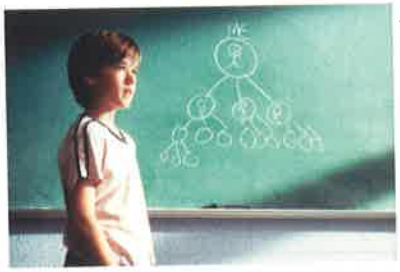
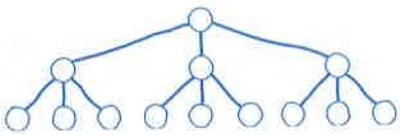


David James/2000 Warner Bros. & Bel Air Pictures, LLC/Newsmaker/Getty Images

The following diagram shows the number of people who have been a beneficiary of a good deed after one round and after two rounds of this project.

Three beneficiaries after one round

A total of 12 beneficiaries after two rounds (3 + 9 = 12)



A mathematical model for the number of pay-it-forward beneficiaries after n rounds is given by $B(n) = \dfrac{3^{n+1} - 3}{2}$. Use this model to determine

a. the number of beneficiaries after 5 rounds and after 10 rounds. Assume that no person is a beneficiary of more than one good deed.

b. how many rounds are required to produce at least 2 million beneficiaries.

58. **Fish Population** The number of bass in a lake is given by

$$P(t) = \frac{3600}{1 + 7e^{-0.05t}}$$

where t is the number of months that have passed since the lake was stocked with bass.

a. How many bass were in the lake immediately after it was stocked?

b. How many bass were in the lake 1 year after the lake was stocked? Round to the nearest bass.

c. What will happen to the bass population as t increases without bound?

59. **A Temperature Model** A cup of coffee is heated to 180°F and placed in a room that maintains a temperature of 65°F. The temperature of the coffee after t minutes is given by $T(t) = 65 + 115e^{-0.042t}$.

a. Find the temperature, to the nearest degree, of the coffee 10 minutes after it is placed in the room.

b. Determine when, to the nearest tenth of a minute, the temperature of the coffee will reach 100°F.

60. **Intensity of Light** The percent $I(x)$ of the original intensity of light striking the surface of a lake that is available x feet below the surface of the lake is given by the equation $I(x) = 100e^{-0.95x}$.

a. What percentage of the light, to the nearest tenth of a percent, is available 2 feet below the surface of the lake?

b. At what depth, to the nearest hundredth of a foot, is the intensity of the light one-half the intensity at the surface?

61. **Musical Scales** Starting on the left side of a standard 88-key piano, the frequency, in vibrations per second, of the nth note is given by $f(n) = (27.5)2^{(n-1)/12}$.

a. Using this formula, determine the frequency, to the nearest hundredth of a vibration per second, of middle C, key number 40 on an 88-key piano.

b. Is the difference in frequency between middle C (key number 40) and D (key number 42) the same as the difference in frequency between D (key number 42) and E (key number 44)? Explain.

In Exercises 62 and 63, verify that the given function is odd or even as requested.

62. Verify that $f(x) = \dfrac{e^x + e^{-x}}{2}$ is an even function.

63. Verify that $f(x) = \dfrac{e^x - e^{-x}}{2}$ is an odd function.

In Exercises 64 and 65, draw the graphs as indicated.

64. Graph $g(x) = 10^x$, and then sketch the graph of g reflected across the line given by $y = x$.

65. Graph $f(x) = e^x$, and then sketch the graph of f reflected across the line given by $y = x$.

Enrichment Exercises

In Exercises 66 to 69, determine the domain of the given function. Write the domain using interval notation.

66. $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

67. $f(x) = \dfrac{e^{|x|}}{1 + e^x}$

68. $f(x) = \sqrt{1 - e^x}$

69. $f(x) = \sqrt{e^x - e^{-x}}$

70. **An Exponential Reward** According to legend, when Sissa Ben Dahir of India invented the game of chess, King Shirham was so impressed with the game that he offered Sissa Ben Dahir the reward of his choosing. Sissa Ben Dahir pointed to the chessboard and requested, for his reward, one grain of wheat on the first square, two grains of wheat on the second square, four grains of wheat on the third square, eight grains on the fourth square, and so on for all 64 squares on the chessboard. The king considered this a very modest reward and said he would grant the inventor's wish.

The following table shows how many grains of wheat are on each of the first 7 squares of a chessboard and the total number of grains of wheat needed to cover squares 1 to n for $n \le 7$.

Square number, n	Number of grains of wheat on square n	Total number of grains of wheat on squares 1 to n
1	1	1
2	2	$1 + 2 = 3$
3	4	$3 + 4 = 7$
4	8	$7 + 8 = 15$
5	16	$15 + 16 = 31$
6	32	$31 + 32 = 63$
7	64	$63 + 64 = 127$

a. How many grains of wheat are needed to cover all 64 squares of the chessboard, as requested by Sissa Ben Dahir?

b. A grain of wheat weighs approximately 0.000008 kilogram. Find the total weight of the wheat requested by Sissa Ben Dahir.

c. In a recent year, a total of 6.5×10^8 metric tons of wheat were produced in the world. At this level, how many years, to the nearest year, of wheat production would be required to fill the request of Sissa Ben Dahir? One metric ton equals 1000 kilograms.

71. **Average Height** Explain why the graph of

$$f(x) = \frac{e^x + e^{-x}}{2}$$

can be produced by plotting the average height of $g(x) = e^x$ and $h(x) = e^{-x}$ for each value of x.

SECTION 4.3

Logarithmic Functions
Graphs of Logarithmic Functions
Domains of Logarithmic Functions
Common and Natural Logarithms

Logarithmic Functions and Their Applications

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A25.

PS1. If $2^x = 16$, determine the value of x. [4.2]

PS2. If $3^{-x} = \dfrac{1}{27}$, determine the value of x. [4.2]

PS3. If $x^4 = 625$, determine the value of x. [4.2]

PS4. Find the inverse of $f(x) = \dfrac{2x}{x + 3}$. [4.1]

PS5. State the domain of $g(x) = \sqrt{x - 2}$. [2.2]

PS6. If the range of $h(x)$ is the set of all positive real numbers, then what is the domain of $h^{-1}(x)$? [4.1]

Logarithmic Functions

Every exponential function of the form $g(x) = b^x$ is a one-to-one function and therefore has an inverse function. Sometimes we can determine the inverse of a function represented by an equation by interchanging the variables of its equation and then solving for the dependent variable. If we attempt to use this procedure for $g(x) = b^x$, we obtain

$$g(x) = b^x$$
$$y = b^x$$
$$x = b^y \quad \text{• Interchange the variables.}$$

None of our previous methods can be used to solve the equation $x = b^y$ for the exponent y. Thus we need to develop a new procedure. One method would be to merely write

$$y = \text{the power of } b \text{ that produces } x$$

Although this would work, it is not concise. We need a compact notation to represent "y is the power of b that produces x." This more compact notation is given in the following definition.

Math Matters

Logarithms were developed by John Napier (1550–1617) as a means of simplifying the calculations of astronomers. One of his ideas was to devise a method by which the product of two numbers could be determined by performing an addition.

Definition of a Logarithm and a Logarithmic Function

If $x > 0$ and b is a positive constant ($b \neq 1$), then

$$y = \log_b x \quad \text{if and only if} \quad b^y = x$$

The notation $\log_b x$ is read "the **logarithm** (or log) base b of x." The function defined by $f(x) = \log_b x$ is a **logarithmic function** with base b. This function is the inverse of the exponential function $g(x) = b^x$.

It is essential to remember that $f(x) = \log_b x$ is the inverse function of $g(x) = b^x$. Because these functions are inverses and because functions that are inverses have the property that $f(g(x)) = x$ and $g(f(x)) = x$, we have the following important relationships.

Composition of Logarithmic and Exponential Functions

Let $g(x) = b^x$ and $f(x) = \log_b x$ ($x > 0, b > 0, b \neq 1$). Then

$$g(f(x)) = b^{\log_b x} = x \quad \text{and} \quad f(g(x)) = \log_b b^x = x$$

As an example of these relationships, let $g(x) = 2^x$ and $f(x) = \log_2 x$. Then

$$2^{\log_2 x} = x \quad \text{and} \quad \log_2 2^x = x$$

The equations

$$y = \log_b x \quad \text{and} \quad b^y = x$$

are different ways of expressing the same concept.

Definition of Exponential Form and Logarithmic Form

The **exponential form** of $y = \log_b x$ is $b^y = x$.

The **logarithmic form** of $b^y = x$ is $y = \log_b x$.

These concepts are illustrated in the next two examples.

Study tip

The notation $\log_b x$ replaces the phrase "the power of b that produces x." For instance, "3 is the power of 2 that produces 8" is abbreviated $3 = \log_2 8$. In your work with logarithms, remember that a logarithm is an *exponent.*

EXAMPLE 1 Change from Logarithmic to Exponential Form

Write each equation in its exponential form.

a. $3 = \log_2 8$ **b.** $2 = \log_{10}(x + 5)$ **c.** $\log_e x = 4$ **d.** $\log_b b^3 = 3$

Solution

Use the definition $y = \log_b x$ if and only if $b^y = x$.

a. $3 = \log_2 8$ if and only if $2^3 = 8$ (Logarithms are exponents. Base)

b. $2 = \log_{10}(x + 5)$ if and only if $10^2 = x + 5$.

c. $\log_e x = 4$ if and only if $e^4 = x$.

d. $\log_b b^3 = 3$ if and only if $b^3 = b^3$.

▶ Try Exercise 8, page 368

EXAMPLE 2 Change from Exponential to Logarithmic Form

Write each equation in its logarithmic form.

a. $3^2 = 9$ **b.** $5^3 = x$

c. $a^b = c$ **d.** $b^{\log_b 5} = 5$

Solution

The logarithmic form of $b^y = x$ is $y = \log_b x$.

a. $3^2 = 9$ if and only if $2 = \log_3 9$ (Exponent. Base)

b. $5^3 = x$ if and only if $3 = \log_5 x$.

c. $a^b = c$ if and only if $b = \log_a c$.

d. $b^{\log_b 5} = 5$ if and only if $\log_b 5 = \log_b 5$.

▶ Try Exercise 18, page 368

The definition of a logarithm and the definition of an inverse function can be used to establish many properties of logarithms. For instance,

- $\log_b b = 1$ because $b = b^1$.
- $\log_b 1 = 0$ because $1 = b^0$.
- $\log_b(b^x) = x$ because $b^x = b^x$.
- $b^{\log_b x} = x$ because $f(x) = \log_b x$ and $g(x) = b^x$ are inverse functions. Thus $g[f(x)] = x$.

We will refer to the preceding properties as the *basic logarithmic properties.*

Basic Logarithmic Properties

1. $\log_b b = 1$ **2.** $\log_b 1 = 0$ **3.** $\log_b(b^x) = x$ **4.** $b^{\log_b x} = x$

EXAMPLE 3 Apply the Basic Logarithmic Properties

Evaluate each of the following logarithmic expressions.

a. $\log_8 1$ **b.** $\log_5 5$ **c.** $\log_2(2^4)$

d. $3^{\log_3 7}$ **e.** $\log_3 81^2$ **f.** $\log_2 \sqrt{8}$

Solution

a. By property 2, $\log_8 1 = 0$.

b. By property 1, $\log_5 5 = 1$.

c. By property 3, $\log_2(2^4) = 4$.

d. By property 4, $3^{\log_3 7} = 7$.

e. $\log_3 81^2 = \log_3(3^4)^2$ • $81 = 3^4$

$= \log_3 3^8$ • $(b^m)^n = b^{mn}$

$= 8$ • Property 3

f. $\log_2 \sqrt{8} = \log_2 8^{1/2}$

$= \log_2(2^3)^{1/2}$ • $8 = 2^3$

$= \log_2 2^{3/2}$ • $(b^m)^n = b^{mn}$

$= \dfrac{3}{2}$ • Property 3

▶ Try Exercise 36, page 368

Some logarithms can be evaluated just by remembering that a logarithm is an exponent. For instance, $\log_5 25$ equals 2 because the base 5 raised to the second power equals 25.

- $\log_{10} 100 = 2$ because $10^2 = 100$.
- $\log_4 64 = 3$ because $4^3 = 64$.
- $\log_7 \dfrac{1}{49} = -2$ because $7^{-2} = \dfrac{1}{7^2} = \dfrac{1}{49}$.

Question • What is the value of $\log_5 625$?

Graphs of Logarithmic Functions

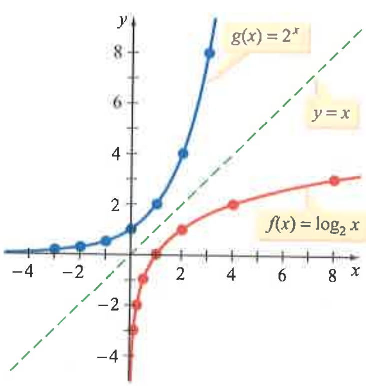


Figure 4.25

Because $f(x) = \log_b x$ is the inverse function of $g(x) = b^x$, the graph of f is a reflection of the graph of g across the line given by $y = x$. The graph of $g(x) = 2^x$ is shown in Figure 4.25. Table 4.7 shows some of the ordered pairs of the graph of g.

Table 4.7

x	−3	−2	−1	0	1	2	3
$g(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

The graph of the inverse of g, which is $f(x) = \log_2 x$, is also shown in Figure 4.25. Some of the ordered pairs of f are shown in Table 4.8. Note that if (x, y) is a point

Answer • $\log_5 625 = 4$ because $5^4 = 625$.

on the graph of g, then (y, x) is a point on the graph of f. Also notice that the graph of f is a reflection of the graph of g across the line given by $y = x$.

Table 4.8

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$f(x) = \log_2 x$	−3	−2	−1	0	1	2	3

The graph of a logarithmic function can be drawn by first rewriting the function in its exponential form. This procedure is illustrated in Example 4.

EXAMPLE 4 Graph a Logarithmic Function

Graph $f(x) = \log_3 x$.

Solution

To graph $f(x) = \log_3 x$, consider the equivalent exponential equation $x = 3^y$. Because this equation is solved for x, choose values of y and calculate the corresponding values of x, as shown in Table 4.9.

Table 4.9

$x = 3^y$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
y	−2	−1	0	1	2

Now plot the ordered pairs and connect the points with a smooth curve, as shown in Figure 4.26.

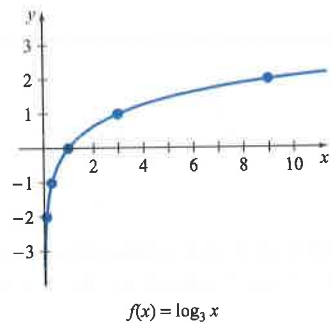


Figure 4.26

▶ Try Exercise 48, page 368

We can use a similar procedure to draw the graph of a logarithmic function with a fractional base. For instance, consider $y = \log_{2/3} x$. Rewriting this in exponential form gives us $\left(\frac{2}{3}\right)^y = x$. Choose values of y and calculate the corresponding x values. See Table 4.10. Plot the points corresponding to the ordered pairs (x, y), and then draw a smooth curve through the points, as shown in Figure 4.27.

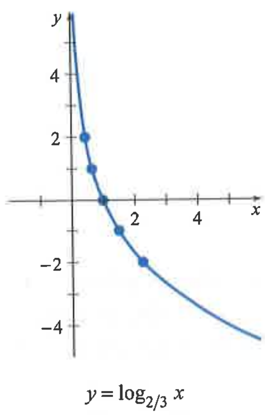


Figure 4.27

Table 4.10

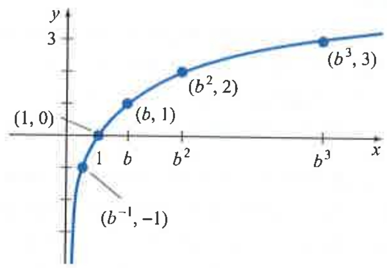
$x = \left(\frac{2}{3}\right)^y$	$\left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$	$\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$	$\left(\frac{2}{3}\right)^{0} = 1$	$\left(\frac{2}{3}\right)^{1} = \frac{2}{3}$	$\left(\frac{2}{3}\right)^{2} = \frac{4}{9}$
y	−2	−1	0	1	2

Properties of $f(x) = \log_b x$

For all positive real numbers b, $b \neq 1$, the function $f(x) = \log_b x$ has the following properties.

- The domain of f consists of the set of positive real numbers, and its range consists of the set of all real numbers.
- The graph of f has an x-intercept of $(1, 0)$ and passes through $(b, 1)$.

- If $b > 1$, f is an increasing function and its graph is asymptotic to the negative y-axis. [As $x \to \infty$, $f(x) \to \infty$, and as $x \to 0$ from the right, $f(x) \to -\infty$.] See Figure 4.28a.
- If $0 < b < 1$, f is a decreasing function and its graph is asymptotic to the positive y-axis. [As $x \to \infty$, $f(x) \to -\infty$, and as $x \to 0$ from the right, $f(x) \to \infty$.] See Figure 4.28b.



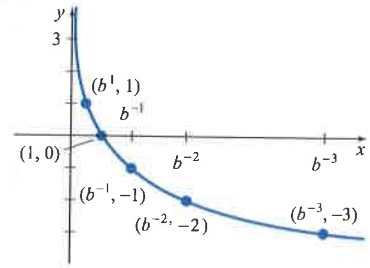


Figure 4.28

Domains of Logarithmic Functions

The function $f(x) = \log_b x$ has as its domain the set of positive real numbers. The function $f(x) = \log_b(g(x))$ has as its domain the set of all x for which $g(x) > 0$. To determine the domain of a function such as $f(x) = \log_b(g(x))$, we must determine the values of x that make $g(x)$ positive. This process is illustrated in Example 5.

EXAMPLE 5 Find the Domain of a Logarithmic Function

Find the domain of each of the following logarithmic functions.

a. $f(x) = \log_6(x - 3)$

b. $F(x) = \log_2|x + 2|$

c. $R(x) = \log_5\left(\dfrac{x}{8 - x}\right)$

Solution

a. Solving $(x - 3) > 0$ for x gives us $x > 3$. The domain of f consists of all real numbers greater than 3. In interval notation, the domain is $(3, \infty)$.

b. The solution set of $|x + 2| > 0$ consists of all real numbers x except $x = -2$. The domain of F consists of all real numbers $x \neq -2$. In interval notation, the domain is $(-\infty, -2) \cup (-2, \infty)$.

c. Solving $\left(\dfrac{x}{8 - x}\right) > 0$ yields the set of all real numbers x between 0 and 8. The domain of R is all real numbers x such that $0 < x < 8$. In interval notation, the domain is $(0, 8)$.

Try Exercise 56, page 369

Some logarithmic functions can be graphed by using horizontal or vertical translations of a previously drawn graph.

EXAMPLE 6 Translations of Logarithmic Functions

Explain how to use the graph of

a. $f(x) = \log_4 x$ to produce the graph of $f(x) = \log_4(x + 3)$.

b. $f(x) = \log_4 x$ to produce the graph of $f(x) = \log_4 x + 3$.

Solution

a. A translation theorem from Section 2.5 states that if c is a positive constant, then the graph of $f(x + c)$ is a horizontal shift c units to the left of the graph of $f(x)$. Thus the graph of $f(x) = \log_4(x + 3)$ can be obtained by shifting the graph of $f(x) = \log_4 x$ to the left 3 units. See Figure 4.29.

b. A translation theorem from Section 2.5 states that if c is a positive constant, then the graph of $f(x) + c$ is a vertical shift c units upward of the graph of $f(x)$. Thus the graph of $f(x) = \log_4 x + 3$ can be obtained by shifting the graph of $f(x) = \log_4 x$ upward 3 units. See Figure 4.30.

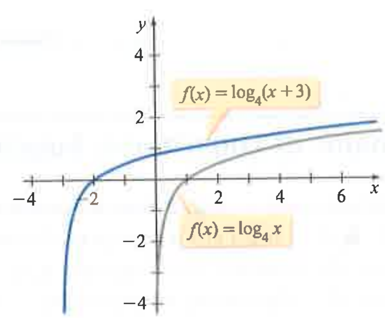


Figure 4.29

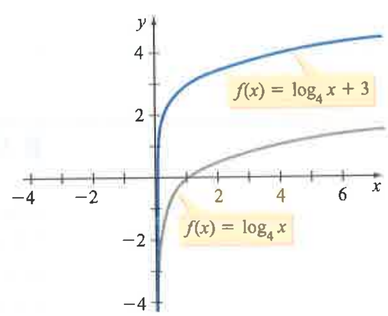


Figure 4.30

▶ Try Exercise 70, page 369

Common and Natural Logarithms

Two of the most frequently used logarithmic functions are *common logarithms,* which have base 10, and *natural logarithms,* which have base e (the base of the natural exponential function).

Definition of Common and Natural Logarithms

The function defined by $f(x) = \log_{10} x$ is called the **common logarithmic function.** It is customarily written as $f(x) = \log x$, without stating the base.

The function defined by $f(x) = \log_e x$ is called the **natural logarithmic function.** It is customarily written as $f(x) = \ln x$.

Most scientific or graphing calculators have a LOG key for evaluating common logarithms and an LN key for evaluating natural logarithms. For instance, using a graphing calculator,

$$\log 24 \approx 1.3802112 \quad \text{and} \quad \ln 81 \approx 4.3944492$$

The graphs of $f(x) = \log x$ and $f(x) = \ln x$ can be drawn using the same techniques we used to draw the graphs in the preceding examples. However, these graphs also can be produced with a graphing calculator by entering log x and ln x into the Y = menu. See Figure 4.31 and Figure 4.32.

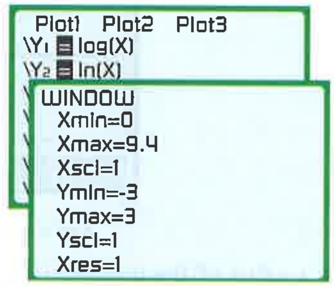


Figure 4.31

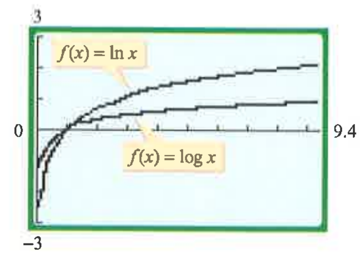


Figure 4.32

Observe that each graph passes through $(1, 0)$. Also note that as $x \to 0$ from the right, the functional values $f(x) \to -\infty$. Thus the y-axis is a vertical asymptote for each of the graphs. The domain of both $f(x) = \log x$ and $f(x) = \ln x$ is the set of positive real numbers. Each of these functions has a range consisting of the set of real numbers.

Many applications can be modeled by logarithmic functions.

EXAMPLE 7 Applied Physiology

Jon Arnold Images/AWL Images/ Getty Images

In the study *The Pace of Life*, M. H. Bornstein and H. G. Bornstein (*Nature*, Vol. 259, pp. 557–558, 1976) reported that as the population of a city increases, the average walking speed of a pedestrian also increases. An approximate relation between the average pedestrian walking speed s, in miles per hour, and the population x, in thousands, of a city is given by the function

$$s(x) = 0.37 \ln x + 0.05$$

a. Determine the average walking speed, to the nearest tenth of a mile per hour, in San Francisco, which has a population of 805,000, and in Green Bay, Wisconsin, which has a population of 101,000.

b. Estimate the population of a city for which the average pedestrian walking speed is 3.1 miles per hour. Round to the nearest hundred thousand.

Math Matters

Although logarithms were originally developed to assist with computations, logarithmic functions have a much broader use today. They are often used in such disciplines as geology, acoustics, chemistry, physics, and economics, to name a few.

Solution

a. The population of San Francisco, in thousands, is 805.

$$\begin{aligned} s(x) &= 0.37 \ln x + 0.05 \\ s(805) &= 0.37 \ln 805 + 0.05 && \text{• Substitute 805 for } x. \\ &\approx 2.5 && \text{• Use a calculator to evaluate.} \end{aligned}$$

The average walking speed in San Francisco is about 2.5 miles per hour.

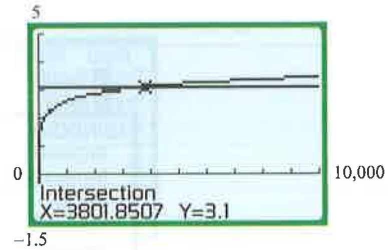
The population of Green Bay, in thousands, is 101.

$$\begin{aligned} s(x) &= 0.37 \ln x + 0.05 \\ s(101) &= 0.37 \ln 101 + 0.05 && \text{• Substitute 101 for } x. \\ &\approx 1.8 && \text{• Use a calculator to evaluate.} \end{aligned}$$

The average walking speed in Green Bay is about 1.8 miles per hour.

(continued)

b. Graph $s(x) = 0.37 \ln x + 0.05$ and $s = 3.1$ in the same viewing window.



Xscl = 1000 Yscl = 1

The x value of the intersection point represents the population in thousands. The graphs indicate that a city with an average pedestrian walking speed of 3.1 miles per hour should have a population of about 3,800,000.

▶ Try Exercise 90, page 369

EXERCISE SET 4.3

Concept Check

1. Is the function $f(x) = \log_b x$ a one-to-one function?
2. Name two characteristics of the graph of $f(x) = \log_b x$, $b > 1$.
3. What is the relationship between the graphs of $f(x) = \log_b x$ and $g(x) = b^x$?
4. Explain how to use the graph of $f(x) = \log_b x$ to produce the graph of $f(x) = \log_b(x + 5)$.

In Exercises 5 to 16, write each equation in its exponential form.

5. $1 = \log 10$
6. $4 = \log 10{,}000$
7. $2 = \log_8 64$
8. $3 = \log_4 64$
9. $\log_5 125 = 3$
10. $-3 = \log_4 64$
11. $\ln x = 7$
12. $\ln 5 = x$
13. $\ln 1 = 0$
14. $\ln x = -3$
15. $3 = \log(5x + 2)$
16. $4 = \ln(3x + 1)$

In Exercises 17 to 28, write each equation in its logarithmic form. Assume $y > 0$ and $b > 0$.

17. $3^2 = 9$
18. $5^3 = 125$
19. $7^2 = 49$
20. $2^5 = 32$
21. $b^x = y$
22. $2^x = y$
23. $y = e^x$
24. $5^1 = 5$
25. $100 = 10^2$
26. $2^{-4} = \dfrac{1}{16}$
27. $e^5 = 3x + 1$
28. $5^{x+2} = 612$

In Exercises 29 to 46, evaluate each logarithmic expression. Do not use a calculator.

29. $\log_4 16$
30. $\log_{3/2} \dfrac{8}{27}$
31. $\log_3 \dfrac{1}{243}$
32. $\log_b 1$
33. $\log_6 1296$
34. $\log_8 \left(\dfrac{1}{512}\right)$
35. $\log \dfrac{1}{100}$
36. $\log_{10}(10^6)$
37. $\log_{0.5} 16$
38. $\log_{0.3} \dfrac{100}{9}$
39. $4 \log 1000$
40. $\log_5 125^2$
41. $3 \log_7 49$
42. $2 \log_8 64$
43. $\log \sqrt{1000}$
44. $10^{\log_{10} 7}$
45. $\ln e^4$
46. $\log_{12} \dfrac{1}{144}$

In Exercises 47 to 54, graph each function by using its exponential form.

47. $f(x) = \log_4 x$
48. $f(x) = \log_6 x$
49. $f(x) = \log_{12} x$
50. $f(x) = \log_8 x$
51. $f(x) = \log_{1/2} x$
52. $f(x) = \log_{1/4} x$
53. $f(x) = \log_{5/2} x$
54. $f(x) = \log_{7/3} x$

Indicates Try It Exercises

In Exercises 55 to 68, find the domain of the function. Write the domain using interval notation.

55. $f(x) = \log_5(x - 3)$

56. $k(x) = \log_4(5 - x)$

57. $k(x) = \log_{2/3}(11 - x)$

58. $H(x) = \log_{1/4}(x^2 + 1)$

59. $P(x) = \ln(x^2 - 4)$

60. $J(x) = \ln\left(\frac{x - 3}{x}\right)$

61. $h(x) = \ln\left(\frac{x^2}{x - 4}\right)$

62. $R(x) = \ln(x^4 - x^2)$

63. $N(x) = \log_2(x^3 - x)$

64. $s(x) = \log_7(x^2 + 7x + 10)$

65. $g(x) = \log \sqrt{2x - 11}$

66. $m(x) = \log|4x - 8|$

67. $t(x) = 2\ln(3x - 7)$

68. $v(x) = \ln(x - 4)^2$

In Exercises 69 to 76, explain how to use the graph of the first function to produce the graph of the second function.

69. $f(x) = \log_4 x$; $f(x) = \log_4(x - 3)$

70. $f(x) = \log_6 x$; $f(x) = \log_6(x + 3)$

71. $f(x) = \log_{12} x$; $f(x) = \log_{12} x + 2$

72. $f(x) = \log_8 x$; $f(x) = \log_8 x - 4$

73. $f(x) = \log_{1/2} x$; $f(x) = 3 + \log_{1/2} x$

74. $f(x) = \log_{1/4} x$; $f(x) = 2 + \log_{1/4} x$

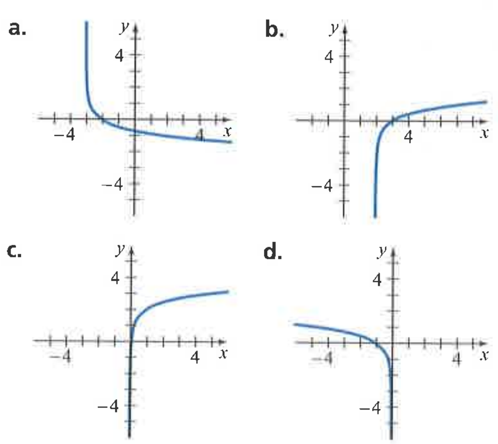
75. $f(x) = \log_{5/2} x$; $f(x) = 1 + \log_{5/2}(x - 4)$

76. $f(x) = \log_{7/3} x$; $f(x) = \log_{7/3}(x - 3) - 1$

In Exercises 77 and 78, examine the four functions and the graphs labeled a, b, c, and d. For each graph, determine which function has been graphed.

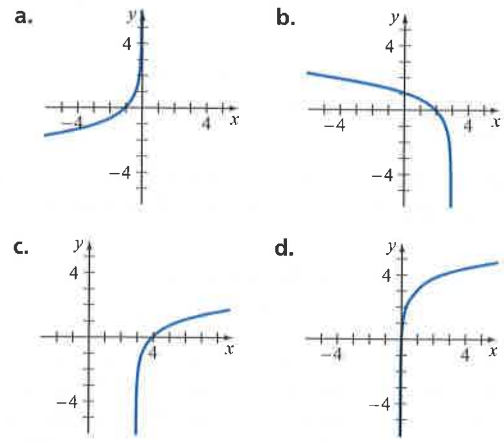
77. $f(x) = \log_5(x - 2)$ $\quad g(x) = 2 + \log_5 x$

$h(x) = \log_5(-x)$ $\quad k(x) = -\log_5(x + 3)$



78. $f(x) = \ln x + 3$ $\quad g(x) = \ln(x - 3)$

$h(x) = \ln(3 - x)$ $\quad k(x) = -\ln(-x)$



In Exercises 79 to 88, use a graphing utility to graph the function.

79. $f(x) = -2\ln x$

80. $f(x) = -\log x$

81. $f(x) = |\ln x|$

82. $f(x) = \ln|x|$

83. $f(x) = \log \sqrt[3]{x}$

84. $f(x) = \ln \sqrt{x}$

85. $f(x) = \log(x + 10)$

86. $f(x) = \ln(x + 3)$

87. $f(x) = 3\log|2x + 10|$

88. $f(x) = \frac{1}{2}\ln|x - 4|$

89. **Money Market Rates** The function

$$r(t) = 0.69607 + 0.60781 \ln t$$

gives the annual interest rate r, as a percent, a bank will pay on its money market accounts, where t is the term (the time the money is invested) in months.

a. What interest rate, to the nearest tenth of a percent, will the bank pay on a money market account with a term of 9 months?

b. What is the minimum number of complete months during which a person must invest to receive an interest rate of at least 3%?

90. **Average Typing Speed** The following function models the average typing speed S, in words per minute, of a student who has been typing for t months.

$$S(t) = 5 + 29\ln(t + 1), \quad 0 \le t \le 16$$

a. What was the student's average typing speed, to the nearest word per minute, when the student first started to type? What was the student's average typing speed, to the nearest word per minute, after 3 months?

b. Use a graph of S to determine how long, to the nearest tenth of a month, it will take the student to achieve an average typing speed of 65 words per minute.

91. **Advertising Costs and Sales** The function

$$N(x) = 2750 + 180 \ln\left(\frac{x}{1000} + 1\right)$$

models the relationship between the dollar amount x spent on advertising a product and the number of units N that a company can sell.

a. Find the number of units that will be sold with advertising expenditures of \$20,000, \$40,000, and \$60,000.

b. How many units will be sold if the company does not pay to advertise the product?

92. **A Horse's Age in Human Years** Many infants start to walk when they are approximately a year old. Most horses are able to run within a few hours after birth. The fact that horses mature much more quickly than humans makes it difficult to estimate the age of a horse in human years during the horse's first year. However, the following function models the age, in human years h, of a horse that is x years old, for $1 < x < 24$.

$$h(x) = -2.18 + 19.27 \ln x$$

Use h to:

a. Estimate the age, in human years, of a 16-year-old horse. Round to the nearest year.

b. Estimate the age of a horse whose age in human years is 32 years. Round to the nearest year.

93. **Men's 100-Meter Freestyle** The function

$$t(x) = 76.42 - 6.25 \ln x$$

models the world record time t, in seconds, in the men's 100-meter freestyle for the years from 1910 to 2010, where x is the number of years after 1910.

Use t to :

a. Estimate the world record time in 2000 ($x = 90$). Round to the nearest hundredth of a second.

b. Predict the world record time in 2030 ($x = 120$). Round to the nearest hundredth of a second.

Enrichment Exercises

94. **Astronomy** Astronomers measure the apparent brightness of a star by a unit called the **apparent magnitude**. This unit was created in the second century B.C. when the Greek astronomer Hipparchus classified the relative brightness of several stars. In his list, he assigned the number 1 to the stars that appeared to be the brightest (Sirius, Vega, and Deneb). They are first-magnitude stars. Hipparchus assigned the number 2 to all the stars in the Big Dipper. They are second-magnitude stars. The following table shows the relationship between a star's brightness relative to a first-magnitude star and the star's apparent magnitude. Notice from the table that a first-magnitude star appears to be about 2.51 times as bright as a second-magnitude star.

Brightness x, relative to a first-magnitude star	Apparent magnitude $M(x)$
1	1
$\frac{1}{2.51}$	2
$\frac{1}{6.31} \approx \frac{1}{2.51^2}$	3
$\frac{1}{15.85} \approx \frac{1}{2.51^3}$	4
$\frac{1}{39.82} \approx \frac{1}{2.51^4}$	5
$\frac{1}{100} \approx \frac{1}{2.51^5}$	6

The following logarithmic function gives the apparent magnitude $M(x)$ of a star as a function of its brightness x.

$$M(x) = -2.51 \log x + 1, \quad 0 < x \le 1$$

a. Use $M(x)$ to find the apparent magnitude of a star that is $\frac{1}{10}$ as bright as a first-magnitude star. Round to the nearest hundredth.

b. Find the apparent magnitude of a star that is $\frac{1}{400}$ as bright as a first-magnitude star. Round to the nearest hundredth.

c. Which star appears brighter: a star with an apparent magnitude of 12 or a star with an apparent magnitude of 15?

d. Is $M(x)$ an increasing function or a decreasing function?

95. **Number of Digits in b^x** The number N of digits in the expansion of b^x, where both b and x are positive integers, is $N = \text{int}(x \log b) + 1$, where $\text{int}(x \log b)$ denotes the greatest integer of $x \log b$. (*Note:* See pages 174–176 for information on the greatest integer function.)

a. Because $2^{10} = 1024$, we know that 2^{10} has four digits. Use the equation $N = \text{int}(x \log b) + 1$ to verify this result.

b. Find the number of digits in 3^{200}.

c. Find the number of digits in 7^{4005}.

d. The largest known prime number as of January 25, 2013, was $2^{57,885,161} - 1$. Find the number of digits in this prime number. (*Hint*: Because $2^{57,885,161}$ is not a power of 10, both $2^{57,885,161}$ and $2^{57,885,161} - 1$ have the same number of digits.)

96. **Number of Digits in $9^{(9^9)}$** A science teacher has offered 10 points extra credit to any student who will write out all the digits in the expansion of $9^{(9^9)}$.

a. Use the formula from Exercise 95 to determine the number of digits in this number.

b. Assume that you can write 1000 digits per page and that 500 sheets of paper are in a ream of paper. How many reams of paper, to the nearest tenth of a ream, are required to write out the expansion of $9^{(9^9)}$? Assume that you write on only one side of each sheet.

97. Use a graphing utility to graph $f(x) = \dfrac{e^x - e^{-x}}{2}$ and $g(x) = \ln(x + \sqrt{x^2 + 1})$ on the same screen. Use a square viewing window. What appears to be the relationship between f and g?

98. Use a graphing utility to graph $f(x) = \dfrac{e^x + e^{-x}}{2}$ for $x \geq 0$ and $g(x) = \ln(x + \sqrt{x^2 - 1})$ for $x \geq 1$ on the same screen. Use a square viewing window. What appears to be the relationship between f and g?

99. Use a graph of $f(x) = \dfrac{2}{e^x + e^{-x}}$ to determine the domain and range of f.

SECTION 4.4

Properties of Logarithms
Change-of-Base Formula
Logarithmic Scales

Properties of Logarithms and Logarithmic Scales

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A26.

In Exercises PS1 to PS6, use a calculator to compare the values of the given expressions.

PS1. $\log 3 + \log 2$; $\log 6$ [4.3]

PS2. $\ln 8 - \ln 3$; $\ln\left(\dfrac{8}{3}\right)$ [4.3]

PS3. $3 \log 4$; $\log(4^3)$ [4.3]

PS4. $2 \ln 5$; $\ln(5^2)$ [4.3]

PS5. $\ln 5$; $\dfrac{\log 5}{\log e}$ [4.3]

PS6. $\log 8$; $\dfrac{\ln 8}{\ln 10}$ [4.3]

Properties of Logarithms

In Section 4.3 we introduced the following basic properties of logarithms.

$$\log_b b = 1 \qquad \text{and} \qquad \log_b 1 = 0$$

Also, because exponential functions and logarithmic functions are inverses of each other, we observed the relationships

$$\log_b(b^x) = x \qquad \text{and} \qquad b^{\log_b x} = x$$

We can use the properties of exponents to establish the following additional logarithmic properties.

Caution

Pay close attention to these properties. Note that

$$\log_b(MN) \neq \log_b M \cdot \log_b N$$

and

$$\log_b \frac{M}{N} \neq \frac{\log_b M}{\log_b N}$$

Also,

$$\log_b(M + N) \neq \log_b M + \log_b N$$

In fact, the expression $\log_b(M + N)$ cannot be expanded.

Properties of Logarithms

In the following properties, b, M, and N are positive real numbers ($b \neq 1$).

Product property	$\log_b(MN) = \log_b M + \log_b N$
Quotient property	$\log_b \dfrac{M}{N} = \log_b M - \log_b N$
Power property	$\log_b(M^p) = p \log_b M$
Logarithm-of-each-side property	$M = N$ implies $\log_b M = \log_b N$
One-to-one property	$\log_b M = \log_b N$ implies $M = N$

A proof of the product property is given on the next page.

Proof

Let $r = \log_b M$ and $s = \log_b N$. These equations can be written in exponential form as

$$M = b^r \qquad \text{and} \qquad N = b^s$$

Now consider the product MN.

$$MN = b^r b^s \qquad \text{• Substitute for } M \text{ and } N.$$
$$MN = b^{r+s} \qquad \text{• Product property of exponents}$$
$$\log_b MN = r + s \qquad \text{• Write in logarithmic form.}$$
$$\log_b MN = \log_b M + \log_b N \qquad \text{• Substitute for } r \text{ and } s.$$

The last equation is our desired result. ■

The quotient property and the power property can be proved in a similar manner. See Exercises 91 and 92 on page 382.

The properties of logarithms are often used to rewrite logarithmic expressions in an equivalent form. The process of using the product or quotient properties to rewrite a single logarithm as the sum or difference of two or more logarithms, or using the power property to rewrite $\log_b(M^p)$ in its equivalent form $p \log_b M$, is called **expanding the logarithmic expression**. We illustrate this process in Example 1.

EXAMPLE 1 Expand Logarithmic Expressions

Use the properties of logarithms to expand the following logarithmic expressions. Assume all variable expressions represent positive real numbers. When possible, evaluate logarithmic expressions.

a. $\log_5(xy^2)$ **b.** $\ln\left(\frac{e\sqrt{y}}{z^3}\right)$

Solution

a. $\log_5(xy^2) = \log_5 x + \log_5 y^2$ • Product property

$= \log_5 x + 2\log_5 y$ • Power property

b. $\ln\left(\frac{e\sqrt{y}}{z^3}\right) = \ln(e\sqrt{y}) - \ln z^3$ • Quotient property

$= \ln e + \ln\sqrt{y} - \ln z^3$ • Product property

$= \ln e + \ln y^{1/2} - \ln z^3$ • Write $\sqrt{y}$ as $y^{1/2}$.

$= \ln e + \frac{1}{2}\ln y - 3\ln z$ • Power property

$= 1 + \frac{1}{2}\ln y - 3\ln z$ • Evaluate $\ln e$.

▶ Try Exercise 6, page 379

The properties of logarithms are also used to *condense* expressions that involve the sum or difference of logarithms into a single logarithm. For instance, we can use the product property to rewrite $\log_b M + \log_b N$ as $\log_b(MN)$, and the quotient property to rewrite $\log_b M - \log_b N$ as $\log_b \frac{M}{N}$. Before applying the product or quotient properties, use the power property to write all expressions of the form $p \log_b M$ in their equivalent $\log_b M^p$ form. See Example 2.