Answer graphs for Exercises 11a, 51a, 52a, 53a, and 54a are on page AA19.

EXERCISE SET 4.6

Concept Check

1. How can you determine whether the exponential function

$$N(t) = N_0 e^{kt}$$

is an exponential growth function or an exponential decay function? If $k > 0$, then N is an exponential growth function and if $k < 0$, then N is an exponential decay function.

2. Explain a difference between the graph of the exponential growth function $P(t) = P_0 e^{kt}$, with $k > 0$, and the logistic function

$$P(t) = \frac{c}{1 + ae^{-bt}}, \quad 0 < P_0 < c$$

See answer at the bottom of page after Exercise 8.

3. What is meant by the terminology "compound continuously"? To compound continuously means to increase the number of compounding periods per year, without bound.

4. What is meant by the term "half-life" in regard to a radioactive substance? The half-life of a radioactive substance is the time required for the disintegration of half of its atoms.

Population Growth In Exercises 5 to 10, solve the given problem related to population growth.

5. The number of bacteria $N(t)$ present in a culture at time t hours is given by

$$N(t) = 2200(2)^t$$

Find the number of bacteria present when

a. $t = 0$ hours 2200 bacteria
b. $t = 3$ hours 17,600 bacteria

6. The population of a city grows exponentially according to the function

$$f(t) = 12{,}400(1.14)^t$$

for $0 \le t \le 5$ years. Find, to the nearest hundred, the population of the city when t is

a. 3 years 18,400
b. 4.25 years 21,600

7. A city had a population of 22,600 in 2007 and a population of 24,200 in 2012.

a. Find the exponential growth function for the city. Use $t = 0$ to represent 2007. $N(t) \approx 22{,}600e^{0.01368t}$

b. Use the growth function to predict the population of the city in 2022. Round to the nearest hundred. 27,700

8. A city had a population of 53,700 in 2008 and a population of 58,100 in 2012.

a. Find the exponential growth function for the city. Use $t = 0$ to represent 2008. $N(t) \approx 53{,}700e^{0.01969t}$

b. Use the growth function to predict the population of the city in 2020. Round to the nearest hundred. 68,000

2. The graph of the exponential growth function increases without bound as $t \to \infty$. The graph of the logistic function also increases; however, its graph approaches a height of c as $t \to \infty$.

9. During the first decade of this century, the population of Irvine, California, grew exponentially. The population of Irvine was 143,110 in 2000 and 212,375 in 2010. Find the exponential growth function that models the population growth of Irvine and use it to predict the population in 2016. Use $t = 0$ to represent 2000, $t = 10$ to represent 2010, and so on. Round to the nearest thousand. $N(t) \approx 143{,}110e^{0.0394740015t}$; 269,000

10. During the first decade of this century, the population of Oklahoma City, Oklahoma, grew exponentially. The population of Oklahoma City was 506,107 in 2000 and 579,999 in 2010. Find the exponential growth function that models the population of Oklahoma City and use it to predict the population in 2015. Use $t = 0$ to represent 2000, $t = 10$ to represent 2010, and so on. Round to the nearest thousand. $N(t) \approx 506{,}107e^{0.013627827t}$; 621,000

11. **Medicine** Sodium-24 is a radioactive isotope of sodium that is used to study circulatory dysfunction. Assuming that 4 micrograms of sodium-24 are injected into a person, the amount A in micrograms remaining in that person after t hours is given by the equation $A = 4e^{-0.046t}$.

a. Graph this equation.

b. What amount of sodium-24 remains after 5 hours? 3.18 μg

c. What is the half-life of sodium-24? ≈15.07 h

d. In how many hours will the amount of sodium-24 be 1 microgram? ≈30.14 h

In Exercises 12 to 16, use the half-life information from Table 4.11, page 394, to work each exercise.

12. **Radioactive Decay** Find the decay function for the amount of polonium (^{210}Po) that remains in a sample after t days. $N(t) \approx N_0 e^{-0.005023t}$

13. **Geology** Geologists have determined that Crater Lake in Oregon was formed by a volcanic eruption. Chemical analysis of a wood chip assumed to be from a tree that died during the eruption has shown that it contains approximately 45% of its original carbon-14. Estimate how long ago the volcanic eruption occurred. ≈6601 years ago

14. **Radioactive Decay** Estimate the percentage of polonium (^{210}Po) that remains in a sample after 2 years. Round to the nearest hundredth of a percent. 2.56%

15. **Archeology** The Rhind papyrus, named after A. Henry Rhind, contains most of what we know today of ancient Egyptian mathematics. A chemical analysis of a sample from the papyrus has shown that it contains approximately 75% of its original carbon-14. Estimate the age of the Rhind papyrus. ≈2378 years old

Indicates Try It Exercises

16. **Archeology** Estimate the age of a bone if it now contains 65% of its original amount of carbon-14. Round to the nearest 100 years. 3600 years old

Compound Interest **In Exercises 17 to 24, solve the given problem related to compound interest.**

17. Find the balance if \$4500 is invested at an annual interest rate of 2.5%, compounded annually, for

    a. 5 years \$5091.34 **b.** 12 years \$6052.00

18. Find the balance if \$17,500 is invested at an annual interest rate of 3.25%, compounded annually, for

    a. 7 years \$21,891.14 **b.** 15 years \$28,274.11

19. If \$22,000 is invested at an annual interest rate of 2.75% for 5 years, find the balance if the interest is compounded

    a. monthly \$25,238.87 **b.** daily \$25,242.71

20. If \$5250 is invested at an annual interest rate of 3.5% for 30 years, find the balance if the interest is compounded

    a. monthly \$14,979.76 **b.** daily \$15,001.91

21. Find the balance if \$3200 is invested at an annual interest rate of 4% for 10 years, compounded continuously. \$4773.84

22. Find the balance if \$55,000 is invested at an annual interest rate of 2.25% for 30 years, compounded continuously. \$108,021.81

23. How long will it take to double your money if it is invested in a certificate of deposit that pays 2.0% annual interest compounded daily? Round to the nearest tenth of a year. 34.7 years

24. How long will it take to triple your money if it is invested in an investment that pays 5.0% annual interest compounded daily? Round to the nearest hundredth of a year. 21.97 years

Continuous Compounding Interest **In Exercises 25 to 28, solve the given problem related to continuous compounding interest.**

25. Use the continuous compounding interest formula to derive an expression for the time it will take money to triple when invested at an annual interest rate of r compounded continuously. $t = \frac{\ln 3}{r}$

26. How long will it take \$1000 to triple if it is invested at an annual interest rate of 5.5% compounded continuously? Round to the nearest year. 20 years

27. How long will it take \$6000 to triple if it is invested in an account that pays 7.6% annual interest compounded continuously? Round to the nearest year. 14 years

28. How long will it take \$10,000 to triple if it is invested in an account that pays 5.5% annual interest compounded continuously? Round to the nearest year. 20 years

In Exercises 29 to 34, determine the following constants for the given logistic growth model.
a. The carrying capacity
b. The growth rate constant
c. The initial population P_0

29. $P(t) = \frac{1900}{1 + 8.5e^{-0.16t}}$ **a.** 1900 **b.** 0.16 **c.** 200

30. $P(t) = \frac{32,550}{1 + 0.75e^{-0.08t}}$ **a.** 32,550 **b.** 0.08 **c.** 18,600

31. $P(t) = \frac{157,500}{1 + 2.5e^{-0.04t}}$ **a.** 157,500 **b.** 0.04 **c.** 45,000

32. $P(t) = \frac{51}{1 + 1.04e^{-0.03t}}$ **a.** 51 **b.** 0.03 **c.** 25

33. $P(t) = \frac{2400}{1 + 7e^{-0.12t}}$ **a.** 2400 **b.** 0.12 **c.** 300

34. $P(t) = \frac{320}{1 + 15e^{-0.12t}}$ **a.** 320 **b.** 0.12 **c.** 20

In Exercises 35 to 38, use algebraic procedures to find the logistic growth model for the data.

35. $P_0 = 400$, $P(2) = 780$, and the carrying capacity is 5500. $P(t) \approx \frac{5500}{1 + 12.75e^{-0.37263t}}$

36. $P_0 = 6200$, $P(8) = 7100$, and the carrying capacity is 9500. $P(t) \approx \frac{9500}{1 + 0.53226e^{-0.05675t}}$

37. $P_0 = 18$, $P(3) = 30$, and the carrying capacity is 100. $P(t) \approx \frac{100}{1 + 4.55556e^{-0.22302t}}$

38. $P_0 = 3200$, $P(22) \approx 5565$, and the growth rate constant is 0.056. $P(t) \approx \frac{8000}{1 + 1.5e^{-0.056t}}$

39. **Revenue** The annual revenue R, in dollars, of a new company can be closely modeled by the logistic function

$$R(t) = \frac{625,000}{1 + 3.1e^{-0.045t}}$$

where the natural number t is the time, in years, since the company was founded.

a. According to the model, what will be the company's annual revenue for its first year and its second year ($t = 1$ and $t = 2$) of operation? Round to the nearest \$1000. \$158,000, \$163,000

b. According to the model, what will the company's annual revenue approach in the long-term future? \$625,000

40. **New Car Sales** The number A of cars sold annually by an automobile dealership can be closely modeled by the logistic function

$$A(t) = \frac{1650}{1 + 2.4e^{-0.055t}}$$

where the natural number t is the time, in years, since the dealership was founded.

a. According to the model, what number of cars will the dealership sell during its first year and its second year ($t = 1$ and $t = 2$) of operation? Round to the nearest unit. 504 cars, 524 cars

b. According to the model, what will the dealership's annual car sales approach in the long-term future? 1650 cars

Population Growth In Exercises 41 to 44, solve the given problem related to population growth.

41. The population of wolves in a preserve satisfies a logistic model in which $P_0 = 312$ in 2008, $c = 1600$, and $P(6) = 416$.

a. Determine the logistic model for this population, where t is the number of years after 2008. $P(t) \approx \dfrac{1600}{1 + 4.12821e^{-0.06198t}}$

b. Use the logistic model from **a** to predict the size of the wolf population in 2018. 497 wolves

42. The population of groundhogs on a ranch satisfies a logistic model in which $P_0 = 240$ in 2010, $c = 3400$, and $P(1) = 310$.

a. Determine the logistic model for this population, where t is the number of years after 2010. $P(t) \approx \dfrac{3400}{1 + 13.16667e^{-0.27833t}}$

b. Use the logistic model from **a** to predict the size of the groundhog population in 2017. About 1182 groundhogs

43. The population of squirrels in a nature preserve satisfies a logistic model in which $P_0 = 1500$ in 2010. The carrying capacity of the preserve is estimated at 8500 squirrels, and $P(2) = 1900$.

a. Determine the logistic model for this population, where t is the number of years after 2010. $P(t) \approx \dfrac{8500}{1 + 4.66667e^{-0.14761t}}$

b. Use the logistic model from **a** to predict the year in which the squirrel population will first exceed 4000. 2019

44. The population of walruses on an island satisfies a logistic model in which $P_0 = 800$ in 2009. The carrying capacity of the island is estimated at 5500 walruses, and $P(1) = 900$.

a. Determine the logistic model for this population, where t is the number of years after 2009. $P(t) \approx \dfrac{5500}{1 + 5.875e^{-0.13929t}}$

b. Use the logistic model from **a** to predict the year in which the walrus population will first exceed 2000. 2017

45. **Physics** Newton's Law of Cooling states that if an object at temperature T_0 is placed into an environment at constant temperature A, then the temperature of the object, $T(t)$ (in degrees Fahrenheit), after t minutes is given by $T(t) = A + (T_0 - A)e^{-kt}$, where k is a constant that depends on the object.

a. Determine the constant k (to the nearest thousandth) for a canned soda drink that takes 5 minutes to cool from 75°F to 65°F after being placed in a refrigerator that maintains a constant temperature of 34°F. 0.056

b. What will be the temperature (to the nearest degree) of the soda drink after 30 minutes? 42°F

c. When (to the nearest minute) will the temperature of the soda drink be 36°F? After 54 min

46. **Psychology** According to a software company, the users of its typing tutorial can expect to type $N(t)$ words per minute after t hours of practice with the product, according to the function $N(t) = 100(1.04 - 0.99^t)$.

a. How many words per minute can a student expect to type after 2 hours of practice? 6 words/min

b. How many words per minute can a student expect to type after 40 hours of practice? 37 words/min

c. According to the function N, how many hours (to the nearest hour) of practice will be required before a student can expect to type 60 words per minute? 82 h

47. **Psychology** In the city of Whispering Palms, which has a population of 80,000 people, the number of people $P(t)$ exposed to a rumor in t hours is given by the function $P(t) = 80{,}000(1 - e^{-0.0005t})$.

a. Find the number of hours until 10% of the population has heard the rumor. 211 h

b. Find the number of hours until 50% of the population has heard the rumor. 1386 h

48. **Law** A lawyer has determined that the number of people $P(t)$ in a city of 1.2 million people who have been exposed to a news item after t days is given by the function

$$P(t) = 1{,}200{,}000(1 - e^{-0.03t})$$

a. How many days after a major crime has been reported has 40% of the population heard of the crime? 17 days

b. A defense lawyer knows it will be difficult to pick an unbiased jury after 80% of the population has heard of the crime. After how many days will 80% of the population have heard of the crime? 54 days

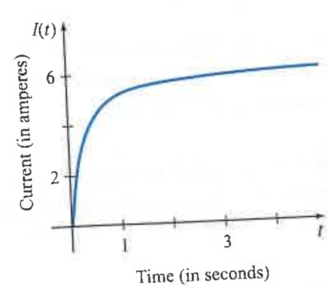
49. **Depreciation** An automobile depreciates according to the function $V(t) = V_0(1 - r)^t$, where $V(t)$ is the value in dollars after t years, V_0 is the original value, and r is the yearly depreciation rate. A car has a yearly depreciation rate of 20%. Determine, to the nearest 0.1 year, in how many years the car will depreciate to half its original value. 3.1 years

50. **Physics** The current $I(t)$ (measured in amperes) of a circuit is given by the function $I(t) = 6(1 - e^{-2.5t})$, where t is the number of seconds after the switch is closed.

a. Find the current when $t = 0$. 0 amps

b. Find the current when $t = 0.5$. ≈4.28 amps

c. Solve the equation for t. $t = -\dfrac{2}{5}\ln\left(1 - \dfrac{I}{6}\right)$



Air Resistance In Exercises 51 to 54, solve the given problems related to air resistance.

51. Assuming that air resistance is proportional to velocity, the velocity v, in feet per second, of a falling object after t seconds is given by $v = 32(1 - e^{-t})$.

a. Graph this equation for $t \geq 0$.

b. Determine algebraically, to the nearest 0.01 second, when the velocity is 20 feet per second. 0.98 s

c. Determine the horizontal asymptote of the graph of v. $v = 32$

d. Write a sentence that explains the meaning of the horizontal asymptote in the context of this application. As time increases, the velocity approaches, but never reaches or exceeds, 32 ft/s.

52. Assuming that air resistance is proportional to velocity, the velocity v, in feet per second, of a falling object after t seconds is given by

$$v = 64(1 - e^{-t/2})$$

a. Graph this equation for $t \geq 0$.

b. Determine algebraically, to the nearest 0.1 second, when the velocity is 50 feet per second. 3.0 s

c. Determine the horizontal asymptote of the graph of v. $v = 64$

d. Write a sentence that explains the meaning of the horizontal asymptote in the context of this application. As time increases, the object's velocity approaches, but never reaches or exceeds, 64 ft/s.

53. The distance s (in feet) that the object in Exercise 51 will fall in t seconds is given by the function

$$s = 32t + 32(e^{-t} - 1)$$

a. Graph this equation for $t \geq 0$.

b. Determine, to the nearest 0.1 second, the time it takes the object to fall 50 feet. 2.5 s

c. Calculate the slope of the secant line through $(1, s(1))$ and $(2, s(2))$. ≈24.56 ft/s

d. Write a sentence that explains the meaning of the slope of the secant line you calculated in **c.** The average speed of the object was approximately 24.56 ft/s during the period from $t = 1$ to $t = 2$ s.

54. The distance s (in feet) that the object in Exercise 52 will fall in t seconds is given by the function

$$s = 64t + 128(e^{-t/2} - 1)$$

a. Graph this equation for $t \geq 0$.

b. Determine, to the nearest 0.1 second, the time it takes the object to fall 50 feet. 2.1 s

c. Calculate the slope of the secant line through $(1, s(1))$ and $(2, s(2))$. ≈33.5 ft/s

d. Write a sentence that explains the meaning of the slope of the secant line you calculated in **c.** The average speed of the object was approximately 33.5 ft/s during the period from $t = 1$ to $t = 2$ s.

55. **Learning Theory** The logistic model is also used in learning theory. Suppose that historical records from employee training at a company show that the percent score on a product information test is given by

$$P = \frac{100}{1 + 25e^{-0.095t}}$$

where t is the number of hours of training. What is the number of hours (to the nearest hour) of training needed before a new employee will answer 75% of the questions correctly? 45 h

56. **Learning Theory** A company provides training in the assembly of a computer circuit to new employees. Past experience has shown that the number of correctly assembled circuits per week can be modeled by

$$N = \frac{250}{1 + 249e^{-0.503t}}$$

where t is the number of weeks of training. What is the number of weeks (to the nearest week) of training needed before a new employee will correctly make 140 circuits per week? 11 weeks

57. **Medication Level** A patient is given three doses of aspirin. Each dose contains 1 gram of aspirin. The second and third doses are each taken 3 hours after the previous dose is administered. The half-life of the aspirin is 2 hours. The amount of aspirin A in the patient's body t hours after the first dose is administered is

$$A(t) = \begin{cases} 0.5^{t/2} & 0 \leq t < 3 \\ 0.5^{t/2} + 0.5^{(t-3)/2} & 3 \leq t < 6 \\ 0.5^{t/2} + 0.5^{(t-3)/2} + 0.5^{(t-6)/2} & t \geq 6 \end{cases}$$

Find, to the nearest hundredth of a gram, the amount of aspirin in the patient's body when

a. $t = 1$ 0.71 g b. $t = 4$ 0.96 g c. $t = 9$ 0.52 g

58. **Medication Level** Use the dosage formula in Exercise 57 to determine when, to the nearest tenth of an hour, the amount of aspirin in the patient's body first reaches 0.25 gram. After 11.1 h

59. **Annual Growth Rate** The exponential growth function for the population of a city is $N(t) = 78{,}245e^{0.0245t}$, where t is in years. Because

$$e^{0.0245t} = (e^{0.0245})^t \approx (1.0248)^t$$

we can write the growth function as

$$N(t) = 78{,}245(1.0248)^t \approx 78{,}245\left(1 + \frac{0.0248}{1}\right)^{1 \cdot t}$$

In this form we can see that the city's population is growing by 2.48% per year.

The population of the city of Lake Tahoe, Nevada, can be modeled by the exponential growth function $N(t) = 22{,}755e^{0.0287t}$. Find the annual growth rate, expressed as a percent, of Lake Tahoe. Round to the nearest hundredth of a percent. 2.91%

60. **Oil Spills** Crude oil leaks from a tank at a rate that depends on the amount of oil that remains in the tank. Because $\frac{1}{8}$ of the oil in the tank leaks out every 2 hours, the volume $V(t)$ of oil in the tank after t hours is given by $V(t) = V_0(0.875)^{t/2}$, where $V_0 = 350{,}000$ gallons, the number of gallons in the tank at the time the tank started to leak ($t = 0$).

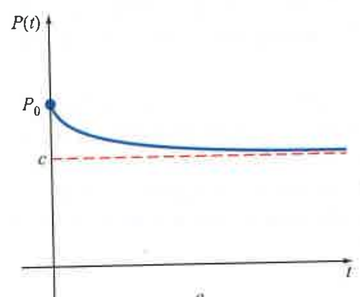
a. How many gallons does the tank hold after 3 hours? 286,471 gal

b. How many gallons does the tank hold after 5 hours? 250,662 gal

c. How long, to the nearest hour, will it take until 90% of the oil has leaked from the tank? 34 h

Enrichment Exercises

If $P_0 > c$ (which implies that $-1 < a < 0$), then the logistic function $P(t) = \dfrac{c}{1 + ae^{-bt}}$ decreases as t increases. Biologists often use this type of logistic function to model populations that decrease over time. See the following figure. Apply this information to Exercises 61 to 63.



$P(t) = \dfrac{c}{1 + ae^{-bt}},\ P_0 > c$

61. **A Declining Fish Population** A biologist finds that the fish population in a small lake can be closely modeled by the logistic function

$$P(t) = \frac{1000}{1 + (-0.3333)e^{-0.05t}}$$

where t is the time, in years, since the lake was first stocked with fish.

a. What was the fish population when the lake was first stocked with fish? 1500

b. According to the logistic model, what will the fish population approach in the long-term future? 1000

62. **A Declining Deer Population** The deer population in a reserve is given by the logistic function

$$P(t) = \frac{1800}{1 + (-0.25)e^{-0.07t}}$$

where t is the time, in years, since July 1, 2010.

a. What was the deer population on July 1, 2010? What was the deer population on July 1, 2012? 2400; 2300

b. According to the logistic model, what will the deer population approach in the long-term future? 1800

63. **Modeling World Record Times in the Men's Mile Race** In the early 1950s, many people speculated that no runner would ever run a mile race in under 4 minutes. During the period from 1913 to 1945, the world record in the mile event had been reduced from 4.14.4 (4 minutes, 14.4 seconds) to 4.01.4, but no one seemed capable of running a sub-4-minute mile. Then, in 1954, Roger Bannister broke through the 4-minute barrier by running a mile in 3.59.6. In 1999, the current record of 3.43.13 was established. It is fun to think about future record times in the mile race. Will they ever go below 3 minutes, 30 seconds? Below 3 minutes, 20 seconds? What about a sub-3-minute mile?

A declining logistic function that closely models the world record times WR, in seconds, in the men's mile run from 1913 ($t = 0$) to 1999 ($t = 86$) is given by

$$WR(t) = \frac{199.13}{1 + (-0.21726)e^{-0.0079889t}}$$

a. Use the above logistic function to predict the world record time for the men's mile run in 2020 and 2050. 3 min, 39.41 s; 3 min, 34.75 s

b. According to the logistic function, what time will the world record in the men's mile event approach but never break through? 3 min, 19.13 s

SECTION 4.7

Analyzing Scatter Plots
Modeling Data
Finding a Logistic Growth Model

Modeling Data with Exponential and Logarithmic Functions

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A28.

PS1. Determine whether $N(t) = 4 - \ln t$ is an increasing or a decreasing function. [4.3] Decreasing

PS2. Determine whether $P(t) = 1 - 2(1.05^t)$ is an increasing or a decreasing function. [4.2] Decreasing

PS3. Evaluate $P(t) = \dfrac{108}{1 + 2e^{-0.1t}}$ for $t = 0$. [4.2] 36

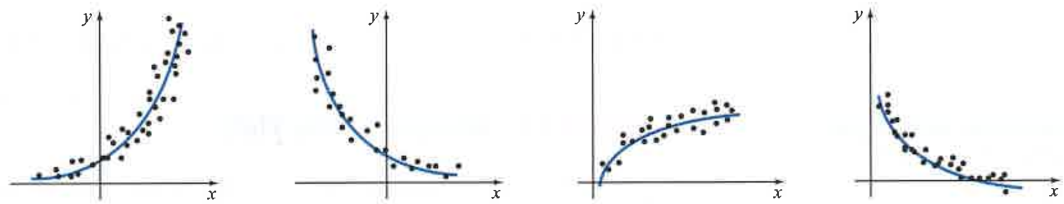
PS4. Evaluate $N(t) = 840e^{1.05t}$ for $t = 0$. [4.2] 840

PS5. Solve $10 = \dfrac{20}{1 + 2.2e^{-0.05t}}$ for t. Round to the nearest tenth. [4.5] 15.8

PS6. Determine the horizontal asymptote of the graph of $P(t) = \dfrac{55}{1 + 3e^{-0.08t}}$, for $t \geq 0$. [4.2] $P = 55$

Analyzing Scatter Plots

In Section 2.7 we used linear and quadratic functions to model several data sets. However, in some applications, data can be modeled more closely by using exponential or logarithmic functions. For instance, Figure 4.43 illustrates some scatter plots that can be modeled effectively by exponential and logarithmic functions.



a. Exponential increasing: $y = ab^x$, $a > 0$, $b > 1$

b. Exponential decreasing: $y = ab^x$, $a > 0$, $0 < b < 1$

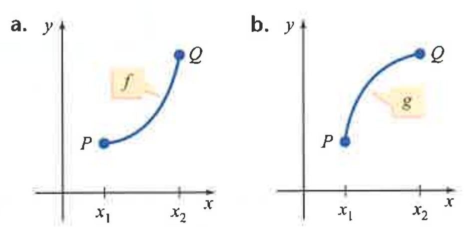
c. Logarithmic increasing: $y = a + b \ln x$, $b > 0$

d. Logarithmic decreasing: $y = a + b \ln x$, $b < 0$

Figure 4.43

Exponential and logarithmic models

The terms *concave upward* and *concave downward* are often used to describe a graph. For instance, Figures 4.44a and 4.44b show the graphs of two increasing functions that join the points P and Q. The graphs of f and g differ in that they bend in different directions. We can distinguish between these two types of "bending" by examining the positions of *tangent lines* to the graphs. In Figures 4.44c and 4.44d, tangent lines (in red) have been drawn to the graphs of f and g. The graph of f lies above its tangent lines, and the graph of g lies below its tangent lines. The function f is said to be concave upward, and g is concave downward.



c. f is concave upward. **d.** g is concave downward.

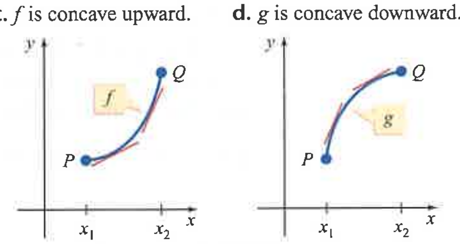


Figure 4.44