

Adding, Subtracting, and Multiplying Polynomials

Learning Target

4.2

Add, subtract, and multiply polynomials.

Success Criteria

Math Practice

Look for Patterns

How can you extend

such as $(2x + 1)^{5}$?

Pascal's Triangle and use it to expand a binomial

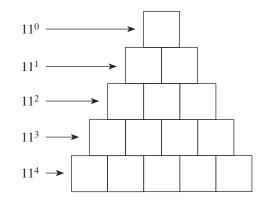
with a greater exponent,

- I can add and subtract polynomials.
- I can multiply polynomials and use special product patterns.
- I can use Pascal's Triangle to expand binomials.

EXPLORE IT Expanding Binomials

Work with a partner.

a. Copy the diagram. Find the value of each expression. Write one digit of the value in each box.



What pattern(s) do you notice?

b. Find each product. Explain your steps.

 $(x+1)^2$ $(x+1)^3$

What pattern do you notice between the values of 11^n and the terms of $(x + 1)^n$ for $0 \le n \le 3$? Does this pattern continue for $(x + 1)^4$? Explain your reasoning.

c. Find each product. Explain your steps.

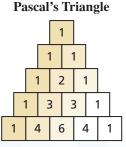
 $(a+b)^3$ $(a-b)^3$

What other pattern(s) do you notice when cubing these binomials?

d. Explain how you can use Pascal's Triangle to find each product. Then find the product.

i. $(x + 2)^3$

ii.
$$(2x - 3)^3$$



AZ Vocabulary VOCAB Pascal's Triangle, p. 165

STUDY TIP

When a power of the variable appears in one polynomial but not the other, leave a space in that column, or write the term with a coefficient of 0.

Adding and Subtracting Polynomials



WATCH

The set of integers is *closed* under addition and subtraction because every sum or difference results in an integer. To add or subtract polynomials, add or subtract the coefficients of like terms. Because adding or subtracting polynomials results in a polynomial, the set of polynomials is also closed under addition and subtraction.

EXAMPLE 1 Adding Polynomials Vertically and Horizontally

- **a.** Add $3x^3 + 2x^2 x 7$ and $x^3 10x^2 + 8$ in a vertical format.
- **b.** Add $9y^3 + 3y^2 2y + 1$ and $-5y^2 + y 4$ in a horizontal format.

SOLUTION

a. Align like terms vertically and add.

 $3x^3 + 2x^2 - x - 7$ $\frac{+ x^3 - 10x^2 + 8}{4x^3 - 8x^2 - x + 1}$

b. Group like terms and simplify.

$$(9y^3 + 3y^2 - 2y + 1) + (-5y^2 + y - 4) = 9y^3 + 3y^2 - 5y^2 - 2y + y + 1 - 4$$
$$= 9y^3 - 2y^2 - y - 3$$

COMMON ERROR

A common mistake is to forget to change signs correctly when subtracting one polynomial from another. Be sure to add the opposite of *every* term of the subtracted polynomial.

EXAMPLE 2

Subtracting Polynomials Vertically and Horizontally



a. Subtract $2x^3 + 6x^2 - x + 1$ from $8x^3 - 3x^2 - 2x + 9$ in a vertical format.

b. Subtract $3z^2 + z - 4$ from $2z^2 + 3z$ in a horizontal format.

SOLUTION

a. Align like terms vertically, then add the opposite of the subtracted polynomial.

 $8x^3 - 3x^2 - 2x + 9$ $8x^3 - 3x^2 - 2x + 9$ $-(2x^3 + 6x^2 - x + 1)$ \rightarrow $+ -2x^3 - 6x^2 + x - 1$ $6r^3 - 9r^2 - r + 8$

b. Write the opposite of the subtracted polynomial, then add like terms.

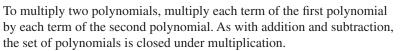
(2z² + 3z) - (3z² + z - 4) = 2z² + 3z - 3z² - z + 4 $= -z^2 + 2z + 4$

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Find the sum or difference.

- 1. $(2x^2 6x + 5) + (7x^2 x 9)$
- **2.** $(6z^4 + 3z^2 10) + (6z^3 4z^2 + z 10)$ **3.** $(3t^3 + 8t^2 - t - 4) - (5t^3 - t^2 + 17)$ **4.** $(p^5 + 2p^3 - 8p^2 + 7) - (9p^5 + 12p^2 - p)$
- 5. OPEN-ENDED Write two trinomials whose sum has (a) four terms, (b) two terms, and (c) one term.
- 6. MP STRUCTURE In your own words, explain why the set of polynomials is closed under addition and subtraction.

Multiplying Polynomials



EXAMPLE 3

Multiplying Polynomials Vertically and Horizontally



a. Multiply $-x^2 + 2x + 4$ and x - 3 in a vertical format.

b. Multiply y + 5 and $3y^2 - 2y + 2$ in a horizontal format.

SOLUTION

--->

a. $-x^2 + 2x + 4$ $\times x - 3$ $3x^2 - 6x - 12$ $-x^3 + 2x^2 + 4x$ $-x^3 + 5x^2 - 2x - 12$ Multiply $-x^2 + 2x + 4$ by -3. Multiply $-x^2 + 2x + 4$ by x. Combine like terms.

b. $(y + 5)(3y^2 - 2y + 2) = (y + 5)3y^2 - (y + 5)2y + (y + 5)2$ = $3y^3 + 15y^2 - 2y^2 - 10y + 2y + 10$ = $3y^3 + 13y^2 - 8y + 10$

EXAMPLE 4 Mu

Multiplying Three Binomials



Multiply x - 1, x + 4, and x + 5 in a horizontal format.

SOLUTION

 $(x-1)(x+4)(x+5) = (x^2 + 3x - 4)(x+5)$ = $(x^2 + 3x - 4)x + (x^2 + 3x - 4)5$ = $x^3 + 3x^2 - 4x + 5x^2 + 15x - 20$ = $x^3 + 8x^2 + 11x - 20$

Some binomial products occur so frequently that it is worth memorizing their patterns. You can verify these polynomial identities by multiplying.

) KEY IDEA

Special Product Patterns

Sum and Difference	Example
$(a+b)(a-b) = a^2 - b^2$	$(x+3)(x-3) = x^2 - 9$
Square of a Binomial	Example
$(a+b)^2 = a^2 + 2ab + b^2$	$(y+4)^2 = y^2 + 8y + 16$
$(a-b)^2 = a^2 - 2ab + b^2$	$(2t-5)^2 = 4t^2 - 20t + 25$
Cube of a Binomial	Example
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	$(z+3)^3 = z^3 + 9z^2 + 27z + 27$
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$	$(m-2)^3 = m^3 - 6m^2 + 12m - 8$

COMMON ERROR

In general,

REMEMBER

The Product of Powers

 $a^m \cdot a^n = a^{m+n}$

where a is a real number and m and n are integers.

Property states that

 $(a \pm b)^2 \neq a^2 \pm b^2$

and

 $(a \pm b)^3 \neq a^3 \pm b^3$.



Proving a Polynomial Identity



WATCH

a. Prove the polynomial identity for the cube of a binomial representing a sum:

 $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$

b. Use the cube of a binomial in part (a) to calculate 11^3 .

SOLUTION

a. Expand and simplify the expression on the left side of the equation.

$$(a + b)^{3} = (a + b)(a + b)(a + b)$$

= $(a^{2} + 2ab + b^{2})(a + b)$
= $(a^{2} + 2ab + b^{2})a + (a^{2} + 2ab + b^{2})b$
= $a^{3} + 2a^{2}b + ab^{2} + a^{2}b + 2ab^{2} + b^{3}$
= $a^{3} + 3a^{2}b + 3ab^{2} + b^{3}$

The simplified left side equals the right side of the original identity. So, the identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ is true.

b. To calculate 11^3 using the cube of a binomial, note that 11 = 10 + 1.

$11^3 = (10 + 1)^3$	Write 11 as 10 + 1.
$= 10^3 + 3(10)^2(1) + 3(10)(1)^2 + 1^3$	Cube of a binomial pattern
= 1000 + 300 + 30 + 1	Simplify.
= 1331	Add.

REMEMBER
The Power of a Product
Property states that
$(ab)^m = a^m b^m$

where *a* and *b* are real numbers and *m* is

an integer.

1	EXAMPLE 6 Using Special Product Pat	terns WATCH
luct	Find each product.	
	a. $(4n + 5)(4n - 5)$ b. $(9y - 2)^2$	c. $(ab + 4)^3$
eal	SOLUTION	
	a. $(4n + 5)(4n - 5) = (4n)^2 - 5^2$	Sum and difference pattern
~~~~>	$= 16n^2 - 25$	Simplify.
	<b>b.</b> $(9y - 2)^2 = (9y)^2 - 2(9y)(2) + 2^2$	Square of a binomial pattern
	$= 81y^2 - 36y + 4$	Simplify.
	<b>c.</b> $(ab + 4)^3 = (ab)^3 + 3(ab)^2(4) + 3(ab)(4)^2 + 4^3$	Cube of a binomial pattern
	$= a^3b^3 + 12a^2b^2 + 48ab + 64$	Simplify.

#### **SELF-ASSESSMENT** 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

#### Find the product.

<b>7.</b> $(4x^2 + x - 5)(2x + 1)$	<b>8.</b> $(y-2)(5y^2+3y-1)$	<b>9.</b> $(m-2)(m-1)(m+3)$
<b>10.</b> $(3t-2)(3t+2)$	<b>11.</b> $(5a+2)^2$	<b>12.</b> $(xy - 3)^3$

- **13. MP STRUCTURE** In your own words, explain why the set of polynomials is closed under multiplication.
- **14.** a. Prove the polynomial identity for the cube of a binomial representing a difference:  $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ .
  - **b.** Use the cube of a binomial in part (a) to calculate  $9^3$ .

## **Pascal's Triangle**



NATCH

Consider the expansion of the binomial  $(a + b)^n$  for whole number values of *n*. When you arrange the coefficients of the variables in the expansion of  $(a + b)^n$ , you will see a special pattern called **Pascal's Triangle**. Pascal's Triangle is named after French mathematician Blaise Pascal (1623–1662).

## **KEY IDEA** Pascal's Triangle

In Pascal's Triangle, the first and last numbers in each row are 1. Every number other than 1 is the sum of the closest two numbers in the row directly above it. The numbers in Pascal's Triangle are the same numbers that are the coefficients of binomial expansions, as shown in the first six rows.

	n	$(a+b)^n$	<b>Binomial Expansion</b>		Pas	scal's [	Trian	gle		
0th row	0	$(a + b)^0 =$	1			1				
1st row	1	$(a + b)^1 =$	1a + 1b			1	1			
2nd row	2	$(a + b)^2 =$	$1a^2 + 2ab + 1b^2$		1	2		1		
3rd row	3	$(a + b)^3 =$	$1a^3 + 3a^2b + 3ab^2 + 1b^3$		1	3	3	1		
4th row	4	$(a + b)^4 =$	$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$		1 4	6	i i	4	1	
5th row	5	$(a+b)^5 = 1a$	$5^{5} + 5a^{4}b + 10a^{3}b^{2} + 10a^{2}b^{3} + 5ab^{4} + 1b^{5}$	1	5	10	10	5		1



Blaise Pascal (1623-1662)

In general, the *n*th row in Pascal's Triangle gives the coefficients of  $(a + b)^n$ . Here are some other observations about the expansion of  $(a + b)^n$ .

- **1.** An expansion has n + 1 terms.
- **2.** The power of *a* begins with *n*, decreases by 1 in each successive term, and ends with 0.
- **3.** The power of *b* begins with 0, increases by 1 in each successive term, and ends with *n*.
- **4.** The sum of the powers of each term is *n*.

#### **EXAMPLE 7** Using Pascal's Triangle to Expand Binomials

Use Pascal's Triangle to expand (a)  $(x - 2)^5$  and (b)  $(3y + 1)^3$ .

#### **SOLUTION**

a. The coefficients from the fifth row of Pascal's Triangle are 1, 5, 10, 10, 5, and 1.

$$(x-2)^5 = 1x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + 1(-2)^5$$

$$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$$

**b.** The coefficients from the third row of Pascal's Triangle are 1, 3, 3, and 1.

$$(3y + 1)^3 = 1(3y)^3 + 3(3y)^2(1) + 3(3y)(1)^2 + 1(1)^3$$
  
=  $27y^3 + 27y^2 + 9y + 1$ 

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

**15.** Use Pascal's Triangle to expand (a)  $(z + 3)^4$  and (b)  $(2t - 1)^5$ .

**16.** WRITING Describe three different methods to expand  $(x + 3)^3$ .

## 4.2 Practice with CalcChat[®] AND CalcView[®]



In Exercises 1−6, find the sum. ≥ *Example 1* 

- **1.**  $(3x^2 + 4x 1) + (-2x^2 3x + 2)$
- **2.**  $(-5x^2 + 4x 2) + (-8x^2 + 2x + 1)$
- **3.**  $(12x^5 3x^4 + 2x 5) + (8x^4 3x^3 + 4x + 1)$
- **4.**  $(8x^4 + 2x^2 1) + (3x^3 5x^2 + 7x + 1)$
- **5.**  $(2x^5 + 7x^6 3x^2 + 9x) + (5x^5 + 8x^3 6x^2 + 2x 5)$
- **6.**  $(9x^4 3x^3 + 4x^2 + 5x + 7) + (11x^4 9 4x^2 11x)$

In Exercises 7–12, find the difference. **D** *Example 2* 

- **7.**  $(3x^3 2x^2 + 4x 8) (5x^3 + 12x^2 3x 4)$
- **8.**  $(7x^4 9x^3 4x^2 + 5x + 6) (2x^4 + 3x^3 x^2 + x 4)$
- **9.**  $(5x^6 2x^4 + 9x^3 + 2x 4) (7x^5 8x^4 + 2x 11)$
- **10.**  $(4x^5 7x^3 9x^2 + 18) (14x^5 8x^4 + 11x^2 + x)$
- **11.**  $(8x^5 + 6x^3 2x^2 + 10x) (4 + 9x^5 x^3 13x^2)$
- **12.**  $(3x 9x^2 + 11x^4 + 11) (2x^4 + 6x^3 + 2x 9)$

#### In Exercises 13–20, find the product. **D** *Example 3*

- **13.**  $7x^3(5x^2 + 3x + 1)$
- **14.**  $-4x^5(11x^3 + 2x^2 + 9x + 1)$
- **15.**  $(5x^2 4x + 6)(-2x + 3)$
- **16.**  $(-x-3)(2x^2+5x+8)$

**17.** 
$$(x^2 - 2x - 4)(x^2 - 3x - 5)$$

- **18.**  $(3x^2 + x 2)(-4x^2 2x 1)$
- **19.**  $(3x^3 9x + 7)(x^2 2x + 1)$
- **20.**  $(4x^2 8x 2)(x^4 + 3x^2 + 4x)$
- In Exercises 21–26, find the product of the binomials. *Example 4*
- **21.** (x-3)(x+2)(x+4)
- **22.** (x-5)(x+2)(x-6)
- **23.** (x-2)(3x+1)(4x-3)

- **24.** (2x + 5)(x 2)(3x + 4)
- **25.** (3x 4)(5 2x)(4x + 1)
- **26.** (4-5x)(1-2x)(3x+2)
- 27. MP REASONING Prove the polynomial identity  $(a + b)(a b) = a^2 b^2$ . Then give an example of two whole numbers greater than 10 that can be multiplied using mental math and the given identity. Justify your answer.  $\triangleright$  *Example 5*
- **28. MP NUMBER SENSE** Your Spanish club wants to order 29 hooded sweatshirts that cost \$31 each. Explain how you can use the polynomial identity  $(a + b)(a b) = a^2 b^2$  and mental math to find the total cost of the hooded sweatshirts.

#### In Exercises 29–38, find the product. **D** *Example 6*

<b>29.</b> $(x - 9)(x + 9)$	<b>30.</b> $(m + 6)^2$
<b>31.</b> $(3c-5)^2$	<b>32.</b> $(2y-5)(2y+5)$
<b>33.</b> $(7h + 4)^2$	<b>34.</b> $(9g - 4)^2$
<b>35.</b> $(2k+6)^3$	<b>36.</b> $(4n - 3)^3$
<b>37.</b> $(pq - 2)^3$	<b>38.</b> $(wz + 8)^3$

**ERROR ANALYSIS** In Exercises 39 and 40, describe and correct the error in performing the operation.

39.  

$$(x^{2}-3x+4) - (x^{3}+7x-2)$$

$$= x^{2}-3x+4 - x^{3}+7x-2$$

$$= -x^{3} + x^{2} + 4x + 2$$
40.  

$$(2x-7)^{3} = (2x)^{3} - 7^{3}$$

$$= 8x^{3} - 343$$

In Exercises 41–46, use Pascal's Triangle to expand the binomial. *Example 7* 

**41.**  $(6m + 2)^2$ **42.**  $(2t + 4)^3$ **43.**  $(2q - 3)^4$ **44.**  $(g + 2)^5$ **45.**  $(yz + 1)^5$ **46.**  $(np - 1)^4$ 



#### 47. MODELING REAL LIFE During a

recent period of time, the numbers (in thousands) of males M and females F who attend degreegranting institutions in the United States can be modeled by

$$M = 0.75t^2 - 79.5t + 9020$$

 $F = 22.44t^2 - 264.1t + 11,971$ 

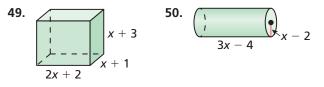
where *t* is time in years. Write a polynomial to model the total number of people attending degree-granting institutions. Interpret its constant term.

- **48.** MODELING REAL LIFE You throw a ball up into the air. The velocity v (in meters per second) of the ball after *t* seconds is given by v = -9.8t + 10. The mass *m* of the ball is 0.5 kilogram.
  - **a.** Use the formula  $K = \frac{1}{2}mv^2$  to write a polynomial in standard form that represents the kinetic energy *K* (in joules) of the ball after *t* seconds.
  - **b.** The potential energy U (in joules) of the ball after t seconds is given by

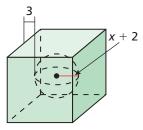
 $U = -24.01t^2 + 49t + 4.9.$ 

Write a polynomial that represents the total kinetic and potential energy. Interpret your result.

**CONNECTING CONCEPTS** In Exercises 49 and 50, write an expression for the volume of the figure as a polynomial in standard form.



- **51. MP REASONING** Is it possible for  $x^2 + 3x + 2x^{-1}$  to be the sum, difference, or product of two polynomials? Explain your reasoning.
- **52. COMPARING METHODS** Find the product  $(a^2 + 4b^2)^2(3a^2 b^2)^2$  using two different methods. Which method do you prefer? Explain.
- **53. MP PROBLEM SOLVING** The sphere is centered in the cube. Write an expression for the volume of the cube outside the sphere as a polynomial in standard form.



**54. MAKING AN ARGUMENT** Is the sum of two binomials always a binomial? Is the product of two binomials always a trinomial? Explain your reasoning.

**55. MODELING REAL LIFE** Two people make three deposits into their bank accounts. The accounts earn interest at the same rate *r* at the end of each year.

http://ww	n ★ Q	
Person A		Account No. 2-5384100608
Date	Transaction	Amount
01/01/2018	Deposit	\$2000.00
01/01/2019	Deposit	\$3000.00
01/01/2020	Deposit	\$1000.00
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Person B		Account No. 1-5233032905
Date	Transaction	Amount
01/01/2018	Deposit	\$5000.00
01/01/2019	Deposit	\$1000.00

On January 1, 2021, Person A's account is worth

 $2000(1 + r)^3 + 3000(1 + r)^2 + 1000(1 + r).$ 

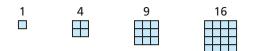
- **a.** Write a polynomial for the value of Person B's account on January 1, 2021.
- **b.** Write the total value of the two accounts as a polynomial in standard form. Then interpret the coefficients of the polynomial.
- **c.** What is the total value of the two accounts on January 1, 2021 when the interest rate is 0.01?

#### 56. HOW DO YOU SEE IT?

What polynomials are being multiplied		2 <i>x</i> ²	- <i>x</i>	4	
in the table? What	3 <i>x</i>	6 <i>x</i> ³	$-3x^{2}$	12x	
is the degree of the product?	-7	$-14x^{2}$	7 <i>x</i>	-28	

- 57. MP **REASONING** Copy Pascal's Triangle and include rows for n = 6, 7, 8, 9, and 10. Use the new rows to expand  $(x + 3)^7$  and  $(x 5)^9$ .
- **58. ABSTRACT REASONING** You are given the function f(x) = (x + a)(x + b)(x + c)(x + d). When f(x) is written in standard form, show that the coefficient of  $x^3$  is the sum of *a*, *b*, *c*, and *d*, and the constant term is the product of *a*, *b*, *c*, and *d*.
- **59. DRAWING CONCLUSIONS** Let  $g(x) = 12x^4 + 8x + 9$ and  $h(x) = 3x^5 + 2x^3 - 7x + 4$ .
  - **a.** What is the degree of the polynomial g(x) + h(x)? g(x) - h(x)?  $g(x) \cdot h(x)$ ?
  - b. In general, if g(x) and h(x) are polynomials such that g(x) has degree m and h(x) has degree n, and m > n, what is the degree of g(x) + h(x)? g(x) h(x)? g(x) h(x)?

**60. MP PATTERNS** The first four square numbers are represented below.



- **a.** Find the differences between consecutive square numbers. What do you notice?
- **b.** Show how the polynomial identity  $(n + 1)^2 n^2 = 2n + 1$  models the differences between consecutive square numbers.
- **c.** Prove the polynomial identity in part (b).

#### In Exercises 61 and 62, simplify the expression.

**61.**  $(1+i)^5$  **62.**  $(3-i)^6$ 

### **REVIEW & REFRESH**

**65.** Solve 
$$x^2 - 7x + 2 = -2x^2 + 10x - 9$$
 by graphing

In Exercises 66–69, perform the operation.

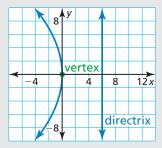
**66.** 
$$(7x^2 - 4) - (3x^2 - 5x + 1)$$

**67.** 
$$(-5x^4 + 6x^3 + x - 12) + (4x^4 - 15x^3 + 2x^2 - 3)$$

**68.** 
$$(x^2 - 3x + 2)(3x - 1)$$

**69.**  $(x-4)^3$ 

**70.** Write an equation of the parabola.



In Exercises 71 and 72, describe the end behavior of the function.

- **71.**  $f(x) = -2x^5 + 4x^3 x^2 8$
- **72.**  $g(x) = 3x^6 x^5 5x^2 + 4x + 1$
- **73. MP REASONING** The vertex of a parabola is (-3, 4) and one *x*-intercept is 2. What is the other *x*-intercept? Explain your reasoning.

## **63. CRITICAL THINKING** Recall that a Pythagorean triple is a set of positive



integers *a*, *b*, and *c* such that  $a^2 + b^2 = c^2$ . You can use the polynomial identity  $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$  to generate Pythagorean triples.

- a. Prove the polynomial identity.
- **b.** Use the identity to generate the Pythagorean triple when x = 6 and y = 5. Then verify that your answer satisfies  $a^2 + b^2 = c^2$ .
- 64. THOUGHT PROVOKING

Is the square of an odd number odd or even? Prove your answer.



**74.** Graph the system of quadratic inequalities.

$$y > x^2 + 4x + 5$$
$$y \le x^2 - 1$$

In Exercises 75 and 76, evaluate the function for the given value of *x*.

**75.** 
$$y = 2(0.5)^x$$
;  $x = 3$  **76.**  $y = -9(3)^x$ ;  $x = -1$ 

In Exercises 77–80, perform the operation. Write the answer in standard form.

- **77.** (3 2i) + (5 + 9i) **78.** (12 + 3i) (7 8i)
- **79.** 7i(-3i) **80.** (4+i)(2-i)
- **81. MODELING REAL LIFE** A contractor is hired to build an apartment complex. Each unit has a bedroom, kitchen, and bathroom. The bedroom will have the same area as the kitchen. The owner orders 980 square feet of tile to completely cover the floors of two kitchens and two bathrooms. Determine how many square feet of carpet is needed for each bedroom.

