4.3 Dividing Polynomials

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Learning Target Divide polynomials by other polynomials and use the Remainder Theorem.

Success Criteria

- I can use long division to divide polynomials by other polynomials.
- I can divide polynomials by binomials of the form $x - k$ using synthetic division.
- I can explain the Remainder Theorem.

EXPLORE IT! Dividing Polynomials

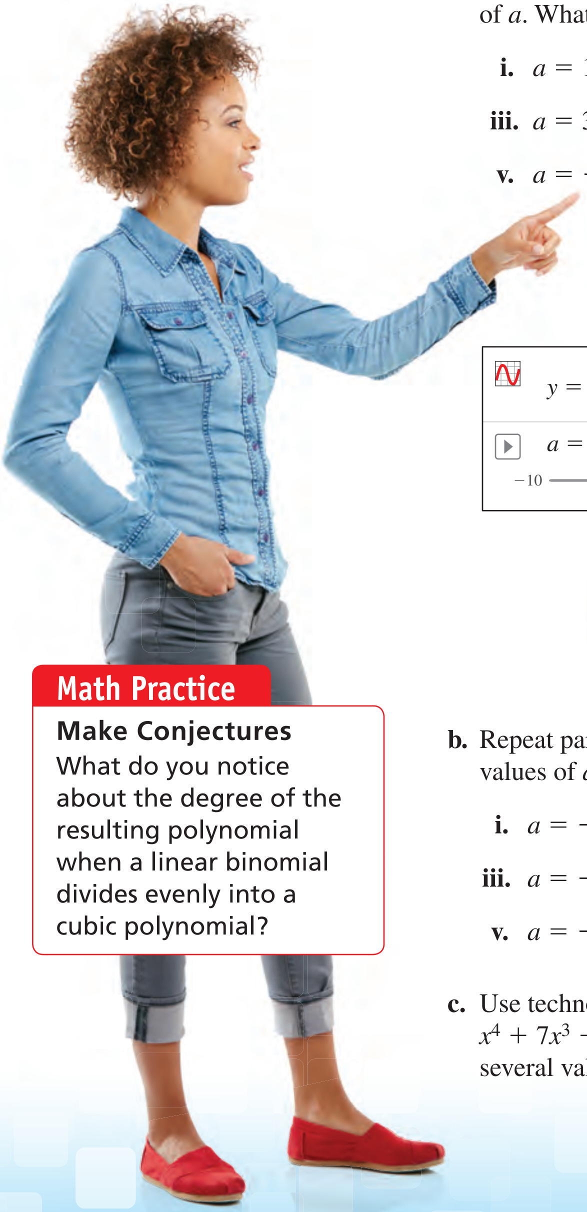
Work with a partner.

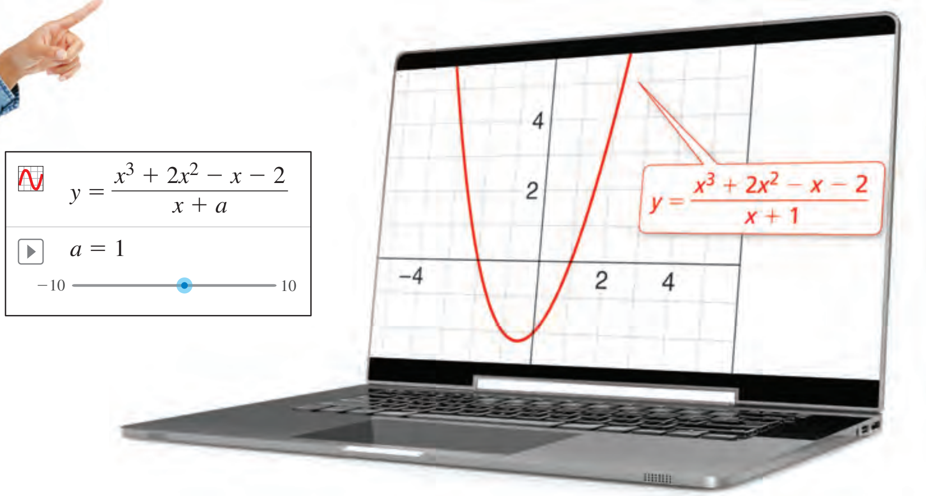
a. Consider the polynomial $x^3 + 2x^2 - x - 2$. Use technology to explore the graph of the polynomial divided by the binomial $x + a$ for the given values of a. What do you notice? What can you conclude?

i. $a = 1$ **ii.** $a = 2$

iii. $a = 3$ **iv.** $a = -1$

v. $a = -2$ **vi.** $a = 4$





Math Practice

Make Conjectures What do you notice about the degree of the resulting polynomial when a linear binomial divides evenly into a cubic polynomial?

b. Repeat part (a) for the polynomial $x^3 - 3x^2 - 10x + 24$ and the given values of a. What do you notice? What can you conclude?

i. $a = -1$ **ii.** $a = 2$

iii. $a = -2$ **iv.** $a = 3$

v. $a = -3$ **vi.** $a = 4$

c. Use technology to explore the graph of the polynomial $x^4 + 7x^3 + 9x^2 - 7x - 10$ divided by the binomial $x + a$ for several values of a. Make several observations about the graphs.

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Long Division of Polynomials

Vocabulary (VOCAB)

polynomial long division, *p. 170*
synthetic division, *p. 171*

When you divide a polynomial $f(x)$ by a nonzero polynomial divisor $d(x)$, you get a quotient polynomial $q(x)$ and a remainder polynomial $r(x)$.

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

The degree of the divisor $d(x)$ is less than or equal to the degree of the dividend $f(x)$. Also, the degree of the remainder $r(x)$ must be less than the degree of the divisor. When the remainder is 0, the divisor *divides evenly* into the dividend. One way to divide polynomials is called **polynomial long division**.

EXAMPLE 1 Using Polynomial Long Division

WATCH

Divide $2x^4 + 3x^3 + 5x - 1$ by $x^2 + 3x + 2$.

SOLUTION

Write polynomial division in the same format you use when dividing numbers. Include a "0" as the coefficient of x^2 in the dividend. At each step, divide the term with the highest power in what is left of the dividend by the first term of the divisor. This gives the next term of the quotient.

$$\begin{array}{rll}
 & 2x^2 - 3x + 5 & \leftarrow \text{quotient} \\
x^2 + 3x + 2 \,\big) & 2x^4 + 3x^3 + 0x^2 + 5x - 1 & \\
 & 2x^4 + 6x^3 + 4x^2 & \text{Multiply divisor by } \frac{2x^4}{x^2} = 2x^2. \\
 & -3x^3 - 4x^2 + 5x & \text{Subtract. Bring down next term.} \\
 & -3x^3 - 9x^2 - 6x & \text{Multiply divisor by } \frac{-3x^3}{x^2} = -3x. \\
 & 5x^2 + 11x - 1 & \text{Subtract. Bring down next term.} \\
 & 5x^2 + 15x + 10 & \text{Multiply divisor by } \frac{5x^2}{x^2} = 5. \\
 & -4x - 11 & \leftarrow \text{remainder}
\end{array}$$

COMMON ERROR

When there is a remainder, add the expression $\frac{r(x)}{d(x)}$, not just $r(x)$.

$$\frac{2x^4 + 3x^3 + 5x - 1}{x^2 + 3x + 2} = 2x^2 - 3x + 5 + \frac{-4x - 11}{x^2 + 3x + 2}$$

Check You can check the result of a division problem by multiplying the quotient by the divisor and adding the remainder. The result should be the dividend.

$$\begin{aligned}
&(2x^2 - 3x + 5)(x^2 + 3x + 2) + (-4x - 11) \\
&\quad = (2x^2)(x^2 + 3x + 2) - (3x)(x^2 + 3x + 2) + (5)(x^2 + 3x + 2) - 4x - 11 \\
&\quad = 2x^4 + 6x^3 + 4x^2 - 3x^3 - 9x^2 - 6x + 5x^2 + 15x + 10 - 4x - 11 \\
&\quad = 2x^4 + 3x^3 + 5x - 1 \ \checkmark
\end{aligned}$$

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Divide using polynomial long division.

1. $(2x^2 - 5x - 3) \div (x - 3)$
2. $(4x^2 + 3x - 11) \div (x + 1)$
3. $(x^3 - x^2 - 2x + 8) \div (x - 1)$
4. $(x^4 + 2x^2 - x + 5) \div (x^2 - x + 1)$
5. MP **REASONING** Write a trinomial that can be divided evenly by $x - 4$. Explain how you found your answer.

Synthetic Division

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Synthetic division is a shortcut for dividing polynomials by binomials of the form $x - k$.

EXAMPLE 2 Using Synthetic Division

WATCH

Divide $-x^3 + 4x^2 + 9$ by $x - 3$.

SOLUTION

Step 1 Write the coefficients of the dividend in order of descending exponents. Include a "0" for the missing x-term. Because the divisor is $x - 3$, $k = 3$. Write the k-value to the left of the vertical bar.

k-value → 3	-1	4	0	9	← coefficients of $-x^3 + 4x^2 + 9$

Step 2 Bring down the leading coefficient. Multiply the leading coefficient by k. Write the product under the second coefficient. Add.

3	-1	4	0	9
		-3		
	-1	1		

Step 3 Multiply the previous sum by k. Write the product under the third coefficient. Add. Repeat this process for the remaining coefficient. The first three numbers in the bottom row are the coefficients of the quotient, and the last number is the remainder.

3	-1	4	0	9	
		-3	3	9	
coefficients of quotient →	-1	1	3	18	← remainder

$$\frac{-x^3 + 4x^2 + 9}{x - 3} = -x^2 + x + 3 + \frac{18}{x - 3}$$

EXAMPLE 3 Using Synthetic Division

WATCH

Divide $3x^3 - 2x^2 + 2x - 5$ by $x + 1$.

SOLUTION

Use synthetic division. Because the divisor is $x + 1 = x - (-1)$, $k = -1$.

-1	3	-2	2	-5
		-3	5	-7
	3	-5	7	-12

$$\frac{3x^3 - 2x^2 + 2x - 5}{x + 1} = 3x^2 - 5x + 7 - \frac{12}{x + 1}$$

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

6. Divide (a) $(x^3 - 3x^2 - 7x + 6) \div (x - 2)$ and (b) $(2x^3 - x - 7) \div (x + 3)$ using synthetic division.

7. **MP REASONING** Explain why you can simply "bring down" the leading coefficient of the dividend when using synthetic division.

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The Remainder Theorem

The remainder in the synthetic division process has an important interpretation. When you divide a polynomial $f(x)$ by $d(x) = x - k$, the result is

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$ Polynomial division

$$\frac{f(x)}{x - k} = q(x) + \frac{r(x)}{x - k}$$ Substitute $x - k$ for $d(x)$.

$$f(x) = (x - k)q(x) + r(x).$$ Multiply both sides by $x - k$.

Because either $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $x - k$, you know that $r(x)$ is a constant function. So, let $r(x) = r$, where r is a real number, and evaluate $f(x)$ when $x = k$.

$$f(k) = (k - k)q(k) + r$$ Substitute k for x and r for $r(x)$.

$$f(k) = r$$ Simplify.

This result is stated in the *Remainder Theorem*.

KEY IDEA

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. So, to evaluate $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$.

EXAMPLE 4 Evaluating a Polynomial

WATCH

Use synthetic division to evaluate $f(x) = 5x^3 - x^2 + 13x + 29$ when $x = -4$.

SOLUTION

-4	5	-1	13	29
		-20	84	-388
	5	-21	97	-359

The remainder is -359. So, you can conclude from the Remainder Theorem that $f(-4) = -359$.

Check Check this by substituting $x = -4$ in the original function.

$$f(-4) = 5(-4)^3 - (-4)^2 + 13(-4) + 29$$
$$= -320 - 16 - 52 + 29$$
$$= -359 \checkmark$$

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Use synthetic division to evaluate the function for the indicated value of *x*.

8. $f(x) = 4x^2 - 10x - 21; x = 5$

9. $f(x) = 5x^4 + 2x^3 - 20x - 6; x = 2$

10. MP **REASONING** Is the set of polynomials closed under division? Justify your answer.

4.3 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–8, divide using polynomial long division. *Example 1*

1. $(x^2 + x - 17) \div (x - 4)$

2. $(3x^2 - 14x - 5) \div (x - 5)$

3. $(x^3 + x^2 + x + 2) \div (x^2 - 1)$

4. $(7x^3 + x^2 + x) \div (x^2 + 1)$

5. $(8x^3 - 3x + 1) \div (4x^3 + x^2 - 2x - 3)$

6. $(10x^3 + 5x^2 - 1) \div (2x^3 - 4x^2 - x + 2)$

7. $(5x^4 - 2x^3 - 7x^2 - 39) \div (x^2 + 2x - 4)$

8. $(4x^4 + 5x - 4) \div (x^2 - 3x - 2)$

In Exercises 9–16, divide using synthetic division. *Examples 2 and 3*

9. $(x^2 + 8x + 1) \div (x - 4)$

10. $(4x^2 - 13x - 5) \div (x - 2)$

11. $(2x^2 - x + 7) \div (x + 5)$

12. $(x^2 + 9) \div (x + 4)$

13. $(x^3 - 4x + 6) \div (x + 3)$

14. $(3x^3 - 5x^2 - 2) \div (x - 1)$

15. $(x^4 - 5x^3 - 8x^2 + 13x - 12) \div (x - 6)$

16. $(x^4 + 4x^3 + 16x - 35) \div (x + 5)$

ANALYZING RELATIONSHIPS In Exercises 17–20, match the equivalent expressions. Justify your answers.

17. $(x^2 + x - 3) \div (x - 2)$

18. $(x^2 - x - 3) \div (x - 2)$

19. $(x^2 - x + 3) \div (x - 2)$

20. $(x^2 + x + 3) \div (x - 2)$

A. $x + 1 - \dfrac{1}{x - 2}$ **B.** $x + 3 + \dfrac{9}{x - 2}$

C. $x + 1 + \dfrac{5}{x - 2}$ **D.** $x + 3 + \dfrac{3}{x - 2}$

ERROR ANALYSIS In Exercises 21 and 22, describe and correct the error in using synthetic division to divide $x^3 - 5x + 3$ by $x - 2$.

21.

2	1	0	−5	3
		2	4	−2
	1	2	−1	1

$$\frac{x^3 - 5x + 3}{x - 2} = x^3 + 2x^2 - x + 1$$

22.

2	1	−5	3
		2	−6
	1	−3	−3

$$\frac{x^3 - 5x + 3}{x - 2} = x^2 - 3x - \frac{3}{x - 2}$$

In Exercises 23–30, use synthetic division to evaluate the function for the indicated value of x. *Example 4*

23. $f(x) = -x^2 - 8x + 30;\ x = -1$

24. $f(x) = 3x^2 + 2x - 20;\ x = 3$

25. $f(x) = x^3 - 2x^2 + 4x + 3;\ x = 2$

26. $f(x) = x^3 + x^2 - 3x + 9;\ x = -4$

27. $f(x) = x^3 - 6x + 1;\ x = 6$

28. $f(x) = x^3 - 9x - 7;\ x = 10$

29. $f(x) = x^4 + 6x^2 - 7x + 1;\ x = 3$

30. $f(x) = -x^4 - x^3 - 2;\ x = 5$

31. **MODELING REAL LIFE** You are making a blanket with a fringe border of equal width on each side. The length of the blanket without the fringe border is 72 inches. The combined area A (in square inches) of the blanket and the fringe border is represented by $A = 4x^2 + 240x + 3456$. What is the width of the blanket without the fringe border?

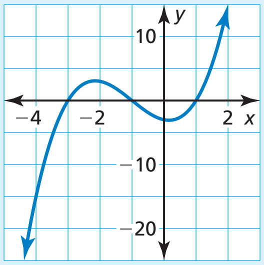
32. **MODELING REAL LIFE** The profit P (in millions of dollars) earned by a company x years since 2014 can be modeled by $P = 0.1x^3 - x^2 + 2.5x + 1.7$, where $0 < x < 6$. Use synthetic division to show that the company earned a profit of \$2.3 million in 2020.

33. **MAKING AN ARGUMENT** You use synthetic division to divide $f(x)$ by $x - a$ and find that the remainder is 25. Your friend concludes that $f(25) = a$. Is your friend correct? Explain your reasoning.

34. **HOW DO YOU SEE IT?**
The graph represents the polynomial function $f(x) = x^3 + 3x^2 - x - 3$.

a. When $f(x)$ is divided by $x - k$, the remainder is -15. What is the value of k?

b. Use the graph to determine the remainders of $(x^3 + 3x^2 - x - 3) \div (x + 3)$ and $(x^3 + 3x^2 - x - 3) \div (x + 1)$.



35. **COLLEGE PREP** What is the value of k such that $(x^3 - x^2 + kx - 30) \div (x - 5)$ has a remainder of 0?

Ⓐ -14 Ⓑ -2
Ⓒ 26 Ⓓ 32

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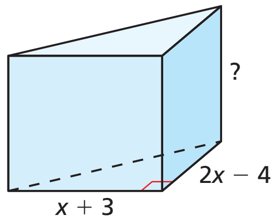
36. **CONNECTING CONCEPTS** The volume V of the triangular prism is given by $V = 2x^3 - 5x^2 - 19x + 42$. Find an expression for the missing dimension.



37. MP **STRUCTURE** You divide two polynomials and obtain the result $5x^2 - 13x + 47 - \frac{102}{x + 2}$. What is the dividend? How did you find it?

38. **THOUGHT PROVOKING**
Explain how to use synthetic division to divide $4x^3 - 3x^2 + 3x - 7$ by $4x + 1$. Then find the quotient.

REVIEW & REFRESH

WATCH

In Exercises 39–42, find the zero(s) of the function.

39. $f(x) = x^2 - 6x + 9$

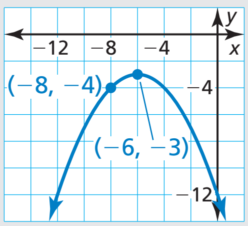
40. $g(x) = 3(x + 6)(x - 2)$

41. $g(x) = x^2 + 14x + 45$

42. $h(x) = 4x^2 + 36$

43. Use synthetic division to evaluate $f(x) = x^3 - 3x + 12$ when $x = -5$.

44. Write an equation of the parabola in vertex form.



45. MP **STRUCTURE** For what value of k is the expression $(x + 3)(kx^2 - 4x + 1) - (2x^3 + 9x^2 - 7)$ equal to $4x^3 + 5x^2 - 11x + 10$?

46. Graph $f(x) = x^3 - 3x + 1$.

47. Divide $4x^4 - 2x^3 + x^2 - 5x + 8$ by $x^2 - 2x - 1$.

In Exercises 48 and 49, solve the equation.

48. $5^{2x} = 5^{x + 4}$

49. $2^{x - 3} = 16$

50. The table shows the results of flipping a penny 50 times. What is the experimental probability of flipping heads?

Heads	Tails
23	27

In Exercises 51–54, simplify the expression.

51. $\sqrt{300x^3}$

52. $\sqrt{\frac{10}{49}}$

53. $2\sqrt{3} - 6\sqrt{5} + 7\sqrt{3}$

54. $5\sqrt{2} - \sqrt{8}$

55. **MODELING REAL LIFE** City officials want to build a rectangular flower bed in a downtown park. The flower bed must have a perimeter of 50 feet and an area of at least 100 square feet. Describe the possible lengths of the flower bed.