**Alternative to Example 6**
Exercise 60, page 359.

## EXAMPLE 6 Use a Mathematical Model

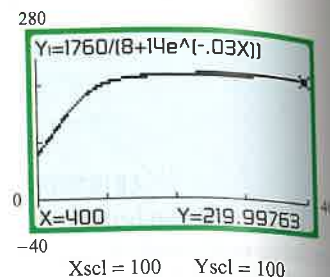
The weekly revenue $R$, in dollars, from the sale of a product varies with time according to the function

$$R(x) = \frac{1760}{8 + 14e^{-0.03x}}$$

where $x$ is the number of weeks that have passed since the product was put on the market. What will the weekly revenue approach as time goes by?

### Solution

***Method 1*** Use a graphing utility to graph $R$ to determine what happens to the revenue as the time increases. The graph on the right appears to show that as the weeks go by, the weekly revenue will increase and approach \$220 per week.




***Method 2*** Write the revenue function in the following form.

$$R(x) = \frac{1760}{8 + \dfrac{14}{e^{0.03x}}} \qquad \bullet\ 14e^{-0.03x} = \frac{14}{e^{0.03x}}$$

As $x$ increases without bound, $e^{0.03x}$ increases without bound, and the fraction $\frac{14}{e^{0.03x}}$ approaches 0. Therefore, as $x \to \infty$, $R(x) \to \frac{1760}{8 + 0} = 220$. As the number of weeks increases, the revenue approaches \$220 per week.

▶ **Try Exercise 58, page 359**

*Answer graphs to Exercises 21–28, 43–50, and 64–65 are on page AA16.*

# EXERCISE SET 4.2

## Concept Check

1. Name two characteristics of the graph of $f(x) = b^x$, $b > 1$. Answers will vary.
2. Is $f(x) = b^x$, $0 < b < 1$, an increasing function or a decreasing function? Decreasing
3. Explain how to use the graph of $f(x) = b^x$ to produce the graph of $g(x) = b^{x-2}$. Shift the graph of $f$ horizontally to the right 2 units.
4. The base of the natural exponential function is denoted by the letter $e$. What is the value of $e$, rounded to the nearest thousandth? 2.718

**In Exercises 5 to 12, evaluate the exponential function for the given $x$ values.**

5. $f(x) = 3^x$; $x = 0$ and $x = 4$ $f(0) = 1$; $f(4) = 81$
6. $f(x) = 5^x$; $x = 3$ and $x = -2$ $f(3) = 125$; $f(-2) = \frac{1}{25}$
7. $g(x) = 10^x$; $x = -3$ and $x = 2$ $g(-3) = \frac{1}{1000}$; $g(2) = 100$
8. $g(x) = 6^x$; $x = 5$ and $x = -4$ $g(5) = 7776$; $g(-4) = \frac{1}{129[illegible]}$
9. $h(x) = \left(\frac{5}{3}\right)^x$; $x = 3$ and $x = -2$ $h(3) = \frac{125}{27}$; $h(-2) =$ [illegible]
10. $h(x) = \left(\frac{4}{5}\right)^x$; $x = -3$ and $x = 4$ $h(-3) = \frac{125}{64}$; $h(4) =$ [illegible]
11. $j(x) = \left(\frac{1}{2}\right)^x$; $x = -2$ and $x = 4$ $j(-2) = 4$; $j(4) = \frac{1}{16}$
12. $j(x) = \left(\frac{1}{4}\right)^x$; $x = -1$ and $x = 5$ $j(-1) = 4$; $j(5) = \frac{1}{102[illegible]}$

■ Indicates Try It Exercises

**In Exercises 13 to 18, use a calculator to evaluate the exponential function for the given $x$ value. Round to the nearest hundredth.**

**13.** $f(x) = 2^x$; $x = 3.2$ 9.19 **14.** $f(x) = 3^x$; $x = -1.5$ 0.19

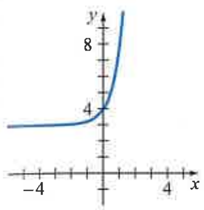
**15.** $g(x) = e^x$; $x = -3$ 0.05 **16.** $g(x) = e^x$; $x = 4.2$ 66.69

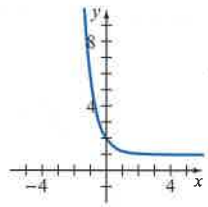
**17.** $h(x) = 3.5^x$; $x = \sqrt{3}$ 8.76 **18.** $h(x) = 2.4^x$; $x = e$ 10.80

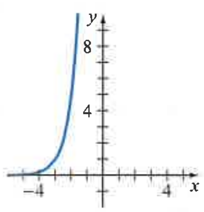
**In Exercises 19 and 20, examine the four functions and the graphs labeled a, b, c, and d. For each graph, determine which function has been graphed.**

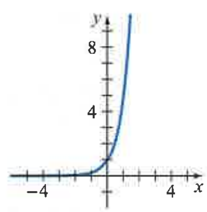
**19.** $f(x) = 5^x$ $g(x) = 1 + 5^{-x}$ $h(x) = 5^{x+3}$ $k(x) = 5^x + 3$

a. $k(x)$
b. $g(x)$
c. $h(x)$
d. $f(x)$

a.


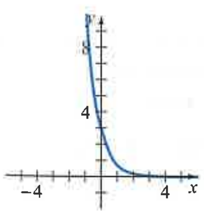

b.


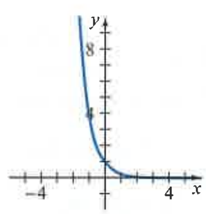

c.


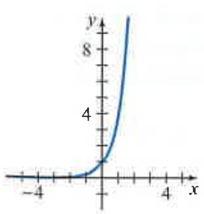

d.


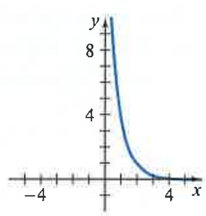

**20.** $f(x) = \left(\frac{1}{4}\right)^x$ $g(x) = \left(\frac{1}{4}\right)^{-x}$ $h(x) = \left(\frac{1}{4}\right)^{x-2}$ $k(x) = 3\left(\frac{1}{4}\right)^x$

a. $k(x)$
b. $f(x)$
c. $g(x)$
d. $h(x)$

a.



b.



c.



d.



**In Exercises 21 to 28, sketch the graph of each function.**

**21.** $f(x) = 3^x$ **22.** $f(x) = 4^x$

**23.** $f(x) = 10^x$ **24.** $f(x) = 6^x$

**25.** $f(x) = \left(\frac{3}{2}\right)^x$ **26.** $f(x) = \left(\frac{5}{2}\right)^x$

**27.** $f(x) = \left(\frac{1}{3}\right)^x$ **28.** $f(x) = \left(\frac{2}{3}\right)^x$

**In Exercises 29 to 42, explain how to use the graph of the first function $f$ to produce the graph of the second function $F$.**

**29.** $f(x) = 2^x$, $F(x) = 2^x + 3$
Shift the graph of $f$ vertically upward 3 units.

**30.** $f(x) = 3^x$, $F(x) = 3^x - 2$
Shift the graph of $f$ vertically downward 2 units.

**31.** $f(x) = e^x$, $F(x) = e^{x-3}$
Shift the graph of $f$ horizontally to the right 3 units.

**32.** $f(x) = 6^x$, $F(x) = 6^{x+5}$
Shift the graph of $f$ horizontally to the left 5 units.

**33.** $f(x) = \left(\frac{3}{2}\right)^x$, $F(x) = \left(\frac{3}{2}\right)^{-x}$ Reflect the graph of $f$ across the $y$-axis.

**34.** $f(x) = \left(\frac{5}{2}\right)^x$, $F(x) = -\left[\left(\frac{5}{2}\right)^x\right]$ Reflect the graph of $f$ across the $x$-axis.

**35.** $f(x) = \left(\frac{1}{3}\right)^x$, $F(x) = 2\left[\left(\frac{1}{3}\right)^x\right]$ Stretch the graph of $f$ vertically away from the $x$-axis by a factor of 2.

**36.** $f(x) = \left(\frac{2}{3}\right)^x$, $F(x) = \frac{1}{2}\left[\left(\frac{2}{3}\right)^x\right]$ Shrink the graph of $f$ vertically toward the $x$-axis by a factor of $\frac{1}{2}$.

**37.** $f(x) = e^x$, $F(x) = e^{x+3} - 2$
Shift the graph of $f$ horizontally 3 units to the left, and then shift this graph vertically downward 2 units.

**38.** $f(x) = e^x$, $F(x) = e^{-x} + 4$
Reflect the graph of $f$ across the $y$-axis, and then shift this graph vertically upward 4 units.

**39.** $f(x) = 2^x$, $F(x) = -(2^{x-4})$ Shift the graph of $f$ horizontally to the right 4 units, and then reflect this graph across the $x$-axis.

**40.** $f(x) = 2^x$, $F(x) = -(2^{-x})$ Reflect the graph of $f$ across the $y$-axis, and then reflect this graph across the $x$-axis.

**41.** $f(x) = 0.5^x$, $F(x) = 3 + 0.5^{-x}$ Reflect the graph of $f$ across the $y$-axis, and then shift this graph vertically upward 3 units.

**42.** $f(x) = 0.5^x$, $F(x) = 3(0.5^{x+2}) - 1$ Shift the graph of $f$ horizontally to the left 2 units, stretch this graph away from the $x$-axis by a factor of 3, and then shift this graph vertically downward 1 unit.

**In Exercises 43 to 50, use a graphing utility to graph each function.**

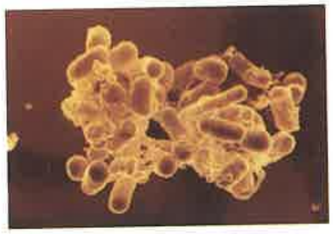
**43.** $f(x) = \dfrac{3^x + 3^{-x}}{2}$ **44.** $f(x) = 4 \cdot 3^{-x^2}$

**45.** $f(x) = \dfrac{e^x - e^{-x}}{2}$ **46.** $f(x) = \dfrac{e^x + e^{-x}}{2}$

**47.** $f(x) = -e^{(x-4)}$ **48.** $f(x) = 0.5e^{-x}$

**49.** $f(x) = \dfrac{10}{1 + 0.4e^{-0.5x}}$, $x \geq 0$ **50.** $f(x) = \dfrac{10}{1 + 1.5e^{-0.5x}}$, $x \geq 0$

**51.** **E. Coli Infection** *Escherichia coli* (*E. coli*) is a bacterium that can reproduce at an exponential rate. The *E. coli* reproduce by dividing. A small number of *E. coli* bacteria in the large intestine of a human can trigger a serious infection within a few hours. Consider a particular *E. coli* infection that starts with 100 *E. coli* bacteria. Each bacterium splits into two parts every half hour. Assuming none of the bacteria die, the size of the *E. coli* population after $t$ hours is given by $P(t) = 100 \cdot 2^{2t}$, where $0 \le t \le 16$.



Charles O'Rear/CORBIS

**a.** Find $P(3)$ and $P(6)$. 6400 bacteria; 409,600 bacteria

**b.** Find the time, to the nearest tenth of an hour, it takes for the *E. coli* population to number 1 billion. 11.6 h

**52.** **Medication in the Bloodstream** The exponential function

$$A(t) = 200e^{-0.014t}$$

gives the amount of medication, in milligrams, in a patient's bloodstream $t$ minutes after the medication has been injected into the patient's bloodstream.

**a.** Find the amount of medication, to the nearest milligram, in the patient's bloodstream after 45 minutes. 107 mg

**b.** Determine how long it will take, to the nearest minute, for the amount of medication in the patient's bloodstream to reach 50 milligrams. 99 min

**53.** **Demand for a Product** The demand $d$ for a specific product, in items per month, is given by

$$d(p) = 880e^{-0.18p}$$

where $p$ is the price, in dollars, of the product.

**a.** What will be the monthly demand, to the nearest unit, when the price of the product is \$10 and when the price is \$18? 145 items per month; 34 items per month

**b.** What will happen to the demand as the price increases without bound? The demand will approach 0 items per month.

**54.** **Sales** The monthly income $I$, in dollars, from a new product is given by

$$I(t) = 8600 - 5500e^{-0.005t}$$

where $t$ is the time, in months, since the product was first put on the market.

**a.** What was the monthly income after the 10th month and after the 100th month? \$3368.24; \$5264.08

**b.** What will the monthly income from the product approach as the time increases without bound? \$8600

**55.** **Photochromatic Eyeglass Lenses** Photochromatic eyeglass lenses contain molecules of silver chloride or silver halide. These molecules are transparent in the absence of ultraviolet (UV) rays. UV rays are normally absent in artificial lighting. However, when the lenses are exposed to UV rays, as in direct sunlight, the molecules take on a new molecular structure, which causes the lenses to darken. The number of molecules affected varies with the intensity of the UV rays. The intensity of UV rays is measured using a scale called the UV index. On this scale, a value near 0 indicates a low UV intensity and a value near 10 indicates a high UV intensity.

For the photochromatic lenses shown below, the function $P(x) = (0.9)^x$ models the transparency $P$ of the lenses as a function of the UV index $x$.

UV index, 0
Lens transparency, 100%

UV index, 5
Lens transparency, 59.0%

UV index, 9
Lens transparency, 38.7%

**a.** Find the transparency of these lenses, to the nearest tenth of a percent, when they are exposed to light rays with a UV index of 3.5. 69.2%

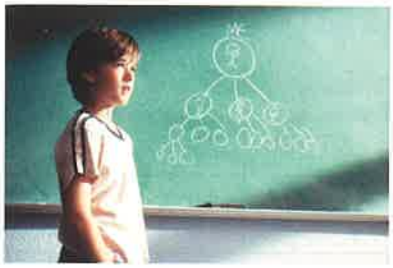
**b.** What is the UV index of light rays that cause these photochromatic lenses to have a transparency of 45%? Round to the nearest tenth. 7.6

**56.** **Radiation** Lead shielding is used to contain radiation. The percentage of a certain radiation that can penetrate $x$ millimeters of lead shielding is given by $I(x) = 100e^{-1.5x}$.

**a.** What percentage of radiation, to the nearest tenth of a percent, will penetrate a lead shield that is 1 millimeter thick? 22.3%

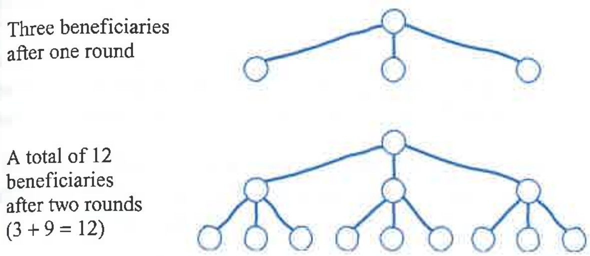
**b.** How many millimeters of lead shielding are required so that less than 0.05% of the radiation penetrates the shielding? Round to the nearest millimeter. 5 mm

**57.** **The Pay It Forward Model** In the movie *Pay It Forward,* Trevor McKinney, played by Haley Joel Osment, is given a school assignment to "think of an idea to change the world—and then put it into action." In response to this assignment, Trevor develops a *pay it forward* project. In this project, anyone who benefits from another person's good deed must do a good deed for three additional people. Each of these three people is then obligated to do a good deed for another three people, and so on.



David James/2000 Warner Bros. & Bel Air Pictures, LLC/Newsmaker/Getty Images

The following diagram shows the number of people who have been a beneficiary of a good deed after one round and after two rounds of this project.




A mathematical model for the number of pay-it-forward beneficiaries after $n$ rounds is given by $B(n) = \dfrac{3^{n+1} - 3}{2}$. Use this model to determine

**a.** the number of beneficiaries after 5 rounds and after 10 rounds. Assume that no person is a beneficiary of more than one good deed. 363 beneficiaries; 88,572 beneficiaries

**b.** how many rounds are required to produce at least 2 million beneficiaries. 13 rounds

**58.** **Fish Population** The number of bass in a lake is given by

$$P(t) = \frac{3600}{1 + 7e^{-0.05t}}$$

where $t$ is the number of months that have passed since the lake was stocked with bass.

**a.** How many bass were in the lake immediately after it was stocked? 450 bass

**b.** How many bass were in the lake 1 year after the lake was stocked? Round to the nearest bass. 744 bass

**c.** What will happen to the bass population as $t$ increases without bound?
The bass population will increase, approaching 3600.

**59.** **A Temperature Model** A cup of coffee is heated to 180°F and placed in a room that maintains a temperature of 65°F. The temperature of the coffee after $t$ minutes is given by $T(t) = 65 + 115e^{-0.042t}$.

**a.** Find the temperature, to the nearest degree, of the coffee 10 minutes after it is placed in the room. 141°F

**b.** Determine when, to the nearest tenth of a minute, the temperature of the coffee will reach 100°F. After 28.3 min

**60.** **Intensity of Light** The percent $I(x)$ of the original intensity of light striking the surface of a lake that is available $x$ feet below the surface of the lake is given by the equation $I(x) = 100e^{-0.95x}$.

**a.** What percentage of the light, to the nearest tenth of a percent, is available 2 feet below the surface of the lake? 15.0%

**b.** At what depth, to the nearest hundredth of a foot, is the intensity of the light one-half the intensity at the surface? 0.73 ft

**61.** **Musical Scales** Starting on the left side of a standard 88-key piano, the frequency, in vibrations per second, of the $n$th note is given by $f(n) = (27.5)2^{(n-1)/12}$.


**a.** Using this formula, determine the frequency, to the nearest hundredth of a vibration per second, of middle C, key number 40 on an 88-key piano. 261.63 vibrations per second

**b.** Is the difference in frequency between middle C (key number 40) and D (key number 42) the same as the difference in frequency between D (key number 42) and E (key number 44)? Explain. No. The function $f(n)$ is not a linear function. Therefore, the graph of $f(n)$ does not increase at a constant rate.

**In Exercises 62 and 63, verify that the given function is odd or even as requested.**

**62.** Verify that $f(x) = \dfrac{e^x + e^{-x}}{2}$ is an even function.

**63.** Verify that $f(x) = \dfrac{e^x - e^{-x}}{2}$ is an odd function.

**In Exercises 64 and 65, draw the graphs as indicated.**

**64.** Graph $g(x) = 10^x$, and then sketch the graph of $g$ reflected across the line given by $y = x$.

**65.** Graph $f(x) = e^x$, and then sketch the graph of $f$ reflected across the line given by $y = x$.

## Enrichment Exercises

**In Exercises 66 to 69, determine the domain of the given function. Write the domain using interval notation.**

**66.** $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ $(-\infty, \infty)$

**67.** $f(x) = \dfrac{e^{|x|}}{1 + e^x}$ $(-\infty, \infty)$

**68.** $f(x) = \sqrt{1 - e^x}$ $(-\infty, 0]$

**69.** $f(x) = \sqrt{e^x - e^{-x}}$ $[0, \infty)$

**70.** **An Exponential Reward** According to legend, when Sissa Ben Dahir of India invented the game of chess, King Shirham was so impressed with the game that he offered Sissa Ben Dahir the reward of his choosing. Sissa Ben Dahir pointed to the chessboard and requested, for his reward, one grain of wheat on the first square, two grains of wheat on the second square, four grains of wheat on the third square, eight grains on the fourth square, and so on for all 64 squares on the chessboard. The king considered this a very modest reward and said he would grant the inventor's wish.

The following table shows how many grains of wheat are on each of the first 7 squares of a chessboard and the total number of grains of wheat needed to cover squares 1 to $n$ for $n \le 7$.

| Square number, $n$ | Number of grains of wheat on square $n$ | Total number of grains of wheat on squares 1 to $n$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | $1 + 2 = 3$ |
| 3 | 4 | $3 + 4 = 7$ |
| 4 | 8 | $7 + 8 = 15$ |
| 5 | 16 | $15 + 16 = 31$ |
| 6 | 32 | $31 + 32 = 63$ |
| 7 | 64 | $63 + 64 = 127$ |

**a.** How many grains of wheat are needed to cover all 64 squares of the chessboard, as requested by Sissa Ben Dahir? $2^{64} - 1 \approx 1.844674407 \times 10^{19}$ grains of wheat

**b.** A grain of wheat weighs approximately 0.000008 kilogram. Find the total weight of the wheat requested by Sissa Ben Dahir. $\approx 1.475739526 \times 10^{14}$ kg

**c.** In a recent year, a total of $6.5 \times 10^8$ metric tons of wheat were produced in the world. At this level, how many years, to the nearest year, of wheat production would be required to fill the request of Sissa Ben Dahir? One metric ton equals 1000 kilograms. 227 years

**71.** **Average Height** Explain why the graph of

$$f(x) = \frac{e^x + e^{-x}}{2}$$

can be produced by plotting the average height of $g(x) = e^x$ and $h(x) = e^{-x}$ for each value of $x$.

By definition, the average of two numbers is their sum divided by 2. The expression $\frac{e^x + e^{-x}}{2}$ shows that $f$ is the average of $g(x) = e^x$ and $h(x) = e^{-x}$.

**SECTION 4.3**

Logarithmic Functions
Graphs of Logarithmic Functions
Domains of Logarithmic Functions
Common and Natural Logarithms

# Logarithmic Functions and Their Applications

### PREPARE FOR THIS SECTION

**Prepare for this section by completing the following exercises. The answers can be found on page A25.**

**PS1.** If $2^x = 16$, determine the value of $x$. [4.2] 4

**PS2.** If $3^{-x} = \dfrac{1}{27}$, determine the value of $x$. [4.2] 3

**PS3.** If $x^4 = 625$, determine the value of $x$. [4.2] 5

**PS4.** Find the inverse of $f(x) = \dfrac{2x}{x + 3}$. [4.1] $f^{-1}(x) = \dfrac{3x}{2 - x}$

**PS5.** State the domain of $g(x) = \sqrt{x - 2}$. [2.2] $\{x \mid x \ge 2\}$

**PS6.** If the range of $h(x)$ is the set of all positive real numbers, then what is the domain of $h^{-1}(x)$? [4.1] The set of all positive real numbers