4.6 The Fundamental Theorem of Algebra

Learning Target Use the Fundamental Theorem of Algebra to find all complex roots of polynomial equations.

Success Criteria
- I can identify the degree of a polynomial.
- I can explain the Fundamental Theorem of Algebra.
- I can find all the zeros of a polynomial function.

EXPLORE IT! Finding Zeros of Functions

Work with a partner.

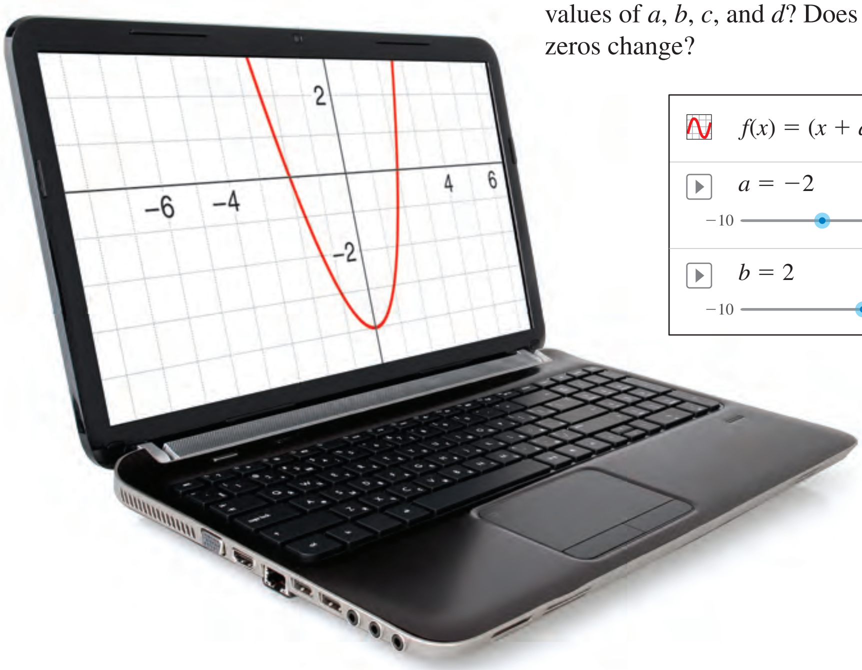
a. Use technology to explore each function for several values of a, b, c, and d.

$$f(x) = (x + a)(x + b)$$

$$g(x) = (x + a)(x + b)(x + c)$$

$$h(x) = (x + a)(x + b)(x + c)(x + d)$$

How does the graph change when you change the values of a, b, c, and d? Does the number of real zeros change?



$f(x) = (x + a)(x + b)$

$a = -2$

−10 10

$b = 2$

−10 10

b. Repeat part (a) for the following functions.

$$m(x) = ax^2 + bx + c$$

$$n(x) = ax^3 + bx^2 + cx + d$$

$$p(x) = ax^4 + bx^3 + cx^2 + dx + e$$

c. Make a conjecture about the number of real zeros of $y = f(x)$ when the degree of $f(x)$ is a positive number n.

Math Practice

Analyze Conjectures
How can you justify your conjecture algebraically?

The Fundamental Theorem of Algebra

The table shows several polynomial equations and their solutions, including repeated solutions. Notice that for the last equation, the repeated solution $x = -1$ is counted twice.

Equation	Degree	Solution(s)	Number of solutions
$2x - 1 = 0$	1	$\frac{1}{2}$	1
$x^2 - 2 = 0$	2	$\pm\sqrt{2}$	2
$x^3 - 8 = 0$	3	$2, -1 \pm i\sqrt{3}$	3
$x^3 + x^2 - x - 1 = 0$	3	$-1, -1, 1$	3

In the table, note the relationship between the degree of the polynomial $f(x)$ and the number of solutions of $f(x) = 0$. This relationship is generalized by the *Fundamental Theorem of Algebra*, first proven by German mathematician Carl Friedrich Gauss (1777−1855).

KEY IDEA

The Fundamental Theorem of Algebra

Theorem If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has at least one solution in the set of complex numbers.

Corollary If $f(x)$ is a polynomial of degree n where $n > 0$, then the equation $f(x) = 0$ has exactly n solutions provided each solution repeated twice is counted as two solutions, each solution repeated three times is counted as three solutions, and so on.

This also means that an nth-degree polynomial function f has exactly n zeros.

Math Practice

Understand Mathematical Terms

Compare the statements "the polynomial equation $f(x) = 0$ has exactly n solutions" and "the polynomial function f has exactly n zeros."

EXAMPLE 1 Finding Solutions of a Polynomial Equation

WATCH

How many solutions does $x^4 + x^3 + 8x + 8 = 0$ have? Find all the solutions.

SOLUTION

Because $x^4 + x^3 + 8x + 8 = 0$ is a polynomial equation of degree 4, it has four solutions. Notice that you can factor by grouping to begin solving the equation.

$(x^4 + x^3) + (8x + 8) = 0$ — Group terms with common factors.

$x^3(x + 1) + 8(x + 1) = 0$ — Factor out GCF of each pair of terms.

$(x + 1)(x^3 + 8) = 0$ — Distributive Property

$(x + 1)(x + 2)(x^2 - 2x + 4) = 0$ — Sum of two cubes pattern

The linear factors indicate that -2 and -1 are solutions. To find the remaining two solutions, solve $x^2 - 2x + 4 = 0$ by using the Quadratic Formula.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)} = 1 \pm i\sqrt{3}$$

▶ The solutions are $-2, -1, 1 - i\sqrt{3}$, and $1 + i\sqrt{3}$.

EXAMPLE 2 Finding the Zeros of a Polynomial Function

WATCH

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Find all the zeros of $f(x) = x^5 + x^3 - 2x^2 - 12x - 8$.

SOLUTION

Step 1 Find the rational zeros of f. Because f is a polynomial function of degree 5, it has five zeros. The possible rational zeros are ± 1, ± 2, ± 4, and ± 8. Using synthetic division, you can determine that -1 is a zero repeated twice and 2 is also a zero.

Step 2 Write $f(x)$ in factored form. Dividing $f(x)$ by its known factors $x + 1$, $x + 1$, and $x - 2$ gives a quotient of $x^2 + 4$. So,

$$f(x) = (x + 1)^2(x - 2)(x^2 + 4).$$

Step 3 Find the imaginary zeros of f. Solving $x^2 + 4 = 0$, you get $x = \pm 2i$. This means $x^2 + 4 = (x + 2i)(x - 2i)$.

$$f(x) = (x + 1)^2(x - 2)(x + 2i)(x - 2i)$$

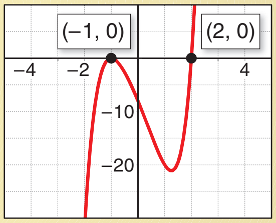
REMEMBER

You can use imaginary numbers to write $x^2 + 4$ as $(x + 2i)(x - 2i)$. In general, $a^2 + b^2 = (a + bi)(a - bi)$.

▶ From the factorization, there are five zeros. The zeros of f are

-1, -1, 2, $-2i$, and $2i$.

Check The graph of f and the real zeros are shown. Notice that only the *real* zeros appear as x-intercepts. Also, the graph of f touches the x-axis at the repeated zero $x = -1$ and crosses the x-axis at $x = 2$.



SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Identify the number of solutions of the polynomial equation. Then find all the solutions.

1. $x^5 - 4x^3 - x^2 + 4 = 0$
2. $x^4 + 7x^2 - 144 = 0$

Find all the zeros of the polynomial function.

3. $f(x) = x^3 + 7x^2 + 16x + 12$
4. $f(x) = x^5 - 3x^4 + 5x^3 - x^2 - 6x + 4$
5. **WRITING** Show why the Fundamental Theorem of Algebra is true for all quadratic equations.

Complex Conjugates

Pairs of complex numbers of the forms $a + bi$ and $a - bi$, where $b \neq 0$, are called complex conjugates. In Example 2, notice that the zeros $2i$ and $-2i$ are complex conjugates. This illustrates the next theorem.

KEY IDEA

The Complex Conjugates Theorem

If f is a polynomial function with real coefficients, and $a + bi$ is an imaginary zero of f, then $a - bi$ is also a zero of f.

EXAMPLE 3 Using Zeros to Write a Polynomial Function

WATCH

GO DIGITAL

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and $3 + i$.

SOLUTION

Because the coefficients are rational and $3 + i$ is a zero, $3 - i$ must also be a zero by the Complex Conjugates Theorem. Use the three zeros and the Factor Theorem to write $f(x)$ as a product of three factors.

$f(x) = (x - 2)[x - (3 + i)][x - (3 - i)]$ Write $f(x)$ in factored form.

$= (x - 2)[(x - 3) - i][(x - 3) + i]$ Regroup terms.

$= (x - 2)[(x - 3)^2 - i^2]$ Multiply.

$= (x - 2)[(x^2 - 6x + 9) - (-1)]$ Expand binomial and use $i^2 = -1$.

$= (x - 2)(x^2 - 6x + 10)$ Simplify.

$= x^3 - 6x^2 + 10x - 2x^2 + 12x - 20$ Multiply.

$= x^3 - 8x^2 + 22x - 20$ Combine like terms.

Check

You can check this result by evaluating f at each of the given zeros.

$f(2) = 2^3 - 8(2)^2 + 22(2) - 20 = 8 - 32 + 44 - 20 = 0$ ✓

$f(3 + i) = (3 + i)^3 - 8(3 + i)^2 + 22(3 + i) - 20$

$= 18 + 26i - 64 - 48i + 66 + 22i - 20$

$= 0$ ✓

Because $f(3 + i) = 0$, by the Complex Conjugates Theorem $f(3 - i) = 0$. ✓

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

6. $-1, 4i$ **7.** $3, 1 + i\sqrt{5}$ **8.** $\sqrt{2}, 1 - 3i$ **9.** $2, 2i, 4 - \sqrt{6}$

Descartes's Rule of Signs

French mathematician René Descartes (1596–1650) found the following relationship between the coefficients of a polynomial function and the number of positive and negative zeros of the function.

KEY IDEA

Descartes's Rule of Signs

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial function with real coefficients.

- The number of *positive real zeros* of f is equal to the number of changes in sign of the coefficients of $f(x)$ or is less than this by an even number.
- The number of *negative real zeros* of f is equal to the number of changes in sign of the coefficients of $f(-x)$ or is less than this by an even number.

EXAMPLE 4 Using Descartes's Rule of Signs

WATCH

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Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for $f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8$.

SOLUTION

$$f(x) = x^6 - 2x^5 + 3x^4 - 10x^3 - 6x^2 - 8x - 8$$

The coefficients in $f(x)$ have 3 sign changes, so f has 3 or 1 positive real zero(s).

$$f(-x) = (-x)^6 - 2(-x)^5 + 3(-x)^4 - 10(-x)^3 - 6(-x)^2 - 8(-x) - 8$$
$$= x^6 + 2x^5 + 3x^4 + 10x^3 - 6x^2 + 8x - 8$$

The coefficients in $f(-x)$ have 3 sign changes, so f has 3 or 1 negative real zero(s).

▶ The possible numbers of zeros for f are summarized in the table below.

Positive real zeros	Negative real zeros	Imaginary zeros	Total zeros
3	3	0	6
3	1	2	6
1	3	2	6
1	1	4	6

EXAMPLE 5 Modeling Real Life

WATCH INFO

A tachometer measures the speed (in revolutions per minute, or RPMs) at which an engine shaft rotates. For a certain boat, the speed x (in hundreds of RPMs) of the engine shaft and the speed s (in miles per hour) of the boat are modeled by

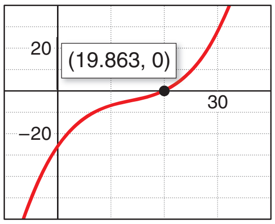
$$s(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 11.0.$$

What is the tachometer reading when the boat travels 15 miles per hour?

SOLUTION

Substitute 15 for $s(x)$ in the function. You can rewrite the resulting equation as

$$0 = 0.00547x^3 - 0.225x^2 + 3.62x - 26.0.$$



The related function is $f(x) = 0.00547x^3 - 0.225x^2 + 3.62x - 26.0$. By Descartes's Rule of Signs, you know f has 3 or 1 positive real zero(s). In the context of speed, negative real zeros and imaginary zeros do not make sense, so you do not need to check for them. To approximate the positive real zeros of f, use technology. From the graph, there is 1 real zero, $x \approx 19.9$.

▶ The tachometer reading is about 1990 RPMs.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function.

10. $f(x) = x^3 + 9x - 25$

11. $f(x) = 3x^4 - 7x^3 + x^2 - 13x + 8$

12. **WHAT IF?** In Example 5, what is the tachometer reading when the boat travels 20 miles per hour?

4.6 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–6, identify the number of solutions of the polynomial equation. Then find all the solutions. *Example 1*

1. $x^3 + 4x^2 - 11x - 30 = 0$

2. $z^3 + 10z^2 + 28z + 24 = 0$

3. $4x^5 - 8x^4 + 6x^3 = 0$

4. $-2x^6 - 8x^5 - x^4 = 0$

5. $t^4 - 2t^3 + t = 2$

6. $y^4 + 5y^3 - 125y = 625$

In Exercises 7–14, find all the zeros of the polynomial function. *Example 2*

7. $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 8$

8. $f(x) = x^4 + 5x^3 - 7x^2 - 29x + 30$

9. $g(x) = x^4 - 9x^2 - 4x + 12$

10. $h(x) = x^3 + 5x^2 - 4x - 20$

11. $g(x) = x^4 + 4x^3 + 7x^2 + 16x + 12$

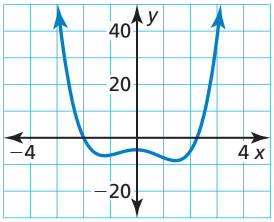
12. $h(x) = x^4 - x^3 + 7x^2 - 9x - 18$

13. $g(x) = x^5 + 3x^4 - 4x^3 - 2x^2 - 12x - 16$

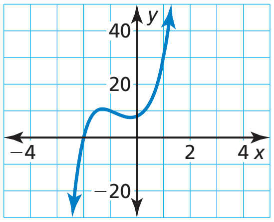
14. $f(x) = x^5 - 20x^3 + 20x^2 - 21x + 20$

ANALYZING RELATIONSHIPS In Exercises 15–18, determine the number of imaginary zeros for the function with the given degree and graph. Explain your reasoning.

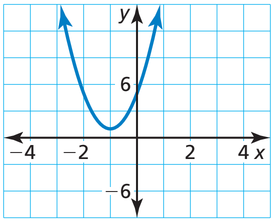
15. Degree: 4



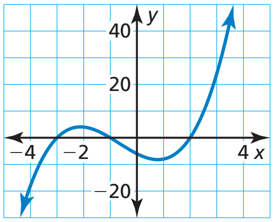
16. Degree: 5



17. Degree: 2



18. Degree: 3



In Exercises 19–26, write a polynomial function *f* of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros. *Example 3*

19. $-5, -1, 2$

20. $-2, 1, 3$

21. $3, 4 + i$

22. $2, 5 - i$

23. $4, -\sqrt{5}$

24. $3, 2 - \sqrt{3}$

25. $2, 1 + i, 2 - \sqrt{3}$

26. $3, 4 + 2i, 1 + \sqrt{7}$

ERROR ANALYSIS In Exercises 27 and 28, describe and correct the error in writing a polynomial function with rational coefficients and the given zero(s).

27. Zeros: $2, 1 + i$

$$f(x) = (x - 2)[x - (1 + i)]$$
$$= x(x - 1 - i) - 2(x - 1 - i)$$
$$= x^2 - x - ix - 2x + 2 + 2i$$
$$= x^2 - (3 + i)x + (2 + 2i)$$

28. Zero: $2 + i$

$$f(x) = [x - (2 + i)][x + (2 + i)]$$
$$= (x - 2 - i)(x + 2 + i)$$
$$= x^2 + 2x + ix - 2x - 4$$
$$- 2i - ix - 2i - i^2$$
$$= x^2 - 4i - 3$$

In Exercises 29–36, determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function. *Example 4*

29. $g(x) = x^4 - x^2 - 6$

30. $g(x) = -x^3 + 5x^2 + 12$

31. $g(x) = x^3 - 4x^2 + 8x + 7$

32. $g(x) = x^5 - 2x^3 - x^2 + 6$

33. $g(x) = x^5 - 3x^3 + 8x - 10$

34. $g(x) = x^5 + 7x^4 - 4x^3 - 3x^2 + 9x - 15$

35. $g(x) = x^6 + x^5 - 3x^4 + x^3 + 5x^2 + 9x - 18$

36. $g(x) = x^7 + 4x^4 - 10x + 25$

37. **MODELING REAL LIFE** From 1900 to 2017, the number P (in thousands) of immigrants residing in the United States can be modeled by

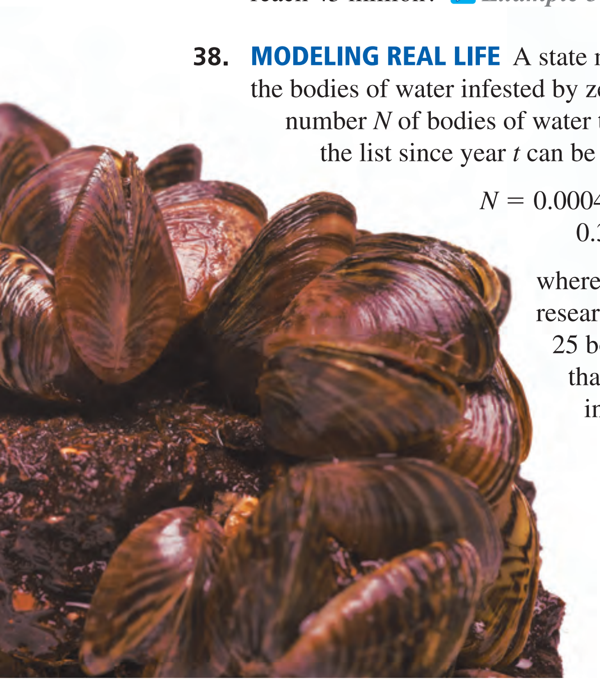
$$P = 0.0757t^3 - 7.893t^2 + 187.10t + 11{,}757.7$$

where t is the number of years since 1900. In which year did the number of immigrants reach 43 million? *Example 5*

38. **MODELING REAL LIFE** A state maintains a list of the bodies of water infested by zebra mussels. The number N of bodies of water that have been on the list since year t can be modeled by

$$N = 0.0004t^4 + 0.042t^3 + 0.35t^2 - 1.0t + 14$$

where $0 \leq t \leq 18$. A researcher studies the 25 bodies of water that have been infested the longest. Since what year have these 25 bodies of water been infested?

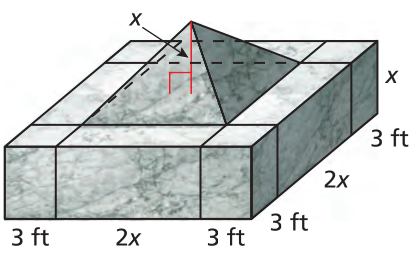


39. **MODELING REAL LIFE** For the 12 years that a grocery store has been open, its annual revenue R (in millions of dollars) can be modeled by the function

$$R = 0.0001(-t^4 + 12t^3 - 77t^2 + 600t + 13{,}650)$$

where t is the number of years since the store opened. In which year(s) was the revenue \$1.5 million?

40. **CONNECTING CONCEPTS** A solid monument with the dimensions shown is to be built using 1000 cubic feet of marble. What is the value of x?



41. **MP STRUCTURE** What is the least number of possible terms of an nth-degree polynomial function with root $4i$? Justify your answer.

42. **OPEN-ENDED** Write a polynomial function of degree 6 with zeros 1, 2, and $-i$. Justify your answer.

43. **COLLEGE PREP** Which is *not* a possible classification of the zeros for $f(x) = x^5 - 4x^3 + 6x^2 + 2x - 6$? Explain.

Ⓐ three positive real zeros, two negative real zeros, and no imaginary zeros

Ⓑ three positive real zeros, no negative real zeros, and two imaginary zeros

Ⓒ one positive real zero, four negative real zeros, and no imaginary zeros

Ⓓ one positive real zero, two negative real zeros, and two imaginary zeros

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44. **COLLEGE PREP** Use Descartes's Rule of Signs to determine which functions could have 1 positive real zero. Select all that apply.

Ⓐ $f(x) = x^4 + 2x^3 - 9x^2 - 2x - 8$

Ⓑ $f(x) = x^4 + 4x^3 + 8x^2 + 16x$

Ⓒ $f(x) = -x^4 + 5x^2 - 4$

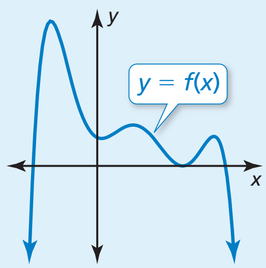
Ⓓ $f(x) = -x^4 + 4x^3 - 7x^2 + 12x + 12$

45. **MP REASONING** Two zeros of

$$f(x) = x^3 - 6x^2 - 16x + 96$$

are 4 and -4. Is the third zero *real* or *imaginary*? Explain your reasoning.

46. **HOW DO YOU SEE IT?**
The graph represents a polynomial function of degree 6.



a. How many positive real zeros does the function have? negative real zeros? imaginary zeros?

b. Use Descartes's Rule of Signs and your answers in part (a) to describe the possible sign changes in the coefficients of $f(x)$.

47. **MAKING AN ARGUMENT** The graph of the constant polynomial function $f(x) = 2$ is a line that does not have any x-intercepts. Does the function contradict the Fundamental Theorem of Algebra? Explain your reasoning.

48. THOUGHT PROVOKING Find the zeros of several polynomial functions with leading coefficients of 1. For functions of this form, make a conjecture about the relationship between (a) the sum of the zeros and the coefficients, and (b) the product of the zeros and the coefficients.

49. MP USING TOOLS Use technology to graph f for $n = 2, 3, 4, 5, 6,$ and 7.

$$f(x) = (x + 3)^n$$

a. Compare the graphs when n is even and n is odd.

b. Describe the behavior of the graph near $x = -3$ as n increases.

c. Use your results from parts (a) and (b) to describe the behavior of the graph of $g(x) = (x - 4)^{20}$ near $x = 4$.

50. DIG DEEPER You want to save money so you can buy a used car in four years. At the end of each summer, you deposit \$1000 earned from summer jobs into your bank account. The table shows the values of your deposits over the four-year period. In the table, g is the growth factor $1 + r$, where r is the annual interest rate expressed as a decimal.

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Deposit	Year 1	Year 2	Year 3	Year 4
1st Deposit	1000	$1000g$	$1000g^2$	$1000g^3$
2nd Deposit	–	1000		
3rd Deposit	–	–	1000	
4th Deposit	–	–	–	1000

a. Copy and complete the table.

b. What annual interest rate do you need in order to pay cash for a car that costs \$4300?

REVIEW & REFRESH

WATCH

In Exercises 51–54, match the function with the correct graph. Explain your reasoning.

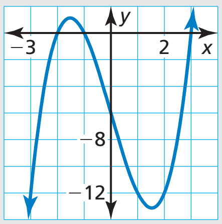
51. $f(x) = x(x - 3)(x + 1)$

52. $g(x) = (x + 3)(x - 1)(x - 2)$

53. $h(x) = x(x + 3)(x - 1)$

54. $k(x) = (x - 3)(x + 1)(x + 2)$

A.



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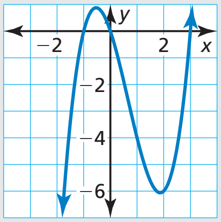

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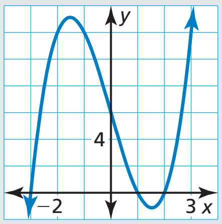

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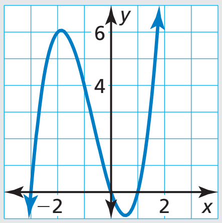


55. MODELING REAL LIFE Solution A is 10% acid and Solution B is 35% acid. How much of each solution should a chemist mix to make 2 cups of a solution that is 15% acid?

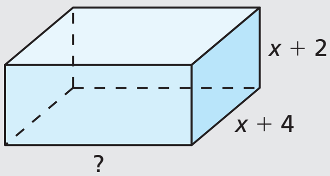
In Exercises 56–58, write a function g whose graph represents the indicated transformation of the graph of f.

56. $f(x) = x$; vertical shrink by a factor of $\frac{1}{3}$ and a reflection in the y-axis

57. $f(x) = |x + 1| - 3$; horizontal stretch by a factor of 9

58. $f(x) = x^2$; reflection in the x-axis, followed by a translation 2 units right and 7 units up

59. The volume V of the rectangular prism is given by $V = 2x^3 + 17x^2 + 46x + 40$. Find an expression for the missing dimension.



In Exercises 60–62, determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function. Then find the zeros.

60. $f(x) = 2x^3 - 5x^2 - x + 6$

61. $h(x) = x^4 - 18x^2 + 81$

62. $g(x) = x^4 + x^3 + 4x^2 + 4x$