • Divide each side by 82.

$$e^{-0.39t} = 1 - \frac{70}{82}$$ • Solve for $e^{-0.39t}$.

$$-0.39t = \ln\frac{6}{41}$$ • Write in logarithmic form.

$$t = \frac{\ln(6/41)}{-0.39} \approx 4.9277246$$ • Solve for t.

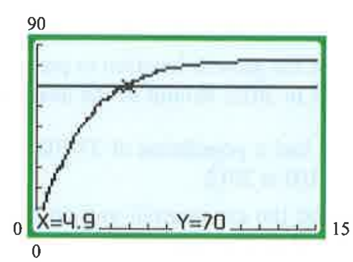
The velocity will be 70 feet per second after approximately 4.9 seconds.

b. The horizontal asymptote $v = 82$ means that, as time increases, the velocity of the object will approach, but never reach or exceed, 82 feet per second.

▶ Try Exercise 52, page 405

Visualize the Solution

a. A graph of $y = 82(1 - e^{-0.39x})$ and $y = 70$ shows that the x-coordinate of the point of intersection is about 4.9.



$y = 82(1 - e^{-0.39x})$

(*Note:* The x value shown is rounded to the nearest tenth.)

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EXERCISE SET 4.6

Concept Check

1. How can you determine whether the exponential function

$$N(t) = N_0 e^{kt}$$

is an exponential growth function or an exponential decay function?

2. Explain a difference between the graph of the exponential growth function $P(t) = P_0 e^{kt}$, with $k > 0$, and the logistic function

$$P(t) = \frac{c}{1 + ae^{-bt}},\ 0 < P_0 < c$$

3. What is meant by the terminology "compound continuously"?

4. What is meant by the term "half-life" in regard to a radioactive substance?

Population Growth In Exercises 5 to 10, solve the given problem related to population growth.

5. The number of bacteria $N(t)$ present in a culture at time t hours is given by

$$N(t) = 2200(2)^t$$

Find the number of bacteria present when

a. $t = 0$ hours **b.** $t = 3$ hours

6. The population of a city grows exponentially according to the function

$$f(t) = 12{,}400(1.14)^t$$

for $0 \le t \le 5$ years. Find, to the nearest hundred, the population of the city when t is

a. 3 years **b.** 4.25 years

7. A city had a population of 22,600 in 2007 and a population of 24,200 in 2012.

a. Find the exponential growth function for the city. Use $t = 0$ to represent 2007.

b. Use the growth function to predict the population of the city in 2022. Round to the nearest hundred.

8. A city had a population of 53,700 in 2008 and a population of 58,100 in 2012.

a. Find the exponential growth function for the city. Use $t = 0$ to represent 2008.

b. Use the growth function to predict the population of the city in 2020. Round to the nearest hundred.

9. During the first decade of this century, the population of Irvine, California, grew exponentially. The population of Irvine was 143,110 in 2000 and 212,375 in 2010. Find the exponential growth function that models the population growth of Irvine and use it to predict the population in 2016. Use $t = 0$ to represent 2000, $t = 10$ to represent 2010, and so on. Round to the nearest thousand.

10. During the first decade of this century, the population of Oklahoma City, Oklahoma, grew exponentially. The population of Oklahoma City was 506,107 in 2000 and 579,999 in 2010. Find the exponential growth function that models the population of Oklahoma City and use it to predict the population in 2015. Use $t = 0$ to represent 2000, $t = 10$ to represent 2010, and so on. Round to the nearest thousand.

11. **Medicine** Sodium-24 is a radioactive isotope of sodium that is used to study circulatory dysfunction. Assuming that 4 micrograms of sodium-24 are injected into a person, the amount A in micrograms remaining in that person after t hours is given by the equation $A = 4e^{-0.046t}$.

a. Graph this equation.

b. What amount of sodium-24 remains after 5 hours?

c. What is the half-life of sodium-24?

d. In how many hours will the amount of sodium-24 be 1 microgram?

In Exercises 12 to 16, use the half-life information from Table 4.11, page 394, to work each exercise.

12. **Radioactive Decay** Find the decay function for the amount of polonium (^{210}Po) that remains in a sample after t days.

13. **Geology** Geologists have determined that Crater Lake in Oregon was formed by a volcanic eruption. Chemical analysis of a wood chip assumed to be from a tree that died during the eruption has shown that it contains approximately 45% of its original carbon-14. Estimate how long ago the volcanic eruption occurred.

14. **Radioactive Decay** Estimate the percentage of polonium (^{210}Po) that remains in a sample after 2 years. Round to the nearest hundredth of a percent.

15. **Archeology** The Rhind papyrus, named after A. Henry Rhind, contains most of what we know today of ancient Egyptian mathematics. A chemical analysis of a sample from the papyrus has shown that it contains approximately 75% of its original carbon-14. Estimate the age of the Rhind papyrus.

Indicates Try It Exercises

16. **Archeology** Estimate the age of a bone if it now contains 65% of its original amount of carbon-14. Round to the nearest 100 years.

Compound Interest In Exercises 17 to 24, solve the given problem related to compound interest.

17. Find the balance if $4500 is invested at an annual interest rate of 2.5%, compounded annually, for

a. 5 years **b.** 12 years

18. Find the balance if $17,500 is invested at an annual interest rate of 3.25%, compounded annually, for

a. 7 years **b.** 15 years

19. If $22,000 is invested at an annual interest rate of 2.75% for 5 years, find the balance if the interest is compounded

a. monthly **b.** daily

20. If $5250 is invested at an annual interest rate of 3.5% for 30 years, find the balance if the interest is compounded

a. monthly **b.** daily

21. Find the balance if $3200 is invested at an annual interest rate of 4% for 10 years, compounded continuously.

22. Find the balance if $55,000 is invested at an annual interest rate of 2.25% for 30 years, compounded continuously.

23. How long will it take to double your money if it is invested in a certificate of deposit that pays 2.0% annual interest compounded daily? Round to the nearest tenth of a year.

24. How long will it take to triple your money if it is invested in an investment that pays 5.0% annual interest compounded daily? Round to the nearest hundredth of a year.

Continuous Compounding Interest In Exercises 25 to 28, solve the given problem related to continuous compounding interest.

25. Use the continuous compounding interest formula to derive an expression for the time it will take money to triple when invested at an annual interest rate of r compounded continuously.

26. How long will it take $1000 to triple if it is invested at an annual interest rate of 5.5% compounded continuously? Round to the nearest year.

27. How long will it take $6000 to triple if it is invested in an account that pays 7.6% annual interest compounded continuously? Round to the nearest year.

28. How long will it take $10,000 to triple if it is invested in an account that pays 5.5% annual interest compounded continuously? Round to the nearest year.

In Exercises 29 to 34, determine the following constants for the given logistic growth model.
a. The carrying capacity
b. The growth rate constant
c. The initial population P_0

29. $P(t) = \dfrac{1900}{1 + 8.5e^{-0.16t}}$

30. $P(t) = \dfrac{32{,}550}{1 + 0.75e^{-0.08t}}$

31. $P(t) = \dfrac{157{,}500}{1 + 2.5e^{-0.04t}}$

32. $P(t) = \dfrac{51}{1 + 1.04e^{-0.03t}}$

33. $P(t) = \dfrac{2400}{1 + 7e^{-0.12t}}$

34. $P(t) = \dfrac{320}{1 + 15e^{-0.12t}}$

In Exercises 35 to 38, use algebraic procedures to find the logistic growth model for the data.

35. $P_0 = 400$, $P(2) = 780$, and the carrying capacity is 5500.

36. $P_0 = 6200$, $P(8) = 7100$, and the carrying capacity is 9500.

37. $P_0 = 18$, $P(3) = 30$, and the carrying capacity is 100.

38. $P_0 = 3200$, $P(22) \approx 5565$, and the growth rate constant is 0.056.

39. **Revenue** The annual revenue R, in dollars, of a new company can be closely modeled by the logistic function

$$R(t) = \frac{625{,}000}{1 + 3.1e^{-0.045t}}$$

where the natural number t is the time, in years, since the company was founded.

a. According to the model, what will be the company's annual revenue for its first year and its second year ($t = 1$ and $t = 2$) of operation? Round to the nearest $1000.

b. According to the model, what will the company's annual revenue approach in the long-term future?

40. **New Car Sales** The number A of cars sold annually by an automobile dealership can be closely modeled by the logistic function

$$A(t) = \frac{1650}{1 + 2.4e^{-0.055t}}$$

where the natural number t is the time, in years, since the dealership was founded.

a. According to the model, what number of cars will the dealership sell during its first year and its second year ($t = 1$ and $t = 2$) of operation? Round to the nearest unit.

b. According to the model, what will the dealership's annual car sales approach in the long-term future?

Population Growth In Exercises 41 to 44, solve the given problem related to population growth.

41. The population of wolves in a preserve satisfies a logistic model in which $P_0 = 312$ in 2008, $c = 1600$, and $P(6) = 416$.

 a. Determine the logistic model for this population, where t is the number of years after 2008.

 b. Use the logistic model from **a** to predict the size of the wolf population in 2018.

42. The population of groundhogs on a ranch satisfies a logistic model in which $P_0 = 240$ in 2010, $c = 3400$, and $P(1) = 310$.

 a. Determine the logistic model for this population, where t is the number of years after 2010.

 b. Use the logistic model from **a** to predict the size of the groundhog population in 2017.

43. The population of squirrels in a nature preserve satisfies a logistic model in which $P_0 = 1500$ in 2010. The carrying capacity of the preserve is estimated at 8500 squirrels, and $P(2) = 1900$.

 a. Determine the logistic model for this population, where t is the number of years after 2010.

 b. Use the logistic model from **a** to predict the year in which the squirrel population will first exceed 4000.

44. The population of walruses on an island satisfies a logistic model in which $P_0 = 800$ in 2009. The carrying capacity of the island is estimated at 5500 walruses, and $P(1) = 900$.

 a. Determine the logistic model for this population, where t is the number of years after 2009.

 b. Use the logistic model from **a** to predict the year in which the walrus population will first exceed 2000.

45. **Physics** Newton's Law of Cooling states that if an object at temperature T_0 is placed into an environment at constant temperature A, then the temperature of the object, $T(t)$ (in degrees Fahrenheit), after t minutes is given by $T(t) = A + (T_0 - A)e^{-kt}$, where k is a constant that depends on the object.

 a. Determine the constant k (to the nearest thousandth) for a canned soda drink that takes 5 minutes to cool from 75°F to 65°F after being placed in a refrigerator that maintains a constant temperature of 34°F.

 b. What will be the temperature (to the nearest degree) of the soda drink after 30 minutes?

 c. When (to the nearest minute) will the temperature of the soda drink be 36°F?

46. **Psychology** According to a software company, the users of its typing tutorial can expect to type $N(t)$ words per minute after t hours of practice with the product, according to the function $N(t) = 100(1.04 - 0.99^t)$.

 a. How many words per minute can a student expect to type after 2 hours of practice?

 b. How many words per minute can a student expect to type after 40 hours of practice?

 c. According to the function N, how many hours (to the nearest hour) of practice will be required before a student can expect to type 60 words per minute?

47. **Psychology** In the city of Whispering Palms, which has a population of 80,000 people, the number of people $P(t)$ exposed to a rumor in t hours is given by the function $P(t) = 80{,}000(1 - e^{-0.0005t})$.

 a. Find the number of hours until 10% of the population has heard the rumor.

 b. Find the number of hours until 50% of the population has heard the rumor.

48. **Law** A lawyer has determined that the number of people $P(t)$ in a city of 1.2 million people who have been exposed to a news item after t days is given by the function

$$P(t) = 1{,}200{,}000(1 - e^{-0.03t})$$

 a. How many days after a major crime has been reported has 40% of the population heard of the crime?

 b. A defense lawyer knows it will be difficult to pick an unbiased jury after 80% of the population has heard of the crime. After how many days will 80% of the population have heard of the crime?

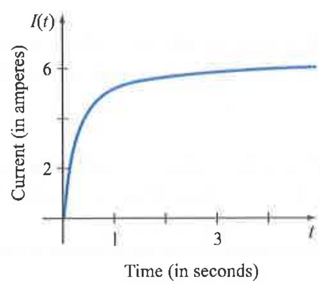
49. **Depreciation** An automobile depreciates according to the function $V(t) = V_0(1 - r)^t$, where $V(t)$ is the value in dollars after t years, V_0 is the original value, and r is the yearly depreciation rate. A car has a yearly depreciation rate of 20%. Determine, to the nearest 0.1 year, in how many years the car will depreciate to half its original value.

50. **Physics** The current $I(t)$ (measured in amperes) of a circuit is given by the function $I(t) = 6(1 - e^{-2.5t})$, where t is the number of seconds after the switch is closed.

 a. Find the current when $t = 0$.

 b. Find the current when $t = 0.5$.

 c. Solve the equation for t.



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Air Resistance In Exercises 51 to 54, solve the given problems related to air resistance.

51. Assuming that air resistance is proportional to velocity, the velocity v, in feet per second, of a falling object after t seconds is given by $v = 32(1 - e^{-t})$.

a. Graph this equation for $t \geq 0$.

b. Determine algebraically, to the nearest 0.01 second, when the velocity is 20 feet per second.

c. Determine the horizontal asymptote of the graph of v.

d. Write a sentence that explains the meaning of the horizontal asymptote in the context of this application.

52. Assuming that air resistance is proportional to velocity, the velocity v, in feet per second, of a falling object after t seconds is given by

$$v = 64(1 - e^{-t/2})$$

a. Graph this equation for $t \geq 0$.

b. Determine algebraically, to the nearest 0.1 second, when the velocity is 50 feet per second.

c. Determine the horizontal asymptote of the graph of v.

d. Write a sentence that explains the meaning of the horizontal asymptote in the context of this application.

53. The distance s (in feet) that the object in Exercise 51 will fall in t seconds is given by the function

$$s = 32t + 32(e^{-t} - 1)$$

a. Graph this equation for $t \geq 0$.

b. Determine, to the nearest 0.1 second, the time it takes the object to fall 50 feet.

c. Calculate the slope of the secant line through $(1, s(1))$ and $(2, s(2))$.

d. Write a sentence that explains the meaning of the slope of the secant line you calculated in **c.**

54. The distance s (in feet) that the object in Exercise 52 will fall in t seconds is given by the function

$$s = 64t + 128(e^{-t/2} - 1)$$

a. Graph this equation for $t \geq 0$.

b. Determine, to the nearest 0.1 second, the time it takes the object to fall 50 feet.

c. Calculate the slope of the secant line through $(1, s(1))$ and $(2, s(2))$.

d. Write a sentence that explains the meaning of the slope of the secant line you calculated in **c.**

55. **Learning Theory** The logistic model is also used in learning theory. Suppose that historical records from employee training at a company show that the percent score on a product information test is given by

$$P = \frac{100}{1 + 25e^{-0.095t}}$$

where t is the number of hours of training. What is the number of hours (to the nearest hour) of training needed before a new employee will answer 75% of the questions correctly?

56. **Learning Theory** A company provides training in the assembly of a computer circuit to new employees. Past experience has shown that the number of correctly assembled circuits per week can be modeled by

$$N = \frac{250}{1 + 249e^{-0.503t}}$$

where t is the number of weeks of training. What is the number of weeks (to the nearest week) of training needed before a new employee will correctly make 140 circuits per week?

57. **Medication Level** A patient is given three doses of aspirin. Each dose contains 1 gram of aspirin. The second and third doses are each taken 3 hours after the previous dose is administered. The half-life of the aspirin is 2 hours. The amount of aspirin A in the patient's body t hours after the first dose is administered is

$$A(t) = \begin{cases} 0.5^{t/2} & 0 \leq t < 3 \\ 0.5^{t/2} + 0.5^{(t-3)/2} & 3 \leq t < 6 \\ 0.5^{t/2} + 0.5^{(t-3)/2} + 0.5^{(t-6)/2} & t \geq 6 \end{cases}$$

Find, to the nearest hundredth of a gram, the amount of aspirin in the patient's body when

a. $t = 1$ **b.** $t = 4$ **c.** $t = 9$

58. **Medication Level** Use the dosage formula in Exercise 57 to determine when, to the nearest tenth of an hour, the amount of aspirin in the patient's body first reaches 0.25 gram.

59. **Annual Growth Rate** The exponential growth function for the population of a city is $N(t) = 78{,}245e^{0.0245t}$, where t is in years. Because

$$e^{0.0245t} = (e^{0.0245})^t \approx (1.0248)^t$$

we can write the growth function as

$$N(t) = 78{,}245(1.0248)^t \approx 78{,}245\left(1 + \frac{0.0248}{1}\right)^{1 \cdot t}$$

In this form we can see that the city's population is growing by 2.48% per year.

The population of the city of Lake Tahoe, Nevada, can be modeled by the exponential growth function $N(t) = 22{,}755e^{0.0287t}$. Find the annual growth rate, expressed as a percent, of Lake Tahoe. Round to the nearest hundredth of a percent.

60. **Oil Spills** Crude oil leaks from a tank at a rate that depends on the amount of oil that remains in the tank. Because $\frac{1}{8}$ of the oil in the tank leaks out every 2 hours, the volume $V(t)$ of oil in the tank after t hours is given by $V(t) = V_0(0.875)^{t/2}$, where $V_0 = 350{,}000$ gallons, the number of gallons in the tank at the time the tank started to leak ($t = 0$).

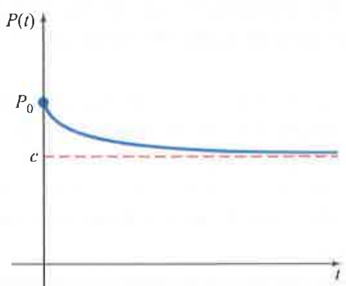
a. How many gallons does the tank hold after 3 hours?

b. How many gallons does the tank hold after 5 hours?

c. How long, to the nearest hour, will it take until 90% of the oil has leaked from the tank?

Enrichment Exercises

If $P_0 > c$ (which implies that $-1 < a < 0$), then the logistic function $P(t) = \dfrac{c}{1 + ae^{-bt}}$ decreases as t increases. Biologists often use this type of logistic function to model populations that decrease over time. See the following figure. Apply this information to Exercises 61 to 63.



$P(t) = \dfrac{c}{1 + ae^{-bt}}, P_0 > c$

61. A Declining Fish Population A biologist finds that the fish population in a small lake can be closely modeled by the logistic function

$$P(t) = \frac{1000}{1 + (-0.3333)e^{-0.05t}}$$

where t is the time, in years, since the lake was first stocked with fish.

a. What was the fish population when the lake was first stocked with fish?

b. According to the logistic model, what will the fish population approach in the long-term future?

62. A Declining Deer Population The deer population in a reserve is given by the logistic function

$$P(t) = \frac{1800}{1 + (-0.25)e^{-0.07t}}$$

where t is the time, in years, since July 1, 2010.

a. What was the deer population on July 1, 2010? What was the deer population on July 1, 2012?

b. According to the logistic model, what will the deer population approach in the long-term future?

63. Modeling World Record Times in the Men's Mile Race In the early 1950s, many people speculated that no runner would ever run a mile race in under 4 minutes. During the period from 1913 to 1945, the world record in the mile event had been reduced from 4.14.4 (4 minutes, 14.4 seconds) to 4.01.4, but no one seemed capable of running a sub-4-minute mile. Then, in 1954, Roger Bannister broke through the 4-minute barrier by running a mile in 3.59.6. In 1999, the current record of 3.43.13 was established. It is fun to think about future record times in the mile race. Will they ever go below 3 minutes, 30 seconds? Below 3 minutes, 20 seconds? What about a sub-3-minute mile?

A declining logistic function that closely models the world record times WR, in seconds, in the men's mile run from 1913 ($t = 0$) to 1999 ($t = 86$) is given by

$$WR(t) = \frac{199.13}{1 + (-0.21726)e^{-0.0079889t}}$$

a. Use the above logistic function to predict the world record time for the men's mile run in 2020 and 2050.

b. According to the logistic function, what time will the world record in the men's mile event approach but never break through?

SECTION 4.7

Analyzing Scatter Plots
Modeling Data
Finding a Logistic Growth Model

Modeling Data with Exponential and Logarithmic Functions

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A28.

PS1. Determine whether $N(t) = 4 - \ln t$ is an increasing or a decreasing function. [4.3]

PS2. Determine whether $P(t) = 1 - 2(1.05^t)$ is an increasing or a decreasing function. [4.2]

PS3. Evaluate $P(t) = \dfrac{108}{1 + 2e^{-0.1t}}$ for $t = 0$. [4.2]

PS4. Evaluate $N(t) = 840e^{1.05t}$ for $t = 0$. [4.2]

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