b. Assume that you can write 1000 digits per page and that 500 sheets of paper are in a ream of paper. How many reams of paper, to the nearest tenth of a ream, are required to write out the expansion of $9^{(9^9)}$? Assume that you write on only one side of each sheet.

97. Use a graphing utility to graph $f(x) = \frac{e^x - e^{-x}}{2}$ and $g(x) = \ln(x + \sqrt{x^2 + 1})$ on the same screen. Use a square viewing window. What appears to be the relationship between f and g?

98. Use a graphing utility to graph $f(x) = \frac{e^x + e^{-x}}{2}$ for $x \ge 0$ and $g(x) = \ln(x + \sqrt{x^2 - 1})$ for $x \ge 1$ on the same screen. Use a square viewing window. What appears to be the relationship between f and g?

99. Use a graph of $f(x) = \frac{2}{e^x + e^{-x}}$ to determine the domain and range of f.

SECTION 4.4

Properties of Logarithms
Change-of-Base Formula
Logarithmic Scales

Properties of Logarithms and Logarithmic Scales

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A26.

In Exercises PS1 to PS6, use a calculator to compare the values of the given expressions.

PS1. $\log 3 + \log 2; \log 6$ [4.3]

PS2. $\ln 8 - \ln 3; \ln\left(\frac{8}{3}\right)$ [4.3]

PS3. $3 \log 4; \log(4^3)$ [4.3]

PS4. $2 \ln 5; \ln(5^2)$ [4.3]

PS5. $\ln 5; \frac{\log 5}{\log e}$ [4.3]

PS6. $\log 8; \frac{\ln 8}{\ln 10}$ [4.3]

Properties of Logarithms

In Section 4.3 we introduced the following basic properties of logarithms.

$$\log_b b = 1 \quad \text{and} \quad \log_b 1 = 0$$

Also, because exponential functions and logarithmic functions are inverses of each other, we observed the relationships

$$\log_b(b^x) = x \quad \text{and} \quad b^{\log_b x} = x$$

We can use the properties of exponents to establish the following additional logarithmic properties.

Properties of Logarithms

In the following properties, b, M, and N are positive real numbers ($b \neq 1$).

Product property	$\log_b(MN) = \log_b M + \log_b N$
Quotient property	$\log_b \frac{M}{N} = \log_b M - \log_b N$
Power property	$\log_b(M^p) = p \log_b M$
Logarithm-of-each-side property	$M = N$ implies $\log_b M = \log_b N$
One-to-one property	$\log_b M = \log_b N$ implies $M = N$

A proof of the product property is given on the next page.

Caution

Pay close attention to these properties. Note that

$$\log_b(MN) \neq \log_b M \cdot \log_b N$$

and

$$\log_b \frac{M}{N} \neq \frac{\log_b M}{\log_b N}$$

Also,

$$\log_b(M + N) \neq \log_b M + \log_b N$$

In fact, the expression $\log_b(M + N)$ cannot be expanded.

Proof

Let $r = \log_b M$ and $s = \log_b N$. These equations can be written in exponential form as

$$M = b^r \qquad \text{and} \qquad N = b^s$$

Now consider the product MN.

$$MN = b^r b^s \qquad \text{• Substitute for } M \text{ and } N.$$
$$MN = b^{r+s} \qquad \text{• Product property of exponents}$$
$$\log_b MN = r + s \qquad \text{• Write in logarithmic form.}$$
$$\log_b MN = \log_b M + \log_b N \qquad \text{• Substitute for } r \text{ and } s.$$

The last equation is our desired result. ■

The quotient property and the power property can be proved in a similar manner. See Exercises 91 and 92 on page 382.

The properties of logarithms are often used to rewrite logarithmic expressions in an equivalent form. The process of using the product or quotient properties to rewrite a single logarithm as the sum or difference of two or more logarithms, or using the power property to rewrite $\log_b(M^p)$ in its equivalent form $p \log_b M$, is called **expanding the logarithmic expression**. We illustrate this process in Example 1.

EXAMPLE 1 Expand Logarithmic Expressions

Use the properties of logarithms to expand the following logarithmic expressions. Assume all variable expressions represent positive real numbers. When possible, evaluate logarithmic expressions.

a. $\log_5(xy^2)$ **b.** $\ln\left(\dfrac{e\sqrt{y}}{z^3}\right)$

Solution

a.
$$\log_5(xy^2) = \log_5 x + \log_5 y^2 \qquad \text{• Product property}$$
$$= \log_5 x + 2\log_5 y \qquad \text{• Power property}$$

b.
$$\ln\left(\frac{e\sqrt{y}}{z^3}\right) = \ln(e\sqrt{y}) - \ln z^3 \qquad \text{• Quotient property}$$
$$= \ln e + \ln\sqrt{y} - \ln z^3 \qquad \text{• Product property}$$
$$= \ln e + \ln y^{1/2} - \ln z^3 \qquad \text{• Write } \sqrt{y} \text{ as } y^{1/2}.$$
$$= \ln e + \frac{1}{2}\ln y - 3\ln z \qquad \text{• Power property}$$
$$= 1 + \frac{1}{2}\ln y - 3\ln z \qquad \text{• Evaluate } \ln e.$$

▶ **Try Exercise 6, page 379**

The properties of logarithms are also used to *condense* expressions that involve the sum or difference of logarithms into a single logarithm. For instance, we can use the product property to rewrite $\log_b M + \log_b N$ as $\log_b(MN)$, and the quotient property to rewrite $\log_b M - \log_b N$ as $\log_b \dfrac{M}{N}$. Before applying the product or quotient properties, use the power property to write all expressions of the form $p \log_b M$ in their equivalent $\log_b M^p$ form. See Example 2.

Question • Does $\log 2 + \log 5 = 1$?

EXAMPLE 2 Condense Logarithmic Expressions

Use the properties of logarithms to rewrite each expression as a single logarithm with a coefficient of 1. Assume all variable expressions represent positive real numbers.

a. $2 \ln x + \frac{1}{2}\ln(x + 4)$ **b.** $\log_5(x^2 - 4) + 3 \log_5 y - \log_5(x - 2)^2$

Solution

a.
$$2 \ln x + \frac{1}{2}\ln(x + 4) = \ln x^2 + \ln(x + 4)^{1/2}$$ • Power property
$$= \ln[x^2(x + 4)^{1/2}]$$ • Product property
$$= \ln[x^2 \sqrt{(x + 4)}]$$ • Rewriting $(x + 4)^{1/2}$ as $\sqrt{x + 4}$ is an optional step.

b.
$$\log_5(x^2 - 4) + 3 \log_5 y - \log_5(x - 2)^2$$
$$= \log_5(x^2 - 4) + \log_5 y^3 - \log_5(x - 2)^2$$ • Power property
$$= [\log_5(x^2 - 4) + \log_5 y^3] - \log_5(x - 2)^2$$ • Order of Operations Agreement
$$= \log_5[(x^2 - 4)y^3] - \log_5(x - 2)^2$$ • Product property
$$= \log_5\left[\frac{(x^2 - 4)y^3}{(x - 2)^2}\right]$$ • Quotient property
$$= \log_5\left[\frac{(x + 2)(x - 2)y^3}{(x - 2)^2}\right]$$ • Factor.
$$= \log_5\left[\frac{(x + 2)y^3}{x - 2}\right]$$ • Simplify.

Try Exercise 22, page 379

Change-of-Base Formula

Recall that to determine the value of y in $\log_3 81 = y$, we ask the question, "What power of 3 is equal to 81?" Because $3^4 = 81$, we have $\log_3 81 = 4$. Now suppose that we need to determine the value of $\log_3 50$. In this case, we need to find the power of 3 that produces 50. Because $3^3 = 27$ and $3^4 = 81$, the value we are seeking is somewhere between 3 and 4. The following procedure can be used to produce an estimate of $\log_3 50$.

The exponential form of $\log_3 50 = y$ is $3^y = 50$. Applying logarithmic properties gives us

$$3^y = 50$$
$$\ln 3^y = \ln 50$$ • Logarithm-of-each-side property
$$y \ln 3 = \ln 50$$ • Power property
$$y = \frac{\ln 50}{\ln 3} \approx 3.56088$$ • Solve for y.

Thus $\log_3 50 \approx 3.56088$. In the preceding procedure we could just as well have used logarithms of any base and arrived at the same value. Thus any logarithm can be expressed in terms of logarithms of any base we wish. This general result is summarized in the following formula.

Answer • Yes. By the product property, $\log 2 + \log 5 = \log(2 \cdot 5) = \log 10 = 1$.

Integrating Technology

Here are some examples that illustrate how to use WolframAlpha to compute logarithms with various bases.

To compute:

- $\log_3 50$, enter log3(50)
- $\log_{10} 50$, enter log10(50)
- $\ln 50 = \log_e 50$, enter log(50) or ln(50)

Caution: WolframAlpha uses the notation log(50) to represent the natural logarithm of 50, not the common logarithm of 50.

Change-of-Base Formula

If x, a, and b are positive real numbers with $a \neq 1$ and $b \neq 1$, then

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Because most calculators use only common logarithms ($a = 10$) or natural logarithms ($a = e$), the change-of-base formula is used most often in the following form.

If x and b are positive real numbers and $b \neq 1$, then

$$\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

EXAMPLE 3 Use the Change-of-Base Formula

Evaluate each logarithm. Round to the nearest ten-thousandth.

a. $\log_3 18$ **b.** $\log_{12} 400$

Solution

To approximate these logarithms, we may use the change-of-base formula with $a = 10$ or $a = e$. For this example, we choose to use the change-of-base formula with $a = e$. That is, we will evaluate these logarithms by using the LN key on a scientific or graphing calculator.

a. $\log_3 18 = \frac{\ln 18}{\ln 3} \approx 2.6309$ **b.** $\log_{12} 400 = \frac{\ln 400}{\ln 12} \approx 2.4111$

▶ Try Exercise 38, page 380

Study tip

If common logarithms had been used for the calculations in Example 3, the final results would have been the same.

$$\log_3 18 = \frac{\log 18}{\log 3} \approx 2.6309$$

$$\log_{12} 400 = \frac{\log 400}{\log 12} \approx 2.4111$$

The change-of-base formula and a graphing calculator can be used to graph logarithmic functions that have a base other than 10 or e. For instance, to graph $f(x) = \log_3(2x + 3)$, we rewrite the function in terms of base 10 or base e. Using base 10 logarithms, we have $f(x) = \log_3(2x + 3) = \frac{\log(2x + 3)}{\log 3}$. The graph is shown in Figure 4.33.

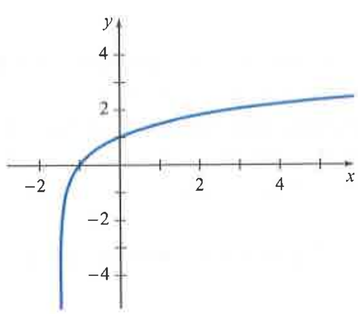


Figure 4.33

EXAMPLE 4 Use the Change-of-Base Formula to Graph a Logarithmic Function

Graph $f(x) = \log_2|x - 3|$.

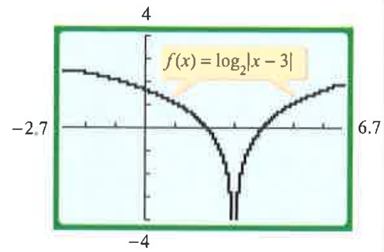
Solution

Rewrite f using the change-of-base formula. We will use the natural logarithm function; however, the common logarithm function could be used instead.

$$f(x) = \log_2|x - 3| = \frac{\ln|x - 3|}{\ln 2}$$

Enter $\frac{\ln|x-3|}{\ln 2}$ into Y1. The graph is shown at the right. Note that the domain of $f(x) = \log_2|x-3|$ is all real numbers except 3 because $|x-3| = 0$ when $x = 3$ and $|x-3|$ is positive for all other values of x.

To produce this graph with WolframAlpha, use the text "plot log2abs(x-3) from −2.7 to 6.7".



▶ Try Exercise 50, page 380

Logarithmic Scales

Logarithmic functions are often used to scale very large (or very small) numbers into numbers that are easier to comprehend. For instance, the *Richter scale* magnitude of an earthquake uses a logarithmic function to convert the intensity of the earthquake's shock waves I into a number M, which for most earthquakes is in the range of 0 to 10. The intensity I of an earthquake is often given in terms of the constant I_0, where I_0 is the intensity of the smallest earthquake (called a **zero-level earthquake**) that can be measured on a seismograph near the earthquake's epicenter. The following formula is used to compute the Richter scale magnitude of an earthquake.

Math Matters

The Richter scale was created by the seismologist Charles F. Richter in 1935. Notice that a tenfold increase in the intensity level of an earthquake increases the Richter scale magnitude of the earthquake by only 1.

Richter Scale Magnitude of an Earthquake

An earthquake with an intensity of I has a **Richter scale magnitude** of

$$M = \log\left(\frac{I}{I_0}\right)$$

where I_0 is the measure of the intensity of a zero-level earthquake.

EXAMPLE 5 Determine the Magnitude of an Earthquake

Find the Richter scale magnitude (to the nearest tenth) of the November 2011 Oklahoma earthquake that had an intensity of $I = 398{,}107I_0$.

Solution

$$M = \log\left(\frac{I}{I_0}\right) = \log\left(\frac{398{,}107 I_0}{I_0}\right) = \log(398{,}107) \approx 5.6$$

The 2011 Oklahoma earthquake had a Richter scale magnitude of 5.6.

▶ Try Exercise 72, page 380

Study tip

Notice in Example 5 that we did not need to know the value of I_0 to determine the Richter scale magnitude of the quake.

If you know the Richter scale magnitude of an earthquake, you can determine the intensity of the earthquake.

EXAMPLE 6 Determine the Intensity of an Earthquake

Find the intensity of the April 2012 Oregon earthquake, which measured 5.9 on the Richter scale.

Solution

$$\log\left(\frac{I}{I_0}\right) = 5.9$$

$$\frac{I}{I_0} = 10^{5.9} \qquad \text{• Write in exponential form.}$$

$$I = 10^{5.9}I_0 \qquad \text{• Solve for } I.$$

$$I \approx 794{,}328I_0$$

The 2012 Oregon earthquake had an intensity that was approximately 794,300 times the intensity of a zero-level earthquake.

▸ Try Exercise 74, page 380

In Example 7 we use the Richter scale magnitudes of two earthquakes to compare the intensities of the earthquakes.

EXAMPLE 7 Compare Intensities of Earthquakes

The 1960 Chile earthquake had a Richter scale magnitude of 9.5. The 1989 San Francisco earthquake had a Richter scale magnitude of 7.1. Compare the intensities of the earthquakes.

Study tip

The results of Example 7 show that if an earthquake has a Richter scale magnitude of M_1 and a smaller earthquake has a Richter scale magnitude of M_2, then the larger earthquake is $10^{M_1 - M_2}$ times as intense as the smaller earthquake.

Solution

Let I_1 be the intensity of the Chilean earthquake, and let I_2 be the intensity of the San Francisco earthquake. Then

$$\log\left(\frac{I_1}{I_0}\right) = 9.5 \qquad \text{and} \qquad \log\left(\frac{I_2}{I_0}\right) = 7.1$$

$$\frac{I_1}{I_0} = 10^{9.5} \qquad\qquad \frac{I_2}{I_0} = 10^{7.1}$$

$$I_1 = 10^{9.5}I_0 \qquad\qquad I_2 = 10^{7.1}I_0$$

To compare the intensities of the earthquakes, we compute the ratio I_1/I_2.

$$\frac{I_1}{I_2} = \frac{10^{9.5}I_0}{10^{7.1}I_0} = \frac{10^{9.5}}{10^{7.1}} = 10^{9.5-7.1} = 10^{2.4} \approx 251$$

The earthquake in Chile was approximately 251 times as intense as the San Francisco earthquake.

▸ Try Exercise 76, page 380

Seismologists generally determine the Richter scale magnitude of an earthquake by examining a **seismogram**. See Figure 4.34.

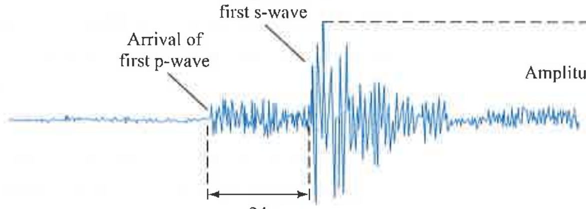


Figure 4.34

The magnitude of an earthquake cannot be determined just by examining the amplitude of a seismogram because this amplitude decreases as the distance between the epicenter of the earthquake and the observation station increases. To account for the distance between the epicenter and the observation station, a seismologist examines a seismogram for small waves called **p-waves** and larger waves called **s-waves**. The Richter scale magnitude M of an earthquake is a function of both the amplitude A of the s-waves and the difference in time t between the occurrence of the s-waves and the occurrence of the p-waves. In the 1950s, Charles Richter developed the following formula to determine the magnitude of an earthquake from the data in a seismogram.

Amplitude–Time–Difference Formula

The Richter scale magnitude M of an earthquake is given by

$$M = \log A + 3\log 8t - 2.92$$

where A is the amplitude, in millimeters, of the s-waves on a seismogram and t is the difference in time, in seconds, between the s-waves and the p-waves.

EXAMPLE 8 Determine the Magnitude of an Earthquake from Its Seismogram

Find the Richter scale magnitude of the earthquake that produced the seismogram in Figure 4.34.

Solution

$$\begin{aligned} M &= \log A + 3\log 8t - 2.92 \\ &= \log 23 + 3\log[8 \cdot 24] - 2.92 \qquad \text{• Substitute 23 for } A \text{ and 24 for } t. \\ &\approx 1.36173 + 6.84990 - 2.92 \\ &\approx 5.3 \end{aligned}$$

The earthquake had a magnitude of about 5.3 on the Richter scale.

Note
The Richter scale magnitude is usually rounded to the nearest tenth.

▸ **Try Exercise 80, page 381**

Logarithmic scales are also used in chemistry. One example concerns the pH of a liquid, which is a measure of the liquid's **acidity** or **alkalinity.** (You may have tested the pH of the water in a swimming pool or an aquarium.) Pure water, which is considered neutral, has a pH of 7.0. The pH scale ranges from 0 to 14, with 0 corresponding to the most acidic solutions and 14 to the most alkaline. Lemon juice has a pH of about 2, whereas household ammonia measures about 11.

Specifically, the pH of a solution is a function of the hydronium-ion concentration of the solution. Because the hydronium-ion concentration of a solution can be very small (with values such as 0.00000001 mole per liter), pH uses a logarithmic scale.

Definition of the pH of a Solution

The **pH of a solution** with a hydronium-ion concentration of $[H^+]$ mole per liter is given by

$$\text{pH} = -\log[H^+]$$

EXAMPLE 9 Find the pH of a Solution

Find the pH of each liquid. Round to the nearest tenth.

a. Orange juice with $[H^+] = 2.8 \times 10^{-4}$ mole per liter

b. Milk with $[H^+] = 3.97 \times 10^{-7}$ mole per liter

c. Rainwater with $[H^+] = 6.31 \times 10^{-5}$ mole per liter

d. A baking soda solution with $[H^+] = 3.98 \times 10^{-9}$ mole per liter

Solution

a. $\text{pH} = -\log[H^+] = -\log(2.8 \times 10^{-4}) \approx 3.6$
The orange juice has a pH of 3.6.

b. $\text{pH} = -\log[H^+] = -\log(3.97 \times 10^{-7}) \approx 6.4$
The milk has a pH of 6.4.

c. $\text{pH} = -\log[H^+] = -\log(6.31 \times 10^{-5}) \approx 4.2$
The rainwater has a pH of 4.2.

d. $\text{pH} = -\log[H^+] = -\log(3.98 \times 10^{-9}) \approx 8.4$
The baking soda solution has a pH of 8.4.

Try Exercise 82, page 381

Math Matters

The pH scale was created by the Danish biochemist Søren Sørensen in 1909 to measure the acidity of water used in the brewing of beer. pH is an abbreviation for *pondus hydrogenii*, which translates as "potential hydrogen."

Figure 4.35 illustrates the pH scale, along with the corresponding hydronium-ion concentrations. A solution on the left half of the scale, with a pH of less than 7, is an **acid,** and a solution on the right half of the scale is an **alkaline solution,** or a **base.** Because the scale is logarithmic, a solution with a pH of 5 is 10 times more acidic than a solution with a pH of 6. From Example 9, we see that the orange juice, milk, and rainwater are acids, whereas the baking soda solution is a base.

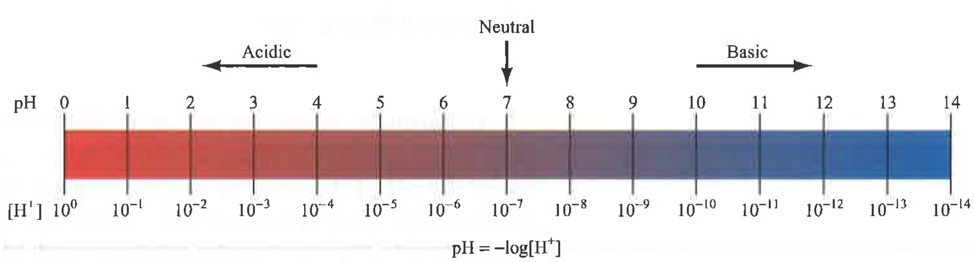


Figure 4.35

EXAMPLE 10 Find the Hydronium-Ion Concentration

A sample of blood has a pH of 7.3. Find the hydronium-ion concentration of the blood.

Solution

$$\text{pH} = -\log[\text{H}^+]$$
$$7.3 = -\log[\text{H}^+] \quad \text{• Substitute 7.3 for pH.}$$
$$-7.3 = \log[\text{H}^+] \quad \text{• Multiply both sides by } -1.$$
$$10^{-7.3} = [\text{H}^+] \quad \text{• Change to exponential form.}$$
$$5.0 \times 10^{-8} \approx [\text{H}^+]$$

The hydronium-ion concentration of the blood is about 5.0×10^{-8} mole per liter.

▶ Try Exercise 84, page 381

EXERCISE SET 4.4

Concept Check

In Exercises 1 to 4, use the given information to complete the sentence. Assume *b*, *M*, and *N* are positive real numbers and $b \neq 1$.

1. The product property of logarithms states that $\log_b(MN)$ equals ________.
2. The quotient property of logarithms states that $\log_b\left(\frac{M}{N}\right)$ equals ________.
3. The power property of logarithms states that $\log_b(M^p)$ equals ________.
4. The one-to-one property of logarithms states that $\log_b M = \log_b N$ implies ________.

In Exercises 5 to 20, expand the given logarithmic expression. Assume all variable expressions represent positive real numbers. When possible, evaluate logarithmic expressions. Do not use a calculator.

5. $\log_b(xyz)$
6. $\ln \frac{z^3}{\sqrt{xy}}$
7. $\log_4\left(\frac{y^2}{16x}\right)$
8. $\ln\left(\frac{e^3}{xy^2}\right)$
9. $\log_2 \frac{\sqrt{x}}{y^3}$
10. $\log_b (x\sqrt[3]{y})$
11. $\log_7 \frac{\sqrt{xz}}{y^2}$
12. $\ln \sqrt[3]{x^2\sqrt{y}}$
13. $\ln(e^2 z)$
14. $\ln(x^{1/2} y^{2/3})$
15. $\log_6\left(\frac{x^3\sqrt[3]{y}}{216z^4}\right)$
16. $\log_5\left(\frac{x\sqrt{z}}{25y^2}\right)$
17. $\log \sqrt{x\sqrt{z}}$
18. $\ln\left(\frac{\sqrt[3]{x^2}}{z^2}\right)$
19. $\ln(\sqrt[3]{z\sqrt{e}})$
20. $\ln\left[\frac{x^2\sqrt{z}}{y^{-3}}\right]$

In Exercises 21 to 36, write each expression as a single logarithm with a coefficient of 1. Assume all variable expressions represent positive real numbers.

21. $\log(x + 5) + 2\log x$
22. $3\log_2 t - \frac{1}{3}\log_2 u + 4\log_2 v$
23. $\ln(x^2 - y^2) - \ln(x - y)$
24. $\frac{1}{2}\log_8(x + 5) - 3\log_8 y$
25. $2\log x + \log y - \frac{1}{2}\log y$
26. $\ln(xy) + 3\ln\left(\frac{y}{z}\right) - 2\ln(xyz)$
27. $\log(xy^2) - \log z$
28. $\ln(y^{1/2}z) - \ln z^{1/2}$

Indicates Try It Exercises

29. $2(\log_6 x + \log_6 y^2) - \log_6(x + 2)$

30. $\frac{1}{2}\log_3 x - \log_3 y + 2\log_3(x + 2)$

31. $2\ln(x + 4) - \ln x - \ln(x^2 - 3)$

32. $\log(3x) - (2\log x - \log y)$

33. $\ln(2x + 5) - \ln y - 2\ln z + \frac{1}{2}\ln w$

34. $\log_b x + \log_b(y + 3) + \log_b(y + 2) - \log_b(y^2 + 5y + 6)$

35. $\ln(x^2 - 9) - 2\ln(x - 3) + 3\ln y$

36. $\log_b(x^2 + 7x + 12) - 2\log_b(x + 4)$

In Exercises 37 to 48, use the change-of-base formula to approximate the logarithm accurate to the nearest ten-thousandth.

37. $\log_7 20$

38. $\log_5 37$

39. $\log_2 14$

40. $\log_{25} 15$

41. $\log_8\left(\frac{3}{5}\right)$

42. $\log_{11}\left(\frac{2}{3}\right)$

43. $\log_9 \sqrt{17}$

44. $\log_4 \sqrt{7}$

45. $\log_{\sqrt{2}} 17$

46. $\log_{\sqrt{3}} 5.5$

47. $\log_\pi e$

48. $\log_\pi \sqrt{15}$

In Exercises 49 to 56, use a graphing utility and the change-of-base formula to graph the logarithmic function.

49. $f(x) = \log_4 x$

50. $g(x) = \log_8(5 - x)$

51. $g(x) = \log_8(x - 3)$

52. $t(x) = \log_9(5 - x)$

53. $h(x) = \log_3(x - 3)^2$

54. $J(x) = \log_{12}(-x)$

55. $F(x) = -\log_5|x - 2|$

56. $n(x) = \log_2\sqrt{x - 8}$

In Exercises 57 to 66, determine whether the statement is true or false for all $x > 0$, $y > 0$. If it is false, write an example that disproves the statement.

57. $\log_b(x + y) = \log_b x + \log_b y$

58. $\log_b(xy) = \log_b x \cdot \log_b y$

59. $\log_b(xy) = \log_b x + \log_b y$

60. $\log_b x \cdot \log_b y = \log_b x + \log_b y$

61. $\log_b x - \log_b y = \log_b(x - y), \quad x > y$

62. $\log_b \frac{x}{y} = \frac{\log_b x}{\log_b y}$

63. $\frac{\log_b x}{\log_b y} = \log_b x - \log_b y$

64. $\log_b(x^n) = n\log_b x$

65. $(\log_b x)^n = n\log_b x$

66. $\log_b \sqrt{x} = \frac{1}{2}\log_b x$

In Exercises 67 and 68, evaluate the given expression *without* using a calculator.

67. $\log_3 5 \cdot \log_5 7 \cdot \log_7 9$

68. $\log_5 20 \cdot \log_{20} 60 \cdot \log_{60} 100 \cdot \log_{100} 125$

69. Which is larger, 500^{501} or 506^{500}? These numbers are too large for most calculators to handle. (They each have 1353 digits!) (*Hint:* Compare the logarithms of each number.)

70. Which is smaller, $\frac{1}{50^{300}}$ or $\frac{1}{151^{233}}$? See the hint in Exercise 69.

71. **Earthquake Magnitude** The Gulf of California earthquake of April 2012 had an intensity of $I = 7{,}943{,}300I_0$. What did this earthquake measure on the Richter scale?

72. **Earthquake Magnitude** The Oaxaca, Mexico, earthquake of March 2012 had an intensity of $I = 25{,}118{,}900I_0$. What did this earthquake measure on the Richter scale?

73. **Earthquake Intensity** The earthquake that occurred off the coast of Northern Sumatra in April 2012 had a Richter scale magnitude of 8.6. Find the intensity of this earthquake.

74. **Earthquake Intensity** The earthquake that occurred just south of Concepción, Chile, in 1960 had a Richter scale magnitude of 9.5. Find the intensity of this earthquake.

75. **Comparison of Earthquakes** Compare the intensity of an earthquake that measures 5.0 on the Richter scale to the intensity of an earthquake that measures 3.0 on the Richter scale by finding the ratio of the larger intensity to the smaller intensity.

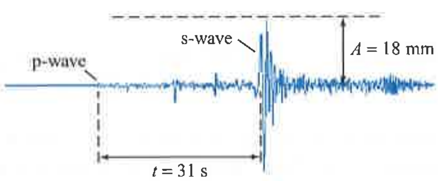
76. **Comparison of Earthquakes** How many times as great was the intensity of the 1960 earthquake in Chile, which measured 9.5 on the Richter scale, than the San Francisco earthquake of 1906, which measured 8.3 on the Richter scale?

77. **Comparison of Earthquakes** The earthquake that occurred near the east coast of Honshu, Japan, on

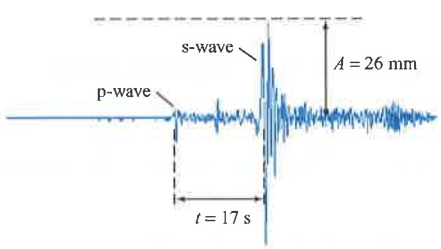
March 11, 2011, measured 9.0 on the Richter scale. In October 1989, an earthquake of magnitude 7.1 on the Richter scale struck the San Francisco Bay area. Compare the intensity of the larger earthquake to the intensity of the smaller earthquake by finding the ratio of the larger intensity to the smaller intensity.

78. **Comparison of Earthquakes** An earthquake that occurred in China in 1978 measured 8.2 on the Richter scale. In 1988, an earthquake in California measured 6.9 on the Richter scale. Compare the intensity of the larger earthquake to the intensity of the smaller earthquake by finding the ratio of the larger intensity to the smaller intensity.

79. **Earthquake Magnitude** Find the Richter scale magnitude of the earthquake that produced the seismogram in the following figure.



80. **Earthquake Magnitude** Find the Richter scale magnitude of the earthquake that produced the seismogram in the following figure.



81. **pH** Milk of magnesia has a hydronium-ion concentration of about 3.97×10^{-11} mole per liter. Determine the pH of milk of magnesia and state whether milk of magnesia is an acid or a base.

82. **pH** Vinegar has a hydronium-ion concentration of 1.26×10^{-3} mole per liter. Determine the pH of vinegar and state whether vinegar is an acid or a base.

83. **Hydronium-Ion Concentration** A morphine solution has a pH of 9.5. Determine the hydronium-ion concentration of the morphine solution.

84. **Hydronium-Ion Concentration** A rainstorm in New York City produced rainwater with a pH of 5.6. Determine the hydronium-ion concentration of the rainwater.

Decibel Level The range of sound intensities that the human ear can detect is so large that a special decibel scale (named after Alexander Graham Bell) is used to measure and compare sound intensities. The decibel level (dB) of a sound is given by

$$dB(I) = 10\log\left(\frac{I}{I_0}\right)$$

where I_0 is the intensity of sound that is barely audible to the human ear. Use the decibel level formula to work Exercises 85 to 88.

85. Find the decibel level for the following sounds. Round to the nearest tenth of a decibel.

Sound	Intensity
a. Automobile traffic	$I = 1.58 \times 10^8 \cdot I_0$
b. Quiet conversation	$I = 10{,}800 \cdot I_0$
c. Fender guitar	$I = 3.16 \times 10^{11} \cdot I_0$
d. Jet engine	$I = 1.58 \times 10^{15} \cdot I_0$

86. A team in Arizona installed in a Ford Bronco a 48,000-watt sound system that it claims can output 175-decibel sound. The human pain threshold for sound is 125 decibels. How many times as great is the intensity of the sound from the Bronco than the human pain threshold for sound?

87. How many times as great is the intensity of a sound that measures 120 decibels than a sound that measures 110 decibels?

88. If the intensity of a sound is doubled, what is the increase in the decibel level? (*Hint:* Find $dB(2I) - dB(I)$.)

Enrichment Exercises

89. **Animated Maps** A software company that creates interactive maps for websites has designed an animated zooming feature such that when a user selects the zoom-in option, the map appears to expand on a location. This is accomplished by displaying several intermediate maps to give the illusion of motion. The company has determined that zooming in on a location is more informative and pleasing to observe when the scale of each step of the animation is determined using the equation

$$S_n = S_0 \cdot 10^{\frac{n}{N}(\log S_f - \log S_0)}$$

where S_n represents the scale of the current step n ($n = 0$ corresponds to the initial scale), S_0 is the starting scale of the map, S_f is the final scale, and N is the number of steps in the animation following the initial scale. (If the initial scale of the map is 1:200, then $S_0 = 200$.) Determine the scales to be used at each intermediate step if a map is to start with a scale of 1:1,000,000, and proceed through five intermediate steps to end with a scale of 1:500,000.

90. **Animated Maps** Use the equation in Exercise 89 to determine the scales for each stage of an animated map zoom that goes from a scale of 1:250,000 to a scale of 1:100,000 in four steps (following the initial scale).

91. Prove the quotient property of logarithms

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

(*Hint:* See the proof of the product property of logarithms on page 372.)

92. Prove the power property of logarithms

$$\log_b(M^p) = p \log_b M$$

See the hint given in Exercise 91.

MID-CHAPTER 4 QUIZ

1. Use composition of functions to verify that

$$f(x) = \frac{500 + 120x}{x} \quad \text{and} \quad g(x) = \frac{500}{x - 120}$$

are inverses of each other.

2. Find the inverse of $f(x) = \dfrac{24x + 5}{x - 4}, x \neq 4$. State any restrictions on the domain of $f^{-1}(x)$.

3. Evaluate $f(x) = e^x$, for $x = -2.4$. Round to the nearest ten-thousandth.

4. Write $\ln x = 6$ in exponential form.

5. Graph $f(x) = \log_3(x + 3)$.

6. Expand $\ln\left(\dfrac{xy^3}{e^2}\right)$. Assume x and y are positive real numbers.

7. Write $\log_3 x^4 - 2\log_3 z + \log_3(xy^2)$ as a single logarithm with a coefficient of 1. Assume all variables are positive real numbers.

8. Use the change-of-base formula to evaluate $\log_8 411$. Round to the nearest ten-thousandth.

9. What is the Richter scale magnitude of an earthquake with an intensity of $789{,}251 I_0$? Round to the nearest tenth.

10. How many times as great is the intensity of an earthquake that measures 7.9 on the Richter scale than the intensity of an earthquake that measures 5.1 on the Richter scale?

SECTION 4.5

Solving Exponential Equations
Solving Logarithmic Equations

Exponential and Logarithmic Equations

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A27.

PS1. Use the definition of a logarithm to write the exponential equation $3^6 = 729$ in logarithmic form. [4.3]

PS2. Use the definition of a logarithm to write the logarithmic equation $\log_5 625 = 4$ in exponential form. [4.3]

PS3. Use the definition of a logarithm to write the exponential equation $a^{x+2} = b$ in logarithmic form. [4.3]

PS4. Solve for x: $4a = 7bx + 2cx$ [1.2]

PS5. Solve for x: $165 = \dfrac{300}{1 + 12x}$ [1.4]

PS6. Solve for x: $A = \dfrac{100 + x}{100 - x}$ [1.4]

Solving Exponential Equations

If a variable appears in the exponent of a term of an equation, such as in $2^{x+1} = 32$, then the equation is called an **exponential equation**. Example 1 uses the following Equality of Exponents Theorem to solve $2^{x+1} = 32$.