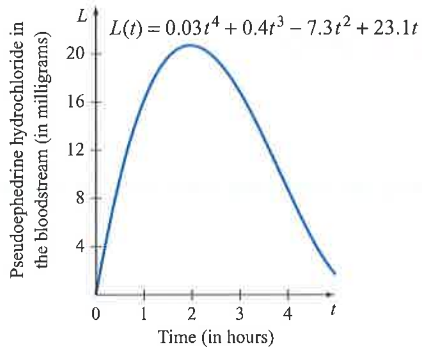


a. Use the graph of L to estimate two different values of t for which the pseudoephedrine hydrochloride levels are the same.

b. Does L have an inverse that is a function? Explain.

65. **Cryptology** Cryptology is the study of making and breaking secret codes. Secret codes are often used to send messages over the Internet. By devising a code that is difficult to break, the sender hopes to prevent the messages from being read by an unauthorized person.

In practice, complicated one-to-one functions and their inverses are used to encode and decode messages. The following procedure uses the simple function $f(x) = 2x - 1$ to illustrate the basic concepts that are involved.

Assign to each letter of the alphabet, and a blank space, a two-digit numerical value, as shown below.

A	10	H	17	O	24	V	31
B	11	I	18	P	25	W	32
C	12	J	19	Q	26	X	33
D	13	K	20	R	27	Y	34
E	14	L	21	S	28	Z	35
F	15	M	22	T	29		36
G	16	N	23	U	30		

Note: A blank space is represented by the numerical value 36.

Using these numerical values, the message MEET YOU AT NOON would be represented by

22 14 14 29 36 34 24 30 36 10 29 36 23 24 24 23

Let $f(x) = 2x - 1$ define a coding function. The above message can be encoded by finding $f(22)$, $f(14)$, $f(14)$, $f(29)$, $f(36)$, $f(34)$, $f(24), \ldots, f(23)$, which yields

43 27 27 57 71 67 47 59 71 19 57 71 45 47 47 45

The inverse of f, which is

$$f^{-1}(x) = \frac{x+1}{2}$$

is used by the receiver of the message to decode the message. For instance,

$$f^{-1}(43) = \frac{43+1}{2} = 22$$

which represents M, and

$$f^{-1}(27) = \frac{27+1}{2} = 14$$

which represents E.

a. Use the preceding coding procedure to encode the message DO YOUR HOMEWORK

b. Use $f^{-1}(x)$ to decode the message

49 33 47 45 27 71 33 47 43 27

c. Explain why it is important to use a one-to-one function to encode a message.

66. **Cryptography** A friend is using the letter–number correspondence in Exercise 65 and the coding function $g(x) = 2x + 3$. Your friend sends you the coded message

59 31 39 73 31 75 61 37 31 75 29 23 71

Use $g^{-1}(x)$ to decode this message.

In Exercises 67 to 70, answer the question without finding the equation of the linear function.

67. Suppose that f is a linear function, $f(2) = 7$, and $f(5) = 12$. If $f(4) = c$, then is c less than 7, between 7 and 12, or greater than 12? Explain your answer.

68. Suppose that f is a linear function, $f(1) = 13$, and $f(4) = 9$. If $f(3) = c$, then is c less than 9, between 9 and 13, or greater than 13? Explain your answer.

69. Suppose that f is a linear function, $f(2) = 3$, and $f(5) = 9$. Between which two numbers is $f^{-1}(6)$?

70. Suppose that f is a linear function, $f(5) = -1$, and $f(9) = -3$. Between which two numbers is $f^{-1}(-2)$?

Only one-to-one functions have inverses that are functions. In Exercises 71 to 78, determine whether the given function is a one-to-one function.

71. $f(x) = x^2 + 1$

72. $v(t) = \sqrt{16 + t}$

73. $F(x) = |x| + x$

74. $T(x) = |x^2 - 6|, \quad x \geq 0$

75. $g(x) = x^3 - 2x$

76. $k(x) = \sqrt{x}$

77. $j(x) = x^3$

78. $n(x) = \dfrac{1}{x}$

79. Use a graph of $f(x) = -x + 3$ to explain why f is its own inverse.

80. Use a graph of $f(x) = \sqrt{16 - x^2}$, with $0 \leq x \leq 4$, to explain why f is its own inverse.

Enrichment Exercises

81. Consider the linear function $f(x) = mx + b$, $m \neq 0$. The graph of f has a slope of m and a y-intercept of $(0, b)$. What are the slope and y-intercept of the graph of f^{-1}?

82. Find the inverse of $f(x) = ax^2 + bx + c$, $a \neq 0$, $x \geq -\dfrac{b}{2a}$.

SECTION 4.2

Exponential Functions
Graphs of Exponential Functions
Natural Exponential Function

Exponential Functions and Their Applications

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A24.

PS1. Evaluate: 2^3 [P.2]

PS2. Evaluate: 3^{-4} [P.2]

PS3. Evaluate: $\frac{2^2 + 2^{-2}}{2}$ [P.2/P.5]

PS4. Evaluate: $\frac{3^2 - 3^{-2}}{2}$ [P.2/P.5]

PS5. Evaluate $f(x) = 10^x$ for $x = -1, 0, 1$, and 2. [P.2]

PS6. Evaluate $f(x) = \left(\frac{1}{2}\right)^x$ for $x = -1, 0, 1$, and 2. [P.2]

Exponential Functions

When a parking facility opened in 1980, it charged $1 for all-day parking. Since then, it has doubled its daily parking fee every 8 years as shown in Table 4.1.

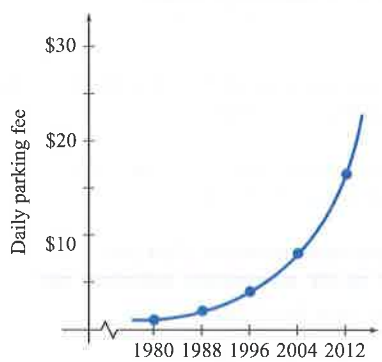


Figure 4.13

Table 4.1

Year	1980	1988	1996	2004	2012
Daily parking fee	$1	$2	$4	$8	$16

In Figure 4.13, we have plotted the data in Table 4.1 and modeled the upward trend in the parking fee by a smooth curve. This model is based on an *exponential function*, which is one of the major topics of this chapter.

The effectiveness of a drug that is used for sedation during a surgical procedure depends on the concentration of the drug in the patient. Through natural body chemistry, the amount of this drug in the body decreases over time. The graph in Figure 4.14 models this decrease. This model is another example of an exponential model.

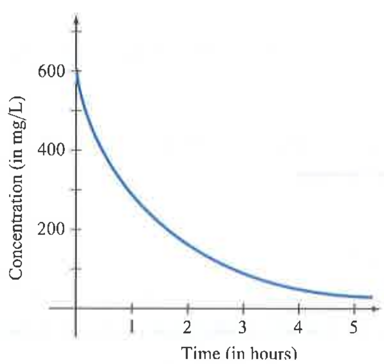


Figure 4.14

Definition of an Exponential Function

The **exponential function with base *b*** is defined by

$$f(x) = b^x$$

where $b > 0$, $b \neq 1$, and x is a real number.

The base b of $f(x) = b^x$ is required to be positive. If the base were a negative number, the value of the function would be a complex number for some values of x. For instance, if $b = -4$ and $x = \frac{1}{2}$, then $f\left(\frac{1}{2}\right) = (-4)^{1/2} = 2i$. To avoid complex number values of a function, the base of any exponential function must be a positive number. Also, b is defined such that $b \neq 1$ because $f(x) = 1^x = 1$ is a constant function.

You may have noticed that in the definition of an exponential function the exponent x is a real number. We have already worked with expressions of the form b^x, where $b > 0$ and x is a *rational* number. For instance,

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$
$$27^{2/3} = (\sqrt[3]{27})^2 = 3^2 = 9$$
$$32^{0.4} = 32^{2/5} = (\sqrt[5]{32})^2 = 2^2 = 4$$

To extend the meaning of b^x to *real* numbers, we need to give meaning to b^x when x is an *irrational* number. For example, what is the meaning of 5^π? To completely answer this question requires concepts from calculus. However, for our purposes, we can think of 5^π as the unique real number that is approached by 5^x as x takes on ever closer rational number approximations of π. For instance, each successive number in the following list is a closer approximation of 5^π than the number to its left.

$$5^3, 5^{3.1}, 5^{3.14}, 5^{3.142}, 5^{3.1416}, 5^{3.14159}, 5^{3.141593}, 5^{3.1415927}, 5^{3.14159265}, \ldots$$

A calculator can be used to show that $5^\pi \approx 156.9925453$. A computer algebra system, such as *Mathematica*, can produce even closer decimal approximations of 5^π by using closer rational number approximations of π. For instance, if you use 3.1415926535897932385 as your approximation of π, then *Mathematica* produces 156.9925453088659076 as an approximation of 5^π.

In a similar manner, we can think of $7^{\sqrt{3}}$ as the number that is approached by ever closer rational number approximations of $\sqrt{3}$. For instance, each successive number in the following list is a closer approximation of $7^{\sqrt{3}}$ than the number to its left.

$$7^1, 7^{1.7}, 7^{1.73}, 7^{1.732}, 7^{1.7321}, 7^{1.73205}, 7^{1.732051}, 7^{1.7320508}, 7^{1.73205081}, \ldots$$

A calculator can be used to show that $7^{\sqrt{3}} \approx 29.0906043$.

It can be shown that the properties of rational number exponents, as stated in Section P.2, hold for real exponents.

EXAMPLE 1 Evaluate an Exponential Function

Evaluate $f(x) = 3^x$ at $x = 2$, $x = -4$, and $x = \pi$.

Solution

$f(2) = 3^2 = 9$

$f(-4) = 3^{-4} = \frac{1}{3^4} = \frac{1}{81}$

$f(\pi) = 3^\pi \approx 3^{3.1415927} \approx 31.54428$ • Evaluate with the aid of a calculator.

Try Exercise 6, page 356

Graphs of Exponential Functions

The graph of $f(x) = 2^x$ is shown in Figure 4.15 on the next page.. The coordinates of some of the points on the curve are given in Table 4.2.

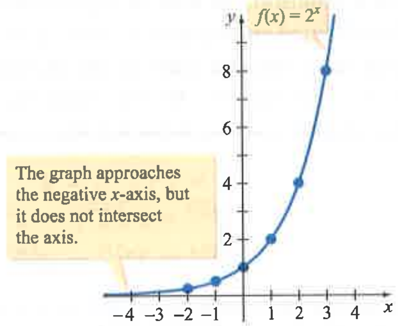


Figure 4.15

Table 4.2

x	$y = f(x) = 2^x$	(x, y)
-2	$f(-2) = 2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$f(-1) = 2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$f(0) = 2^0 = 1$	$(0, 1)$
1	$f(1) = 2^1 = 2$	$(1, 2)$
2	$f(2) = 2^2 = 4$	$(2, 4)$
3	$f(3) = 2^3 = 8$	$(3, 8)$

Note the following properties of the graph of the exponential function $f(x) = 2^x$.

- The y-intercept is $(0, 1)$.
- The graph passes through $(1, 2)$.
- As x decreases without bound (that is, as $x \to -\infty$), $f(x) \to 0$.
- The graph is a smooth, continuous increasing curve.

Now consider the graph of an exponential function for which the base is between 0 and 1. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ is shown in Figure 4.16. The coordinates of some of the points on the curve are given in Table 4.3.

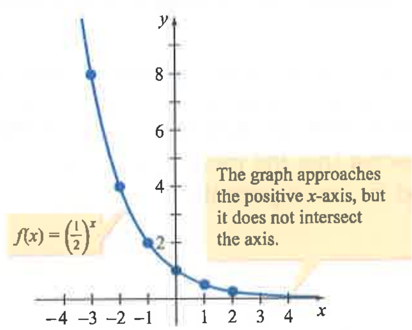


Figure 4.16

Table 4.3

x	$y = f(x) = \left(\frac{1}{2}\right)^x$	(x, y)
-3	$f(-3) = \left(\frac{1}{2}\right)^{-3} = 8$	$(-3, 8)$
-2	$f(-2) = \left(\frac{1}{2}\right)^{-2} = 4$	$(-2, 4)$
-1	$f(-1) = \left(\frac{1}{2}\right)^{-1} = 2$	$(-1, 2)$
0	$f(0) = \left(\frac{1}{2}\right)^0 = 1$	$(0, 1)$
1	$f(1) = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$
2	$f(2) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$

Note the following properties of the graph of $f(x) = \left(\frac{1}{2}\right)^x$.

- The y-intercept is $(0, 1)$.
- The graph passes through $\left(1, \frac{1}{2}\right)$.

- As x increases without bound (that is, as $x \to \infty$), $f(x) \to 0$.
- The graph is a smooth, continuous decreasing curve.

The basic properties of exponential functions are provided in the following summary.

Properties of $f(x) = b^x$

For positive real numbers b, $b \neq 1$, the exponential function defined by $f(x) = b^x$ has the following properties:

- The function f is a one-to-one function. It has the set of real numbers as its domain and the set of positive real numbers as its range.
- The graph of f is a smooth, continuous curve with a y-intercept of (0, 1), and the graph passes through $(1, b)$.
- If $b > 1$, f is an increasing function and the graph of f is asymptotic to the negative x-axis. [As $x \to \infty$, $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to 0$.] See Figure 4.17a.
- If $0 < b < 1$, f is a decreasing function and the graph of f is asymptotic to the positive x-axis. [As $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to 0$.] See Figure 4.17b.

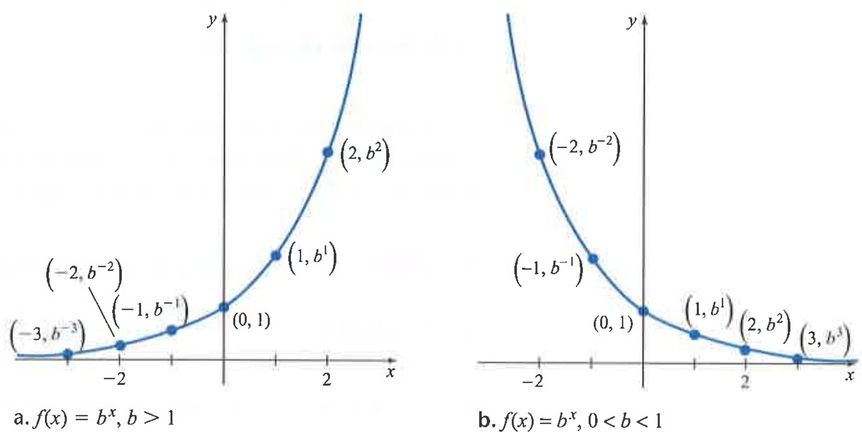


Figure 4.17

Question • What is the x-intercept of the graph of $f(x) = \left(\frac{1}{3}\right)^x$?

EXAMPLE 2 Graph an Exponential Function

Graph: $g(x) = \left(\frac{3}{4}\right)^x$

Solution

Because the base $\frac{3}{4}$ is less than 1, we know that the graph of g is a decreasing function that is asymptotic to the positive x-axis. The y-intercept of the graph

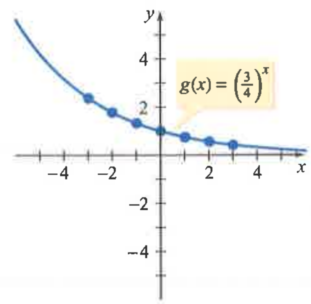
(continued)

Answer • The graph does not have an x-intercept. As x increases without bound, the graph approaches, but does not intersect, the x-axis.

is the point (0, 1), and the graph passes through $\left(1, \frac{3}{4}\right)$. Plot a few additional points (see Table 4.4), and then draw a smooth curve through the points, as in Figure 4.18.

Table 4.4

x	$y = g(x) = \left(\frac{3}{4}\right)^x$	(x, y)
-3	$\left(\frac{3}{4}\right)^{-3} = \frac{64}{27}$	$\left(-3, \frac{64}{27}\right)$
-2	$\left(\frac{3}{4}\right)^{-2} = \frac{16}{9}$	$\left(-2, \frac{16}{9}\right)$
-1	$\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$	$\left(-1, \frac{4}{3}\right)$
2	$\left(\frac{3}{4}\right)^{2} = \frac{9}{16}$	$\left(2, \frac{9}{16}\right)$
3	$\left(\frac{3}{4}\right)^{3} = \frac{27}{64}$	$\left(3, \frac{27}{64}\right)$



▶ Try Exercise 26, page 357

Figure 4.18

Consider the functions $F(x) = 2^x - 3$ and $G(x) = 2^{x-3}$. You can construct the graphs of these functions by plotting points; however, it is easier to construct their graphs by using translations of the graph of $f(x) = 2^x$, as shown in Example 3.

EXAMPLE 3 Use a Translation to Produce a Graph

a. Explain how to use the graph of $f(x) = 2^x$ to produce the graph of $F(x) = 2^x - 3$.

b. Explain how to use the graph of $f(x) = 2^x$ to produce the graph of $G(x) = 2^{x-3}$.

Solution

a. $F(x) = 2^x - 3 = f(x) - 3$. The graph of F is a vertical translation of f down 3 units, as shown in Figure 4.19.

b. $G(x) = 2^{x-3} = f(x - 3)$. The graph of G is a horizontal translation of f to the right 3 units, as shown in Figure 4.20.

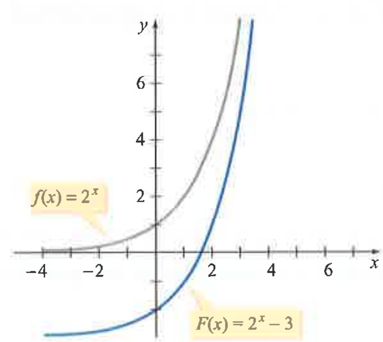


Figure 4.19

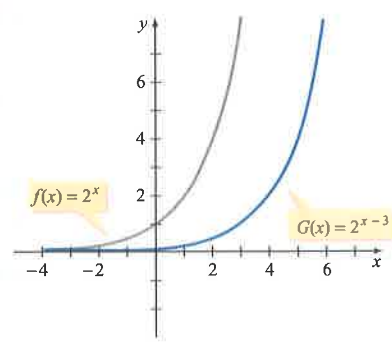


Figure 4.20

▶ Try Exercise 32, page 357

The graphs of some functions can be constructed by stretching, compressing, or reflecting the graph of an exponential function.

EXAMPLE 4 Use Stretching or Reflecting Procedures to Produce a Graph

a. Explain how to use the graph of $f(x) = 2^x$ to produce the graph of $M(x) = 2(2^x)$.

b. Explain how to use the graph of $f(x) = 2^x$ to produce the graph of $N(x) = 2^{-x}$.

Solution

a. $M(x) = 2(2^x) = 2f(x)$. The graph of M is a vertical stretching of f away from the x-axis by a factor of 2, as shown in Figure 4.21. (*Note:* If (x, y) is a point on the graph of $f(x) = 2^x$, then $(x, 2y)$ is a point on the graph of M.)

b. $N(x) = 2^{-x} = f(-x)$. The graph of N is the graph of f reflected across the y-axis, as shown in Figure 4.22. (*Note:* If (x, y) is a point on the graph of $f(x) = 2^x$, then $(-x, y)$ is a point on the graph of N.)

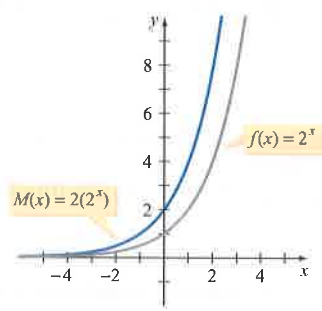


Figure 4.21

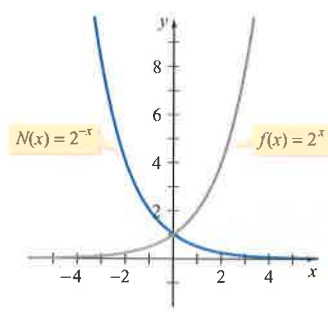


Figure 4.22

▶ **Try Exercise 34, page 357**

▶ Natural Exponential Function

The irrational number π is often used in applications that involve circles. Another irrational number, denoted by the letter e, is useful in many applications that involve growth or decay.

Definition of e

The letter e represents the number that

$$\left(1 + \frac{1}{n}\right)^n$$

approaches as n increases without bound.

The letter e was chosen in honor of the Swiss mathematician Leonhard Euler. He was able to compute the value of e to several decimal places by evaluating $\left(1 + \frac{1}{n}\right)^n$ for large values of n, as shown in Table 4.5 on the next page.

Math Matters

Leonhard Euler (1707–1783) Some mathematicians consider Euler to be the greatest mathematician of all time. He certainly was the most prolific writer of mathematics of all time. He made substantial contributions in the areas of number theory, geometry, calculus, differential equations, differential geometry, topology, complex variables, and analysis, to name but a few. Euler was the first to introduce many of the mathematical notations that we use today. For instance, he introduced the symbol i for the square root of -1, the symbol π for pi, the functional notation $f(x)$, and the letter e for the base of the natural exponential function. Euler's computational skills were truly amazing. The mathematician François Arago remarked, "Euler calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind."

Source: wikiquote.org/wiki/leonhard_Euler.

Table 4.5

Value of n	Value of $\left(1 + \frac{1}{n}\right)^n$
1	2
10	2.59374246
100	2.704813829
1000	2.716923932
10,000	2.718145927
100,000	2.718268237
1,000,000	2.718280469
10,000,000	2.718281693

The value of e accurate to eight decimal places is 2.71828183.

The base of an exponential function can be any positive real number other than 1. The number 10 is a convenient base to use for some situations, but we will see that the number e is often the best base to use in real-life applications. The exponential function with e as the base is known as the *natural exponential function.*

CALCULUS CONNECTION

Definition of the Natural Exponential Function

For all real numbers x, the function defined by

$$f(x) = e^x$$

is called the **natural exponential function.**

A calculator can be used to evaluate e^x for specific values of x. For instance,

$$e^2 \approx 7.389056, \quad e^{3.5} \approx 33.115452, \quad \text{and} \quad e^{-1.4} \approx 0.246597$$

On a TI-83/TI-83 Plus/TI-84 Plus calculator, the e^x function is located above the LN key.

To graph $f(x) = e^x$, use a calculator to find the range values for a few domain values. The range values in Table 4.6 have been rounded to the nearest tenth.

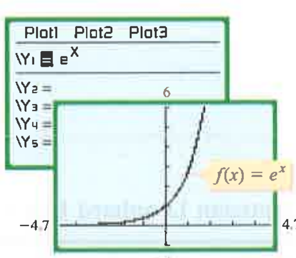
Table 4.6

x	−2	−1	0	1	2
$f(x) = e^x$	0.1	0.4	1.0	2.7	7.4

Plot the points given in Table 4.6, and then connect the points with a smooth curve. Because $e > 1$, we know that the graph is an increasing function. To the far left, the graph will approach the x-axis. The y-intercept is (0, 1). See Figure 4.23. Note in Figure 4.24 how the graph of $f(x) = e^x$ compares with the graphs of $g(x) = 2^x$ and $h(x) = 3^x$. You may have anticipated that the graph of $f(x) = e^x$ would lie between the two other graphs because e is between 2 and 3.

Integrating Technology

The graph of $f(x) = e^x$ below was produced on a TI-83/TI-83 Plus/ TI-84 Plus graphing calculator by entering e^x in the Y = menu.



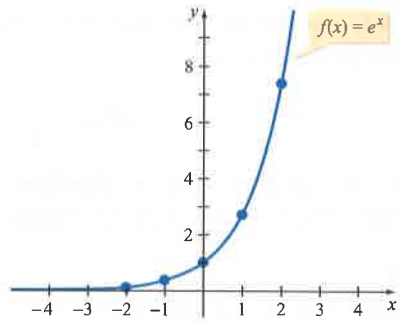


Figure 4.23

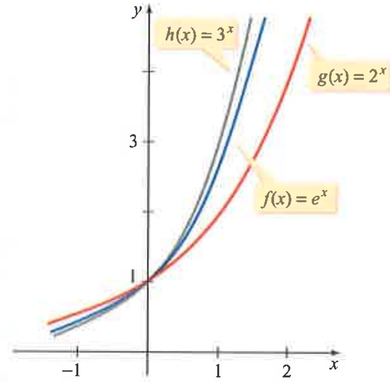


Figure 4.24

Many applications can be modeled effectively by functions that involve an exponential function. For instance, in Example 5 we use a function that involves an exponential function to model the temperature of a cup of coffee.

EXAMPLE 5 Use a Mathematical Model

A cup of coffee is heated to 160°F and placed in a room that maintains a temperature of 70°F. The temperature T of the coffee, in degrees Fahrenheit, after t minutes is given by

$$T = 70 + 90e^{-0.0485t}$$

a. Find the temperature of the coffee, to the nearest degree, 20 minutes after it is placed in the room.

b. Determine when the temperature of the coffee will reach 90°F.

Solution

a. $T = 70 + 90e^{-0.0485t}$

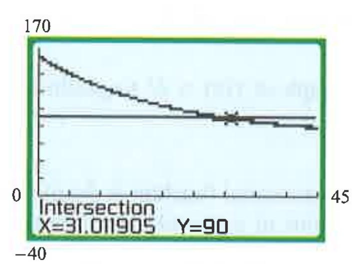
$= 70 + 90e^{-0.0485\cdot(20)}$ • Substitute 20 for t.

$\approx 70 + 34.1$

≈ 104.1

After 20 minutes the temperature of the coffee is about 104°F.

b. Graph $T = 70 + 90e^{-0.0485t}$ and $T = 90$. See the following figure.



Xscl = 5 Yscl = 20

The graphs intersect near (31.01, 90). It takes the coffee about 31 minutes to cool to 90°F.

▸ Try Exercise 52, page 358

Integrating Technology

WolframAlpha can also be used to solve Example 5b by entering the text shown in the following input field.

70 + 90e^(−0.0485t) = 90

Analytic methods for solving exponential equations without the use of technology are presented in Section 4.5.

EXAMPLE 6 Use a Mathematical Model

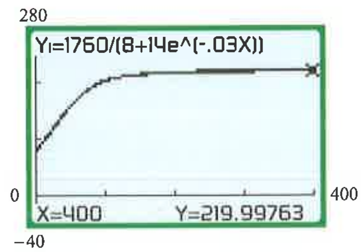
The weekly revenue R, in dollars, from the sale of a product varies with time according to the function

$$R(x) = \frac{1760}{8 + 14e^{-0.03x}}$$

where x is the number of weeks that have passed since the product was put on the market. What will the weekly revenue approach as time goes by?

Solution

Method 1 Use a graphing utility to graph R to determine what happens to the revenue as the time increases. The graph on the right appears to show that as the weeks go by, the weekly revenue will increase and approach \$220 per week.



Xscl = 100 Yscl = 100

Method 2 Write the revenue function in the following form.

$$R(x) = \frac{1760}{8 + \frac{14}{e^{0.03x}}}$$

• $14e^{-0.03x} = \frac{14}{e^{0.03x}}$

As x increases without bound, $e^{0.03x}$ increases without bound, and the fraction $\frac{14}{e^{0.03x}}$ approaches 0. Therefore, as $x \to \infty$, $R(x) \to \frac{1760}{8 + 0} = 220$. As the number of weeks increases, the revenue approaches \$220 per week.

▶ Try Exercise 58, page 359

EXERCISE SET 4.2

Concept Check

1. Name two characteristics of the graph of $f(x) = b^x$, $b > 1$.
2. Is $f(x) = b^x$, $0 < b < 1$, an increasing function or a decreasing function?
3. Explain how to use the graph of $f(x) = b^x$ to produce the graph of $g(x) = b^{x-2}$.
4. The base of the natural exponential function is denoted by the letter e. What is the value of e, rounded to the nearest thousandth?

In Exercises 5 to 12, evaluate the exponential function for the given x values.

5. $f(x) = 3^x$; $x = 0$ and $x = 4$
6. $f(x) = 5^x$; $x = 3$ and $x = -2$
7. $g(x) = 10^x$; $x = -3$ and $x = 2$
8. $g(x) = 6^x$; $x = 5$ and $x = -4$
9. $h(x) = \left(\frac{5}{3}\right)^x$; $x = 3$ and $x = -2$
10. $h(x) = \left(\frac{4}{5}\right)^x$; $x = -3$ and $x = 4$
11. $j(x) = \left(\frac{1}{2}\right)^x$; $x = -2$ and $x = 4$
12. $j(x) = \left(\frac{1}{4}\right)^x$; $x = -1$ and $x = 5$

Indicates Try It Exercises

In Exercises 13 to 18, use a calculator to evaluate the exponential function for the given *x* value. Round to the nearest hundredth.

13. $f(x) = 2^x;\ x = 3.2$

14. $f(x) = 3^x;\ x = -1.5$

15. $g(x) = e^x;\ x = -3$

16. $g(x) = e^x;\ x = 4.2$

17. $h(x) = 3.5^x;\ x = \sqrt{3}$

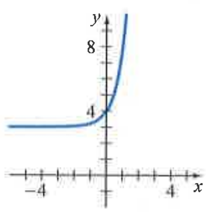
18. $h(x) = 2.4^x;\ x = e$

In Exercises 19 and 20, examine the four functions and the graphs labeled a, b, c, and d. For each graph, determine which function has been graphed.

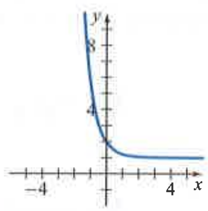
19. $f(x) = 5^x \qquad g(x) = 1 + 5^{-x}$

$h(x) = 5^{x+3} \qquad k(x) = 5^x + 3$

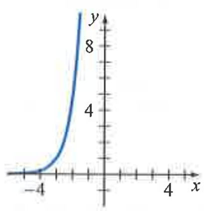
a.



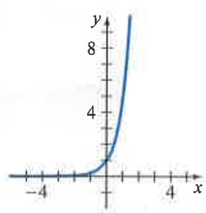
b.



c.



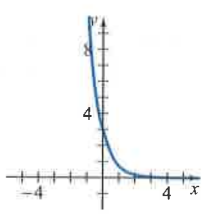
d.



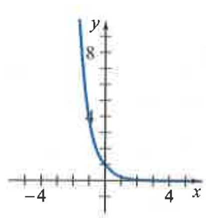
20. $f(x) = \left(\frac{1}{4}\right)^x \qquad g(x) = \left(\frac{1}{4}\right)^{-x}$

$h(x) = \left(\frac{1}{4}\right)^{x-2} \qquad k(x) = 3\left(\frac{1}{4}\right)^x$

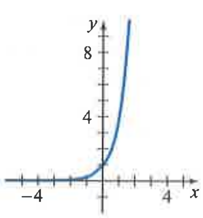
a.



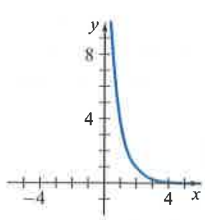
b.



c.



d.



In Exercises 21 to 28, sketch the graph of each function.

21. $f(x) = 3^x$

22. $f(x) = 4^x$

23. $f(x) = 10^x$

24. $f(x) = 6^x$

25. $f(x) = \left(\frac{3}{2}\right)^x$

26. $f(x) = \left(\frac{5}{2}\right)^x$

27. $f(x) = \left(\frac{1}{3}\right)^x$

28. $f(x) = \left(\frac{2}{3}\right)^x$

In Exercises 29 to 42, explain how to use the graph of the first function *f* to produce the graph of the second function *F*.

29. $f(x) = 2^x,\ F(x) = 2^x + 3$

30. $f(x) = 3^x,\ F(x) = 3^x - 2$

31. $f(x) = e^x,\ F(x) = e^{x-3}$

32. $f(x) = 6^x,\ F(x) = 6^{x+5}$

33. $f(x) = \left(\frac{3}{2}\right)^x,\ F(x) = \left(\frac{3}{2}\right)^{-x}$

34. $f(x) = \left(\frac{5}{2}\right)^x,\ F(x) = -\left[\left(\frac{5}{2}\right)^x\right]$

35. $f(x) = \left(\frac{1}{3}\right)^x,\ F(x) = 2\left[\left(\frac{1}{3}\right)^x\right]$

36. $f(x) = \left(\frac{2}{3}\right)^x,\ F(x) = \frac{1}{2}\left[\left(\frac{2}{3}\right)^x\right]$

37. $f(x) = e^x,\ F(x) = e^{x+3} - 2$

38. $f(x) = e^x,\ F(x) = e^{-x} + 4$

39. $f(x) = 2^x,\ F(x) = -(2^{x-4})$

40. $f(x) = 2^x,\ F(x) = -(2^{-x})$

41. $f(x) = 0.5^x,\ F(x) = 3 + 0.5^{-x}$

42. $f(x) = 0.5^x,\ F(x) = 3(0.5^{x+2}) - 1$

In Exercises 43 to 50, use a graphing utility to graph each function.

43. $f(x) = \dfrac{3^x + 3^{-x}}{2}$

44. $f(x) = 4 \cdot 3^{-x^2}$

45. $f(x) = \dfrac{e^x - e^{-x}}{2}$

46. $f(x) = \dfrac{e^x + e^{-x}}{2}$

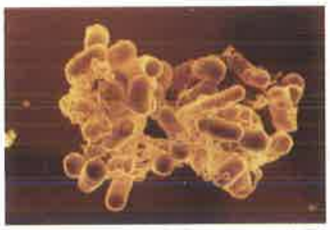
47. $f(x) = -e^{(x-4)}$

48. $f(x) = 0.5e^{-x}$

49. $f(x) = \dfrac{10}{1 + 0.4e^{-0.5x}},\ x \geq 0$

50. $f(x) = \dfrac{10}{1 + 1.5e^{-0.5x}},\ x \geq 0$

51. ***E. Coli* Infection** *Escherichia coli* (*E. coli*) is a bacterium that can reproduce at an exponential rate. The *E. coli* reproduce by dividing. A small number of *E. coli* bacteria in the large intestine of a human can trigger a serious infection within a few hours. Consider a particular *E. coli* infection that starts with 100 *E. coli* bacteria. Each bacterium splits into two parts every half hour. Assuming none of the bacteria die, the size of the *E. coli* population after t hours is given by $P(t) = 100 \cdot 2^{2t}$, where $0 \le t \le 16$.



Charles O'Rear/CORBIS

a. Find $P(3)$ and $P(6)$.

b. Find the time, to the nearest tenth of an hour, it takes for the *E. coli* population to number 1 billion.

52. **Medication in the Bloodstream** The exponential function

$$A(t) = 200e^{-0.014t}$$

gives the amount of medication, in milligrams, in a patient's bloodstream t minutes after the medication has been injected into the patient's bloodstream.

a. Find the amount of medication, to the nearest milligram, in the patient's bloodstream after 45 minutes.

b. Determine how long it will take, to the nearest minute, for the amount of medication in the patient's bloodstream to reach 50 milligrams.

53. **Demand for a Product** The demand d for a specific product, in items per month, is given by

$$d(p) = 880e^{-0.18p}$$

where p is the price, in dollars, of the product.

a. What will be the monthly demand, to the nearest unit, when the price of the product is \$10 and when the price is \$18?

b. What will happen to the demand as the price increases without bound?

54. **Sales** The monthly income I, in dollars, from a new product is given by

$$I(t) = 8600 - 5500e^{-0.005t}$$

where t is the time, in months, since the product was first put on the market.

a. What was the monthly income after the 10th month and after the 100th month?

b. What will the monthly income from the product approach as the time increases without bound?

55. **Photochromatic Eyeglass Lenses** Photochromatic eyeglass lenses contain molecules of silver chloride or silver halide. These molecules are transparent in the absence of ultraviolet (UV) rays. UV rays are normally absent in artificial lighting. However, when the lenses are exposed to UV rays, as in direct sunlight, the molecules take on a new molecular structure, which causes the lenses to darken. The number of molecules affected varies with the intensity of the UV rays. The intensity of UV rays is measured using a scale called the UV index. On this scale, a value near 0 indicates a low UV intensity and a value near 10 indicates a high UV intensity.

For the photochromatic lenses shown below, the function $P(x) = (0.9)^x$ models the transparency P of the lenses as a function of the UV index x.

UV index, 0
Lens transparency, 100%

UV index, 5
Lens transparency, 59.0%

UV index, 9
Lens transparency, 38.7%

a. Find the transparency of these lenses, to the nearest tenth of a percent, when they are exposed to light rays with a UV index of 3.5.

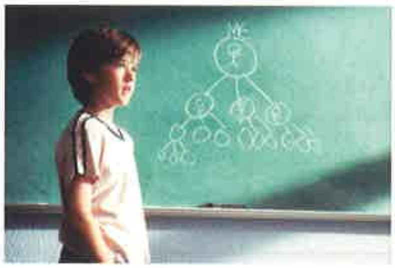
b. What is the UV index of light rays that cause these photochromatic lenses to have a transparency of 45%? Round to the nearest tenth.

56. **Radiation** Lead shielding is used to contain radiation. The percentage of a certain radiation that can penetrate x millimeters of lead shielding is given by $I(x) = 100e^{-1.5x}$.

a. What percentage of radiation, to the nearest tenth of a percent, will penetrate a lead shield that is 1 millimeter thick?

b. How many millimeters of lead shielding are required so that less than 0.05% of the radiation penetrates the shielding? Round to the nearest millimeter.

57. **The Pay It Forward Model** In the movie *Pay It Forward*, Trevor McKinney, played by Haley Joel Osment, is given a school assignment to "think of an idea to change the world—and then put it into action." In response to this assignment, Trevor develops a *pay it forward* project. In this project, anyone who benefits from another person's good deed must do a good deed for three additional people. Each of these three people is then obligated to do a good deed for another three people, and so on.

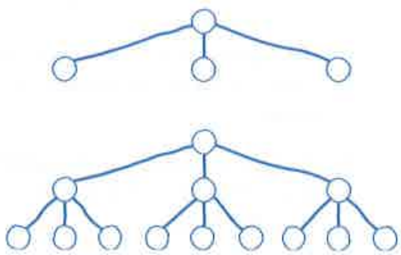


David James/2000 Warner Bros. & Bel Air Pictures, LLC/Newsmaker/Getty Images

The following diagram shows the number of people who have been a beneficiary of a good deed after one round and after two rounds of this project.

Three beneficiaries after one round

A total of 12 beneficiaries after two rounds (3 + 9 = 12)



A mathematical model for the number of pay-it-forward beneficiaries after n rounds is given by $B(n) = \dfrac{3^{n+1} - 3}{2}$. Use this model to determine

a. the number of beneficiaries after 5 rounds and after 10 rounds. Assume that no person is a beneficiary of more than one good deed.

b. how many rounds are required to produce at least 2 million beneficiaries.

58. **Fish Population** The number of bass in a lake is given by

$$P(t) = \frac{3600}{1 + 7e^{-0.05t}}$$

where t is the number of months that have passed since the lake was stocked with bass.

a. How many bass were in the lake immediately after it was stocked?

b. How many bass were in the lake 1 year after the lake was stocked? Round to the nearest bass.

c. What will happen to the bass population as t increases without bound?

59. **A Temperature Model** A cup of coffee is heated to 180°F and placed in a room that maintains a temperature of 65°F. The temperature of the coffee after t minutes is given by $T(t) = 65 + 115e^{-0.042t}$.

a. Find the temperature, to the nearest degree, of the coffee 10 minutes after it is placed in the room.

b. Determine when, to the nearest tenth of a minute, the temperature of the coffee will reach 100°F.

60. **Intensity of Light** The percent $I(x)$ of the original intensity of light striking the surface of a lake that is available x feet below the surface of the lake is given by the equation $I(x) = 100e^{-0.95x}$.

a. What percentage of the light, to the nearest tenth of a percent, is available 2 feet below the surface of the lake?

b. At what depth, to the nearest hundredth of a foot, is the intensity of the light one-half the intensity at the surface?

61. **Musical Scales** Starting on the left side of a standard 88-key piano, the frequency, in vibrations per second, of the nth note is given by $f(n) = (27.5)2^{(n-1)/12}$.

a. Using this formula, determine the frequency, to the nearest hundredth of a vibration per second, of middle C, key number 40 on an 88-key piano.

b. Is the difference in frequency between middle C (key number 40) and D (key number 42) the same as the difference in frequency between D (key number 42) and E (key number 44)? Explain.

In Exercises 62 and 63, verify that the given function is odd or even as requested.

62. Verify that $f(x) = \dfrac{e^x + e^{-x}}{2}$ is an even function.

63. Verify that $f(x) = \dfrac{e^x - e^{-x}}{2}$ is an odd function.

In Exercises 64 and 65, draw the graphs as indicated.

64. Graph $g(x) = 10^x$, and then sketch the graph of g reflected across the line given by $y = x$.

65. Graph $f(x) = e^x$, and then sketch the graph of f reflected across the line given by $y = x$.

Enrichment Exercises

In Exercises 66 to 69, determine the domain of the given function. Write the domain using interval notation.

66. $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

67. $f(x) = \dfrac{e^{|x|}}{1 + e^x}$

68. $f(x) = \sqrt{1 - e^x}$

69. $f(x) = \sqrt{e^x - e^{-x}}$

70. **An Exponential Reward** According to legend, when Sissa Ben Dahir of India invented the game of chess, King Shirham was so impressed with the game that he offered Sissa Ben Dahir the reward of his choosing. Sissa Ben Dahir pointed to the chessboard and requested, for his reward, one grain of wheat on the first square, two grains of wheat on the second square, four grains of wheat on the third square, eight grains on the fourth square, and so on for all 64 squares on the chessboard. The king considered this a very modest reward and said he would grant the inventor's wish.

The following table shows how many grains of wheat are on each of the first 7 squares of a chessboard and the total number of grains of wheat needed to cover squares 1 to n for $n \le 7$.

Square number, n	Number of grains of wheat on square n	Total number of grains of wheat on squares 1 to n
1	1	1
2	2	$1 + 2 = 3$
3	4	$3 + 4 = 7$
4	8	$7 + 8 = 15$
5	16	$15 + 16 = 31$
6	32	$31 + 32 = 63$
7	64	$63 + 64 = 127$

a. How many grains of wheat are needed to cover all 64 squares of the chessboard, as requested by Sissa Ben Dahir?

b. A grain of wheat weighs approximately 0.000008 kilogram. Find the total weight of the wheat requested by Sissa Ben Dahir.

c. In a recent year, a total of 6.5×10^8 metric tons of wheat were produced in the world. At this level, how many years, to the nearest year, of wheat production would be required to fill the request of Sissa Ben Dahir? One metric ton equals 1000 kilograms.

71. **Average Height** Explain why the graph of

$$f(x) = \frac{e^x + e^{-x}}{2}$$

can be produced by plotting the average height of $g(x) = e^x$ and $h(x) = e^{-x}$ for each value of x.

SECTION 4.3

Logarithmic Functions and Their Applications

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A25.

PS1. If $2^x = 16$, determine the value of x. [4.2]

PS2. If $3^{-x} = \dfrac{1}{27}$, determine the value of x. [4.2]

PS3. If $x^4 = 625$, determine the value of x. [4.2]

PS4. Find the inverse of $f(x) = \dfrac{2x}{x + 3}$. [4.1]

PS5. State the domain of $g(x) = \sqrt{x - 2}$. [2.2]

PS6. If the range of $h(x)$ is the set of all positive real numbers, then what is the domain of $h^{-1}(x)$? [4.1]