4.4 Factoring Polynomials

Learning Target Factor polynomials and use the Factor Theorem.

Success Criteria
- I can find common monomial factors of polynomials.
- I can factor polynomials.
- I can use the Factor Theorem.

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EXPLORE IT! Factoring Polynomials

Work with a partner.

a. When sketching the graph of the function $f(x) = x^3 + 7x^2 + 7x - 15$, you notice that the sum of the coefficients of the terms is zero.

$$1 + 7 + 7 + (-15) = 0$$

What does this tell you about $f(1)$?

b. How can you use the result of part (a) to algebraically find other zeros of the function?

c. Find all the zeros of $f(x) = x^3 + 7x^2 + 7x - 15$. Use the zeros to sketch a graph of f.

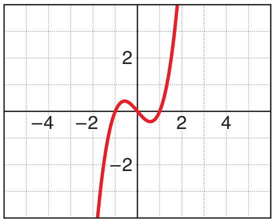
d. Use technology to match each polynomial function with its graph. Then write each polynomial function in factored form. Explain your reasoning.

i. $f(x) = x^2 + 5x + 4$ **ii.** $f(x) = x^3 - 2x^2 - x + 2$

iii. $f(x) = x^3 + x^2 - 2x$ **iv.** $f(x) = x^3 - x$

v. $f(x) = x^4 - 5x^2 + 4$ **vi.** $f(x) = x^4 - 2x^3 - x^2 + 2x$

A.



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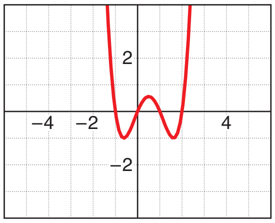

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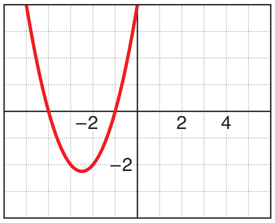

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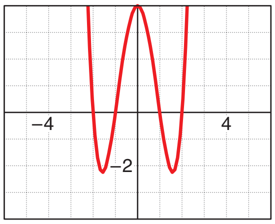

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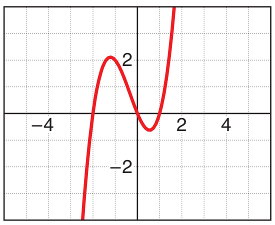

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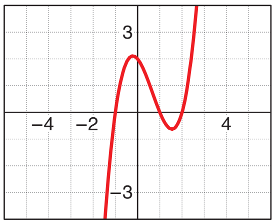

e. What information can you obtain about the graph of a polynomial function written in factored form?

Math Practice

Justify Conclusions
Given that $x - k$ is a factor of a polynomial $f(x)$, what can you determine about the value $f(k)$? Justify your answer.

Factoring Polynomials

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Vocabulary (VOCAB)

factored completely, *p. 176*
factor by grouping, *p. 177*
quadratic form, *p. 177*

Previously, you factored quadratic polynomials. You can also factor polynomials with degree greater than 2. Some of these polynomials can be *factored completely* using techniques you have previously learned. A factorable polynomial with integer coefficients is **factored completely** when it is written as a product of unfactorable polynomials with integer coefficients.

EXAMPLE 1 Finding a Common Monomial Factor

WATCH

Factor each polynomial completely.

a. $x^3 - 4x^2 - 5x$ **b.** $3y^5 - 48y^3$ **c.** $5z^4 + 30z^3 + 45z^2$

SOLUTION

a. $x^3 - 4x^2 - 5x = x(x^2 - 4x - 5)$ Factor common monomial.

$= x(x - 5)(x + 1)$ Factor trinomial.

b. $3y^5 - 48y^3 = 3y^3(y^2 - 16)$ Factor common monomial.

$= 3y^3(y - 4)(y + 4)$ Difference of two squares pattern

c. $5z^4 + 30z^3 + 45z^2 = 5z^2(z^2 + 6z + 9)$ Factor common monomial.

$= 5z^2(z + 3)^2$ Perfect square trinomial pattern

In part (b) of Example 1, the special factoring pattern for the difference of two squares was used to factor the expression completely. There are also factoring patterns that you can use to factor the sum or difference of two *cubes*.

KEY IDEA

Special Factoring Patterns

Sum of Two Cubes

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Example

$64x^3 + 1 = (4x)^3 + 1^3$

$= (4x + 1)(16x^2 - 4x + 1)$

Difference of Two Cubes

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example

$27x^3 - 8 = (3x)^3 - 2^3$

$= (3x - 2)(9x^2 + 6x + 4)$

EXAMPLE 2 Factoring the Sum or Difference of Two Cubes

Factor (a) $x^3 - 125$ and (b) $16s^5 + 54s^2$ completely.

WATCH

SOLUTION

a. $x^3 - 125 = x^3 - 5^3$ Write as $a^3 - b^3$.

$= (x - 5)(x^2 + 5x + 25)$ Difference of two cubes pattern

b. $16s^5 + 54s^2 = 2s^2(8s^3 + 27)$ Factor common monomial.

$= 2s^2[(2s)^3 + 3^3]$ Write $8s^3 + 27$ as $a^3 + b^3$.

$= 2s^2(2s + 3)(4s^2 - 6s + 9)$ Sum of two cubes pattern

For some polynomials, you can **factor by grouping** pairs of terms that have a common monomial factor. The pattern for factoring by grouping is shown below.

$$ra + rb + sa + sb = r(a + b) + s(a + b)$$
$$= (r + s)(a + b)$$

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EXAMPLE 3 Factoring by Grouping

WATCH

Factor $z^3 + 5z^2 - 4z - 20$ completely.

SOLUTION

$z^3 + 5z^2 - 4z - 20 = z^2(z + 5) - 4(z + 5)$ — Factor by grouping.

$= (z^2 - 4)(z + 5)$ — Distributive Property

$= (z - 2)(z + 2)(z + 5)$ — Difference of two squares pattern

An expression of the form $au^2 + bu + c$, where u is an algebraic expression, is said to be in **quadratic form**. The factoring techniques you have studied can sometimes be used to factor such expressions.

EXAMPLE 4 Factoring Polynomials in Quadratic Form

WATCH

Factor (a) $16x^4 - 81$ and (b) $3p^8 + 15p^5 + 18p^2$ completely.

SOLUTION

a. $16x^4 - 81 = (4x^2)^2 - 9^2$ — Write as $a^2 - b^2$.

$= (4x^2 + 9)(4x^2 - 9)$ — Difference of two squares pattern

$= (4x^2 + 9)(2x - 3)(2x + 3)$ — Difference of two squares pattern

b. $3p^8 + 15p^5 + 18p^2 = 3p^2(p^6 + 5p^3 + 6)$ — Factor common monomial.

$= 3p^2(p^3 + 3)(p^3 + 2)$ — Factor trinomial in quadratic form.

Math Practice

Look for Structure

The expression $16x^4 - 81$ is in quadratic form because it can be written as $u^2 - 81$ where $u = 4x^2$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Factor the polynomial completely.

1. $x^3 - 7x^2 + 10x$

2. $3n^7 - 75n^5$

3. $8m^5 - 16m^4 + 8m^3$

4. $a^3 + 27$

5. $6z^5 - 750z^2$

6. $x^3 + 4x^2 - x - 4$

7. $3y^3 + y^2 + 9y + 3$

8. $-16n^4 + 625$

9. $5w^6 - 25w^4 + 30w^2$

10. WRITING How do you know when a polynomial is factored completely?

11. VOCABULARY Explain when you should try factoring a polynomial by grouping.

12. WHICH ONE DOESN'T BELONG? Which expression does *not* belong with the other three? Explain your reasoning.

$12x^4 - 3x^2$ | $3x^2(4x^2 - 1)$ | $3x^2(2x + 1)(2x - 1)$ | $3x^2(2x - 1)^2$

The Factor Theorem

When dividing polynomials in the previous section, the examples had nonzero remainders. Suppose the remainder is 0 when a polynomial $f(x)$ is divided by $x - k$. Then,

$$\frac{f(x)}{x-k} = q(x) + \frac{0}{x-k} = q(x)$$

where $q(x)$ is the quotient polynomial. Therefore, $f(x) = (x - k) \cdot q(x)$, so that $x - k$ is a factor of $f(x)$. This result is summarized by the *Factor Theorem*, which is a special case of the Remainder Theorem.

READING

In other words, $x - k$ is a factor of $f(x)$ if and only if k is a zero of f.

KEY IDEA

The Factor Theorem

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

EXAMPLE 5 Determining Whether a Linear Binomial is a Factor

Determine whether (a) $x - 2$ is a factor of $f(x) = x^2 + 2x - 4$ and (b) $x + 5$ is a factor of $f(x) = 3x^4 + 15x^3 - x^2 + 25$.

SOLUTION

a. Find $f(2)$ by direct substitution.

$$\begin{aligned} f(2) &= 2^2 + 2(2) - 4 \\ &= 4 + 4 - 4 \\ &= 4 \end{aligned}$$

Because $f(2) \neq 0$, the binomial $x - 2$ is not a factor of $f(x) = x^2 + 2x - 4$.

b. Find $f(-5)$ by synthetic division.

−5	3	15	−1	0	25
		−15	0	5	−25
	3	0	−1	5	0

Because $f(-5) = 0$, the binomial $x + 5$ is a factor of $f(x) = 3x^4 + 15x^3 - x^2 + 25$.

STUDY TIP

In part (b), notice that direct substitution would have resulted in more difficult computations than synthetic division.

EXAMPLE 6 Factoring a Polynomial

Show that $x + 3$ is a factor of $f(x) = x^4 + 3x^3 - x - 3$. Then factor $f(x)$ completely.

SOLUTION

Show that $f(-3) = 0$ by synthetic division.

−3	1	3	0	−1	−3
		−3	0	0	3
	1	0	0	−1	0

Because $f(-3) = 0$, you can conclude that $x + 3$ is a factor of $f(x)$ by the Factor Theorem. Use the result to write $f(x)$ as a product of two factors and then factor completely.

$f(x) = x^4 + 3x^3 - x - 3$ — Write original polynomial.

$= (x + 3)(x^3 - 1)$ — Write as a product of two factors.

$= (x + 3)(x - 1)(x^2 + x + 1)$ — Difference of two cubes pattern

Because the x-intercepts of the graph of a function are the zeros of the function, you can use the graph to approximate the zeros. You can check the approximations using the Factor Theorem.

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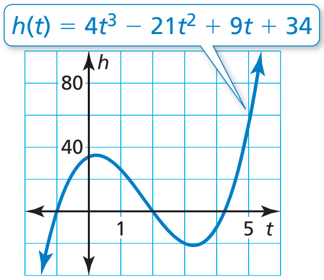
EXAMPLE 7 Modeling Real Life

WATCH INFO



A roller coaster starts at a height of 34 feet and then goes through an underground tunnel. The function $h(t) = 4t^3 - 21t^2 + 9t + 34$ represents the coaster's height h (in feet) after t seconds, where $0 \le t \le 5$. How long is the coaster in the tunnel?

SOLUTION

1. **Understand the Problem** You are given a function rule and a graph that represent the height of a roller coaster. You are asked to determine how long the roller coaster is in an underground tunnel.
2. **Make a Plan** Use the graph to estimate the zeros of the function and check using the Factor Theorem. Then use the zeros to describe where the graph lies below the t-axis.
3. **Solve and Check** From the graph, two of the zeros appear to be -1 and 2. The third zero is between 4 and 5.

STUDY TIP

You could also check that 2 is a zero using the original function, but using the resulting quotient polynomial helps you find the remaining factor.

Step 1 Determine whether -1 is a zero using synthetic division.

-1	4	-21	9	34
		-4	25	-34
	4	-25	34	0

$h(-1) = 0$, so -1 is a zero of h and $t + 1$ is a factor of $h(t)$.

Step 2 Determine whether 2 is a zero. If 2 is also a zero, then $t - 2$ is a factor of the resulting quotient polynomial. Check using synthetic division.

2	4	-25	34
		8	-34
	4	-17	0

The remainder is 0, so $t - 2$ is a factor of $h(t)$ and 2 is a zero of h.

So, $h(t) = (t + 1)(t - 2)(4t - 17)$. The factor $4t - 17$ indicates that the zero between 4 and 5 is $\frac{17}{4}$, or 4.25.

▶ The zeros are -1, 2, and 4.25. Only $t = 2$ and $t = 4.25$ are in the given domain. The graph shows that the roller coaster is in the underground tunnel for $4.25 - 2 = 2.25$ seconds.

Check

Use technology to analyze function values.

t	$h(t)$
0.5	33.75
1.25	20.25
2	0
2.75	−16.88
3.5	−20.25
4.25	0
5	54

zero (at $t = 2$); negative (at $t = 2.75$ and $t = 3.5$); zero (at $t = 4.25$)

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

13. Determine whether $x - 4$ is a factor of $f(x) = 2x^2 + 5x - 12$.

14. Show that $x - 6$ is a factor of $f(x) = x^3 - 5x^2 - 6x$. Then factor $f(x)$ completely.

15. In Example 7, does your answer change when you first determine whether 2 is a zero and then whether -1 is a zero? Justify your answer.

16. WRITING Explain the Factor Theorem and why it is useful.

4.4 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–8, factor the polynomial completely. *Example 1*

1. $x^3 - 2x^2 - 24x$
2. $4k^5 - 100k^3$
3. $3p^5 - 192p^3$
4. $2m^6 - 24m^5 + 64m^4$
5. $2q^4 + 9q^3 - 18q^2$
6. $3r^6 - 11r^5 - 20r^4$
7. $10w^{10} - 19w^9 + 6w^8$
8. $18v^9 + 33v^8 + 14v^7$

In Exercises 9–16, factor the polynomial completely. *Example 2*

9. $x^3 + 64$
10. $y^3 + 512$
11. $g^3 - 343$
12. $c^3 - 27$
13. $4h^9 - 256h^6$
14. $9n^6 - 6561n^3$
15. $16t^7 + 250t^4$
16. $270z^{11} - 80z^8$

ERROR ANALYSIS In Exercises 17 and 18, describe and correct the error in factoring the polynomial completely.

17. ✗
$$3x^3 + 27x = 3x(x^2 + 9)$$
$$= 3x(x + 3)(x - 3)$$

18. ✗
$$x^9 + 8x^3 = (x^3)^3 + (2x)^3$$
$$= (x^3 + 2x)[(x^3)^2 - (x^3)(2x) + (2x)^2]$$
$$= (x^3 + 2x)(x^6 - 2x^4 + 4x^2)$$

In Exercises 19–26, factor the polynomial completely. *Example 3*

19. $y^3 - 5y^2 + 6y - 30$
20. $m^3 - m^2 + 7m - 7$
21. $3a^3 + 18a^2 + 8a + 48$
22. $2k^3 - 20k^2 + 5k - 50$
23. $x^3 - 8x^2 - 4x + 32$
24. $z^3 - 5z^2 - 9z + 45$
25. $4q^3 - 16q^2 - 9q + 36$
26. $16n^3 + 32n^2 - n - 2$

In Exercises 27–34, factor the polynomial completely. *Example 4*

27. $49k^4 - 9$
28. $4m^4 - 25$
29. $c^4 + 9c^2 + 20$
30. $y^4 - 3y^2 - 28$
31. $16z^4 - 625$
32. $81a^4 - 256$
33. $3r^8 + 3r^5 - 60r^2$
34. $4n^{12} - 32n^7 + 48n^2$

In Exercises 35–40, determine whether the binomial is a factor of the polynomial. *Example 5*

35. $f(x) = 2x^3 + 5x^2 - 37x - 60;\ x - 4$
36. $g(x) = 3x^3 - 28x^2 + 29x + 140;\ x + 7$
37. $h(x) = 6x^5 - 15x^4 - 9x^3;\ x + 3$
38. $g(x) = 8x^5 - 58x^4 + 60x^3 + 140;\ x - 6$
39. $h(x) = 6x^4 - 6x^3 - 84x^2 + 144x;\ x + 4$
40. $t(x) = 48x^4 + 36x^3 - 138x^2 - 36x;\ x + 2$

In Exercises 41–46, show that the binomial is a factor of the polynomial. Then factor the polynomial completely. *Example 6*

41. $g(x) = x^3 - x^2 - 20x;\ x + 4$
42. $t(x) = x^3 - 5x^2 - 9x + 45;\ x - 5$
43. $f(x) = x^4 - 6x^3 - 8x + 48;\ x - 6$
44. $s(x) = x^4 + 4x^3 - 64x - 256;\ x + 4$
45. $r(x) = x^3 - 37x + 84;\ x + 7$
46. $h(x) = x^3 - x^2 - 24x - 36;\ x + 2$

MP STRUCTURE In Exercises 47–54, use the method of your choice to factor the polynomial completely. Explain your reasoning.

47. $a^6 + a^5 - 30a^4$
48. $8m^3 - 343$
49. $z^3 - 7z^2 - 9z + 63$
50. $2p^8 - 12p^5 + 16p^2$
51. $64r^3 + 729$
52. $5x^5 - 10x^4 - 40x^3$
53. $16n^4 - 1$
54. $9k^3 - 24k^2 + 3k - 8$

ANALYZING RELATIONSHIPS **In Exercises 55–58, match the function with the correct graph. Explain your reasoning.**

55. $f(x) = (x - 2)(x - 3)(x + 1)$

56. $g(x) = x(x + 2)(x + 1)(x - 2)$

57. $h(x) = (x + 2)(x + 3)(x - 1)$

58. $k(x) = x(x - 2)(x - 1)(x + 2)$

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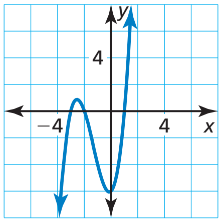

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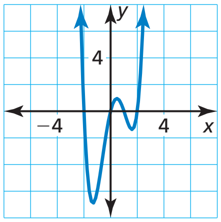

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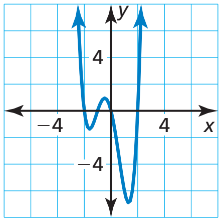

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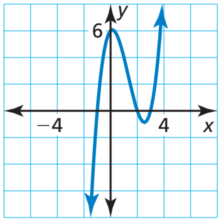
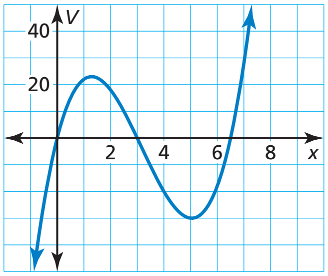


59. **MODELING REAL LIFE** The volume (in cubic inches) of a shipping box is modeled by $V = 2x^3 - 19x^2 + 39x$, where x is the length (in inches). Determine the values of x for which the model makes sense. Explain your reasoning.

Example 7



60. **MODELING REAL LIFE** The profit P (in millions of dollars) for a smart speaker manufacturer can be modeled by $P = -21x^3 + 46x$, where x is the number (in millions) of speakers produced. The company now produces 1 million speakers and makes a profit of \$25 million, but it would like to cut back production. What lesser number of speakers could the company produce and still make the same profit?

61. **MP** **REASONING** Determine whether each polynomial is factored completely. If not, factor completely.

a. $7z^4(2z^2 - z - 6)$

b. $(2 - n)(n^2 + 6n)(3n - 11)$

c. $3(4y - 5)(9y^2 - 6y - 4)$

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62. **COLLEGE PREP** Consider the function $f(x) = x^3 - 3x^2 - 4x$. For what value of k does $\dfrac{f(x)}{x - k}$ have a remainder not equal to 0? Explain.

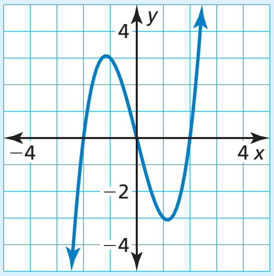
Ⓐ -1 Ⓑ 0

Ⓒ 1 Ⓓ 4

63. **MAKING AN ARGUMENT** You divide $f(x)$ by $x - a$ and find that the remainder does not equal 0. Does this mean that $f(x)$ cannot be factored? Explain.

64. **HOW DO YOU SEE IT?**

Use the graph to write an equation of the cubic function in factored form. Explain your reasoning.



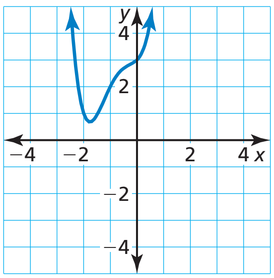
65. **ABSTRACT REASONING** Factor each polynomial completely.

a. $7ac^2 + bc^2 - 7ad^2 - bd^2$

b. $x^{2n} - 2x^n + 1$

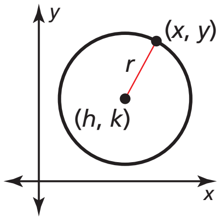
c. $a^5b^2 - a^2b^4 + 2a^4b - 2ab^3 + a^3 - b^2$

66. **MP** **REASONING** The graph of the function $f(x) = x^4 + 3x^3 + 2x^2 + x + 3$ is shown. Can you use the Factor Theorem to factor $f(x)$? Explain.



67. DIG DEEPER What is the value of k such that $x - 7$ is a factor of $h(x) = 2x^3 - 13x^2 - kx + 105$? Justify your answer.

68. **CONNECTING CONCEPTS** The standard equation of a circle with radius r and center (h, k) is $(x - h)^2 + (y - k)^2 = r^2$. Rewrite each equation of a circle in standard form. Identify the center and radius of the circle. Then graph the circle.



a. $x^2 + 6x + 9 + y^2 = 25$

b. $x^2 - 4x + 4 + y^2 = 9$

c. $x^2 - 8x + 16 + y^2 + 2y + 1 = 36$

69. **CRITICAL THINKING** Use the diagram to complete parts (a)–(c).

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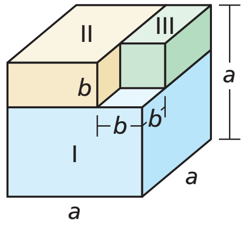
a. Explain why $a^3 - b^3$ is equal to the sum of the volumes of the solids I, II, and III.

b. Write an algebraic expression for the volume of each of the three solids. Leave your expressions in factored form.

c. Use the results from parts (a) and (b) to derive the difference of two cubes pattern $a^3 - b^3$.



70. **THOUGHT PROVOKING**

Consider the equation $\frac{a^3 - b^3}{(a - b)^3} = \frac{31}{3}$, where a and b are real numbers and $a > b > 0$. Find $\frac{b}{a}$.

REVIEW & REFRESH

WATCH

In Exercises 71–74, solve the equation using any method. Explain your choice of method.

71. $x^2 - x - 30 = 0$

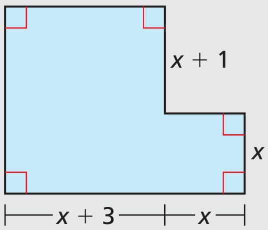
72. $9x^2 - 28x + 3 = 0$

73. $x^2 - 8x = 11$

74. $4x^2 + 36x - 4 = 0$

75. Divide $-x^3 + x^2 - 2x - 16$ by $x + 2$.

76. Write an expression for the area and perimeter of the figure shown.



77. Determine whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

$$f(x) = 5 + 2x^2 - 3x^4 - 2x - x^3$$

In Exercises 78 and 79, factor the polynomial completely.

78. $z^3 + z^2 - 4z - 4$

79. $49b^4 - 64$

80. Determine whether the table represents an *exponential growth function*, an *exponential decay function*, or *neither*. Explain.

x	−1	0	1	2
y	80	20	5	1.25

In Exercises 81–84, graph the function. Label the vertex and axis of symmetry.

81. $f(x) = -(x - 4)^2 + 3$

82. $g(x) = x^2 - 10x + 11$

83. $h(x) = -2(x - 7)(x + 3)$

84. $j(x) = \frac{1}{4}x(x + 5)$

85. **MODELING REAL LIFE** Determine whether the data show a linear relationship. If so, write an equation of a line of fit. Then estimate y when $x = 60$ and explain its meaning in the context of the situation.

Minutes skating, x	10	15	25	30	45
Distance (miles), y	2.2	3.1	5.0	5.8	8.5

86. Write the sentence as an inequality.

A number n plus 8.5 is no more than 17.