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CHAPTER 4 Exponential and Logarithmic Functions

4.1 Inverse Functions
4.2 Exponential Functions and Their Applications
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4.4 Properties of Logarithms and Logarithmic Scales
4.5 Exponential and Logarithmic Equations
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Applications of Exponential and Logarithmic Functions

Exponential and logarithmic functions are often used to model data and make predictions. For instance, in Exercise 18, page 426, an exponential function is used to model the value of several diamonds that have different weights, measured in carats, but are similar in quality.

Logarithmic functions can be used to scale very large or very small numbers so that they are easier to comprehend. In Exercise 71, page 380, a logarithmic function is used to determine the Richter scale magnitude of an earthquake.

JIJI PRESS/AFP/Getty Images

The photo to the right shows a tsunami striking the coast of Japan. This tsunami was caused by the Richter-scale-magnitude 9.0 earthquake that struck off the coast of Honshu, Japan, on March 11, 2011. It ranks as one of the five most powerful earthquakes recorded during the last 100 years.

SECTION 4.1

Introduction to Inverse Functions
Graphs of Inverse Functions
Composition of a Function and Its Inverse
Finding an Inverse Function

Inverse Functions

Introduction to Inverse Functions

Consider the "doubling function" $f(x) = 2x$ that doubles every input. Some of the ordered pairs of this function are

$$\left\{(-4, -8), (-1.5, -3), (1, 2), \left(\frac{5}{3}, \frac{10}{3}\right), (7, 14)\right\}$$

Now consider the "halving function" $g(x) = \frac{1}{2}x$ that takes one-half of every input. Some of the ordered pairs of this function are

$$\left\{(-8, -4), (-3, -1.5), (2, 1), \left(\frac{10}{3}, \frac{5}{3}\right), (14, 7)\right\}$$

Observe that the ordered pairs of g are the ordered pairs of f with the order of the coordinates reversed. The following two examples illustrate this concept.

$f(5) = 2(5) = 10$ $\qquad$ $g(10) = \frac{1}{2}(10) = 5$

Ordered pair: (5, 10) $\qquad$ Ordered pair: (10, 5)

$f(a) = 2(a) = 2a$ $\qquad$ $g(2a) = \frac{1}{2}(2a) = a$

Ordered pair: $(a, 2a)$ $\qquad$ Ordered pair: $(2a, a)$

The function g is said to be the *inverse function* of f.

Note

In this section, our primary interest is finding the inverse of a function; however, we can also find the inverse of a relation. Recall that a relation r is any set of ordered pairs. The inverse of r is the set of ordered pairs formed by reversing the order of the coordinates of the ordered pairs in r.

Definition of an Inverse Function

If the ordered pairs of a function g are the ordered pairs of a function f with the order of the coordinates reversed, then g is the **inverse function** of f.

Consider a function f and its inverse function g. Because the ordered pairs of g are the ordered pairs of f with the order of the coordinates reversed, the domain of the inverse function g is the range of f, and the range of g is the domain of f.

Not all functions have an inverse that is a function. Consider, for instance, the "square function" $S(x) = x^2$. Some of the ordered pairs of S are

$$\{(-3, 9), (-1, 1), (0, 0), (1, 1), (3, 9), (5, 25)\}$$

If we reverse the coordinates of the ordered pairs, we have

$$\{(9, -3), (1, -1), (0, 0), (1, 1), (9, 3), (25, 5)\}$$

This set of ordered pairs is not a function because there are ordered pairs, for instance (9, −3) and (9, 3), with the same first coordinate and different second coordinates. In this case, S has an inverse *relation* but not an inverse *function.*

The definition of a one-to-one function and the horizontal line test were discussed in Section 2.2; however, because of the important role they play in this chapter, we have restated each of them as shown below.

Definition of a One-to-One Function

A function f is a **one-to-one function** if and only if $f(a) = f(b)$ impli

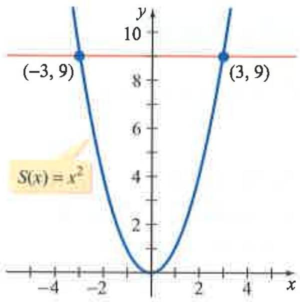


Figure 4.1

Horizontal Line Test for a One-to-One Function

If every horizontal line intersects the graph of a function at most once, then the graph is the graph of a one-to-one function.

A graph of S is shown in Figure 4.1. Note that $x = -3$ and $x = 3$ produce the same value of y. Thus the graph of S fails the horizontal line test; therefore, S is not a one-to-one function. This observation is used in the following theorem.

Condition for an Inverse Function

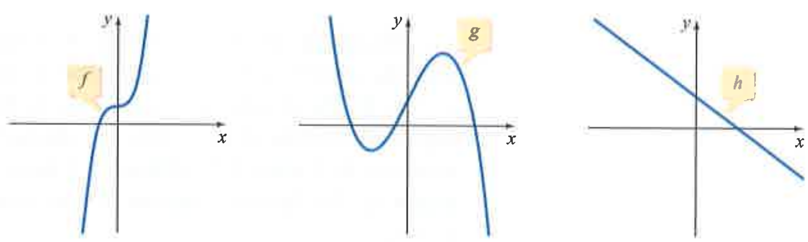
A function f has an inverse function if and only if f is a one-to-one function.

Recall that increasing functions and decreasing functions are one-to-one functions. Thus we can state the following theorem.

A Property of Increasing Functions and Decreasing Functions

If f is an increasing function or a decreasing function, then f has an inverse function.

Question • Which of the functions graphed below has an inverse function?



Caution

$f^{-1}(x)$ does not mean $\frac{1}{f(x)}$. For $f(x) = 2x$, $f^{-1}(x) = \frac{1}{2}x$, but $\frac{1}{f(x)} = \frac{1}{2x}$.

If a function g is the inverse of a function f, we usually denote the inverse function by f^{-1} rather than g. For the doubling and halving functions f and g discussed on page 336, we write

$$f(x) = 2x \qquad f^{-1}(x) = \frac{1}{2}x$$

Graphs of Inverse Functions

Because the coordinates of the ordered pairs of the inverse of a function f are the ordered pairs of f with the order of the coordinates reversed, we can use them to create a graph of f^{-1}.

Answer • The graph of f is the graph of an increasing function. Therefore, f is a one-to-one function and has an inverse function. The graph of h is the graph of a decreasing function. Therefore, h is a one-to-one function and has an inverse function. The graph of g is not the graph of a one-to-one function. g does not have an inverse function.

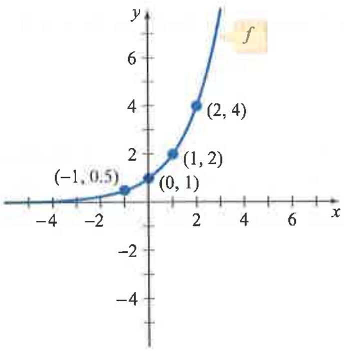


Figure 4.2

EXAMPLE 1 Sketch the Graph of the Inverse of a Function

Sketch the graph of f^{-1} given that f is the function shown in Figure 4.2.

Solution

Because the graph of f passes through $(-1, 0.5)$, $(0, 1)$, $(1, 2)$, and $(2, 4)$, the graph of f^{-1} must pass through $(0.5, -1)$, $(1, 0)$, $(2, 1)$, and $(4, 2)$. Plot the points and then draw a smooth curve through the points, as shown in Figure 4.3.

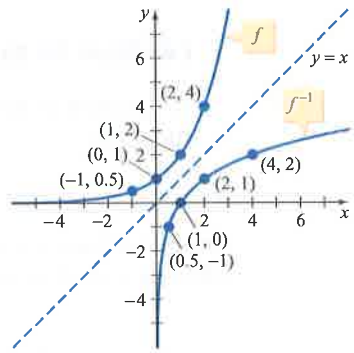


Figure 4.3

▸ **Try Exercise 14, page 344**

Question • If f is a one-to-one function and $f(4) = 5$, what is $f^{-1}(5)$?

The graph from the solution to Example 1 is shown again in Figure 4.4. Note that the graph of f^{-1} is symmetric to the graph of f with respect to the graph of $y = x$. If the graph were folded along the dashed line, the graph of f would lie on top of the graph of f^{-1}. This is a characteristic of all graphs of functions and their inverses. In Figure 4.5, although S does not have an inverse that is a function, the graph of the inverse relation S^{-1} is symmetric to S with respect to the graph of $y = x$.

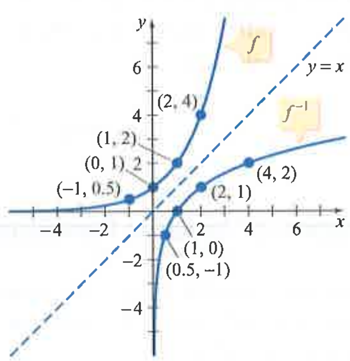


Figure 4.4

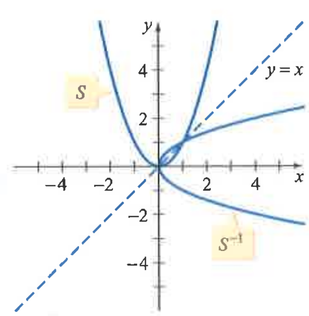


Figure 4.5

Answer • Because $(4, 5)$ is an ordered pair of f, $(5, 4)$ must be an ordered pair of f^{-1}. Therefore, $f^{-1}(5) = 4$.

Composition of a Function and Its Inverse

Observe the effect of forming the composition of $f(x) = 2x$ and $g(x) = \frac{1}{2}x$.

$$f(x) = 2x \qquad\qquad g(x) = \frac{1}{2}x$$

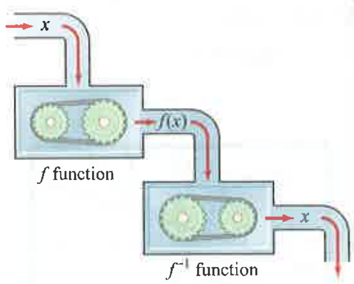
$$f[g(x)] = 2\left[\frac{1}{2}x\right] \quad \text{• Replace } x \text{ with } g(x). \qquad g[f(x)] = \frac{1}{2}[2x] \quad \text{• Replace } x \text{ with } f(x).$$

$$f[g(x)] = x \qquad\qquad g[f(x)] = x$$

This property of the composition of inverse functions always holds true. When taking the composition of inverse functions, the inverse function reverses the effect of the original function. For the two functions above, f doubles a number, and g halves a number. If you double a number and then take one-half of the result, you are back to the original number.

Study tip

If we think of a function as a machine, then the composition of inverse functions property can be represented as shown below. Take any input x for f. Use the output of f as the input for f^{-1}. The result is the original input, x.



Composition of Inverse Functions Property

If f is a one-to-one function, then f^{-1} is the inverse function of f if and only if

$$(f \circ f^{-1})(x) = f[f^{-1}(x)] = x \quad \text{for all } x \text{ in the domain of } f^{-1}$$

and

$$(f^{-1} \circ f)(x) = f^{-1}[f(x)] = x \quad \text{for all } x \text{ in the domain of } f.$$

EXAMPLE 2 Use the Composition of Inverse Functions Property

Use composition of functions to show that $f^{-1}(x) = 3x - 6$ is the inverse function of $f(x) = \frac{1}{3}x + 2$.

Solution

We must show that $f[f^{-1}(x)] = x$ and $f^{-1}[f(x)] = x$.

$$f(x) = \frac{1}{3}x + 2 \qquad\qquad f^{-1}(x) = 3x - 6$$

$$f[f^{-1}(x)] = \frac{1}{3}[3x - 6] + 2 \qquad\qquad f^{-1}[f(x)] = 3\left[\frac{1}{3}x + 2\right] - 6$$

$$f[f^{-1}(x)] = x \qquad\qquad f^{-1}[f(x)] = x$$

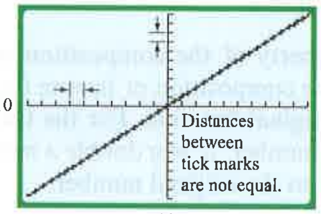
Try Exercise 24, page 345

Integrating Technology

In the standard viewing window of a calculator, the distance between two tick marks on the x-axis is not equal to the distance between two tick marks on the y-axis. As a result, the graph of $y = x$ does not appear to bisect the first and third quadrants. See Figure 4.6. This anomaly is important

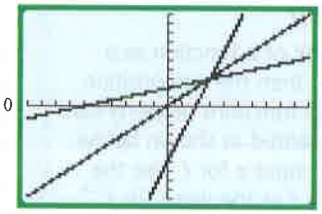
(continued)

if a graphing calculator is being used to check whether two functions are inverses of one another. Because the graph of $y = x$ does not appear to bisect the first and third quadrants, the graphs of f and f^{-1} will not appear to be symmetric about the graph of $y = x$. The graphs of $f(x) = \frac{1}{3}x + 2$ and $f^{-1}(x) = 3x - 6$ from Example 2 are shown in Figure 4.7. Notice that the graphs do not appear to be quite symmetric about the graph of $y = x$.



$y = x$ in the standard viewing window

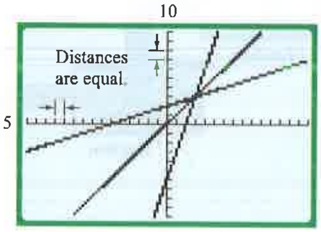
Figure 4.6



f, f^{-1}, and $y = x$ in the standard viewing window

Figure 4.7

To get a better view of a function and its inverse, it is necessary to use the SQUARE viewing window, as in Figure 4.8. In this window, the distance between two tick marks on the x-axis is equal to the distance between two tick marks on the y-axis.



f, f^{-1}, and $y = x$ in a square viewing window

Figure 4.8

Finding an Inverse Function

If a one-to-one function f is defined by an equation, then we can use the following method to find the equation for f^{-1}.

Study tip

If the ordered pairs of f are given by (x, y), then the ordered pairs of f^{-1} are given by (y, x). That is, x and y are interchanged. This is the reason for step 2 at the right.

Steps for Finding the Inverse of a Function

To find the equation of the inverse f^{-1} of the one-to-one function f, follow these steps.

1. Substitute y for $f(x)$.
2. Interchange x and y.
3. Solve, if possible, for y in terms of x.
4. Substitute $f^{-1}(x)$ for y.

EXAMPLE 3 Find the Inverse of a Function

Find the inverse of $f(x) = 3x + 8$.

Solution

$$\begin{aligned} f(x) &= 3x + 8 \\ y &= 3x + 8 && \text{• Replace } f(x) \text{ with } y. \\ x &= 3y + 8 && \text{• Interchange } x \text{ and } y. \\ x - 8 &= 3y && \text{• Solve for } y. \\ \frac{x-8}{3} &= y \\ \frac{1}{3}x - \frac{8}{3} &= f^{-1}(x) && \text{• Replace } y \text{ with } f^{-1}. \end{aligned}$$

The inverse function is given by $f^{-1}(x) = \frac{1}{3}x - \frac{8}{3}$.

▶ Try Exercise 36, page 345

In the next example, we find the inverse of a rational function.

EXAMPLE 4 Find the Inverse of a Function

Find the inverse of $f(x) = \frac{2x+1}{x}$, $x \neq 0$.

Solution

$$\begin{aligned} f(x) &= \frac{2x+1}{x} \\ y &= \frac{2x+1}{x} && \text{• Replace } f(x) \text{ with } y. \\ x &= \frac{2y+1}{y} && \text{• Interchange } x \text{ and } y. \\ xy &= 2y + 1 && \text{• Solve for } y. \\ xy - 2y &= 1 \\ y(x - 2) &= 1 && \text{• Factor the left side.} \\ y &= \frac{1}{x-2} \\ f^{-1}(x) &= \frac{1}{x-2}, \ x \neq 2 && \text{• Replace } y \text{ with } f^{-1}. \end{aligned}$$

▶ Try Exercise 42, page 345

The graph of $f(x) = x^2 + 4x + 3$ is shown in Figure 4.9a on the next page. The function f is not a one-to-one function and therefore does not have an inverse function. However, the function given by $G(x) = x^2 + 4x + 3$, shown in Figure 4.9b, for which the domain is restricted to $\{x \mid x \geq -2\}$, is a one-to-one function and has an inverse function G^{-1}. This is shown in Example 5.

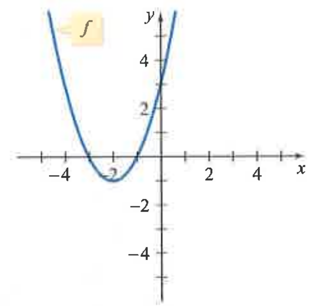


Figure 4.9a

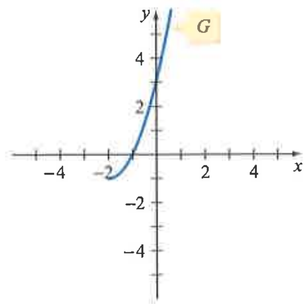


Figure 4.9b

EXAMPLE 5 Find the Inverse of a Function with a Restricted Domain

Find the inverse of $G(x) = x^2 + 4x + 3$, where the domain of G is $\{x \mid x \geq -2\}$.

> ***Recall***
>
> The range of a function f is the domain of f^{-1}, and the domain of f is the range of f^{-1}.

Solution

$$G(x) = x^2 + 4x + 3$$

$$y = x^2 + 4x + 3$$ • Replace $G(x)$ with y.

$$x = y^2 + 4y + 3$$ • Interchange x and y.

$$x = (y^2 + 4y + 4) - 4 + 3$$ • Solve for y by completing the square of $y^2 + 4y$.

$$x = (y + 2)^2 - 1$$ • Factor.

$$x + 1 = (y + 2)^2$$ • Add 1 to each side of the equation.

$$\sqrt{x + 1} = \sqrt{(y + 2)^2}$$ • Take the square root of each side of the equation.

$$\pm\sqrt{x + 1} = y + 2$$ • Recall that if $a^2 = b$, then $a = \pm\sqrt{b}$.

$$\pm\sqrt{x + 1} - 2 = y$$

Because the domain of G is $\{x \mid x \geq -2\}$, the range of G^{-1} is $\{y \mid y \geq -2\}$. This means that we must choose the positive value of $\pm\sqrt{x + 1}$. Thus $G^{-1}(x) = \sqrt{x + 1} - 2$. See Figure 4.10.

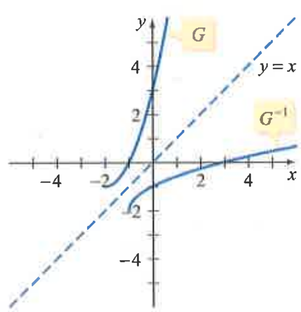


Figure 4.10

▶ **Try Exercise 48, page 345**

In Example 6, we use an inverse function to determine the wholesale price of a gold bracelet for which we know the retail price.

EXAMPLE 6 Solve an Application

A merchant uses the function

$$S(x) = \frac{4}{3}x + 100$$

to determine the retail selling price S, in dollars, of a gold bracelet for which she has paid a wholesale price of x dollars.

a. The merchant paid a wholesale price of $672 for a gold bracelet. Use S to determine the retail selling price of this bracelet.

b. Find S^{-1} and use it to determine the merchant's wholesale price for a gold bracelet that retails at $1596.

Solution

a. $S(672) = \frac{4}{3}(672) + 100 = 896 + 100 = 996$

The merchant charges $996 for a bracelet that has a wholesale price of $672.

b. To find S^{-1}, begin by substituting y for $S(x)$.

$$S(x) = \frac{4}{3}x + 100$$

$$y = \frac{4}{3}x + 100 \qquad \text{• Replace } S(x) \text{ with } y.$$

$$x = \frac{4}{3}y + 100 \qquad \text{• Interchange } x \text{ and } y.$$

$$x - 100 = \frac{4}{3}y \qquad \text{• Solve for } y.$$

$$\frac{3}{4}(x - 100) = y$$

$$\frac{3}{4}x - 75 = y$$

Use inverse notation to write the above equation as

$$S^{-1}(x) = \frac{3}{4}x - 75$$

Substitute 1596 for x to determine the wholesale price.

$$S^{-1}(1596) = \frac{3}{4}(1596) - 75$$

$$= 1197 - 75$$

$$= 1122$$

A gold bracelet that the merchant retails at $1596 has a wholesale price of $1122.

▶ Try Exercise 58, page 346

Integrating Technology

Some graphing utilities can be used to draw the graph of the inverse of a function without the user having to find the inverse function. For instance, Figure 4.11 shows the graph of $f(x) = 0.1x^3 - 4$. The graphs of f and f^{-1} are both shown in Figure 4.12, along with the graph of $y = x$. Note that the graph of f^{-1} is the reflection of the graph of f with respect to the graph of $y = x$. The display shown in Figure 4.12 was produced on a TI-83/TI-83 Plus/TI-84 Plus graphing calculator by using the DrawInv command, which is in the DRAW menu.

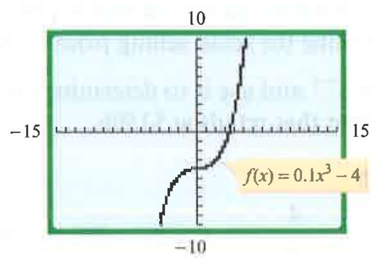


Figure 4.11

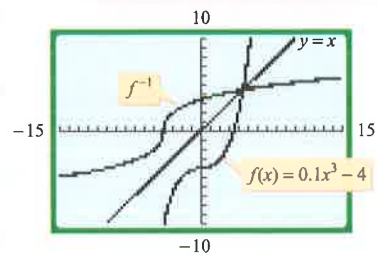


Figure 4.12

EXERCISE SET 4.1

Concept Check

1. What is a one-to-one function?

2. What is the horizontal line test?

3. How is the graph of the inverse of a function related to the graph of the function?

4. What are the steps for finding the inverse of a one-to-one function that is defined by an equation?

In Exercises 5 to 8, assume that the given function has an inverse function.

5. Given $f(3) = 7$, find $f^{-1}(7)$.

6. Given $g(-3) = 5$, find $g^{-1}(5)$.

7. Given $h^{-1}(-3) = -4$, find $h(-4)$.

8. Given $f^{-1}(7) = 0$, find $f(0)$.

9. If 3 is in the domain of f^{-1}, find $f[f^{-1}(3)]$.

10. If f is a one-to-one function and $f(0) = 5$, $f(1) = 2$, and $f(2) = 7$, find the following.

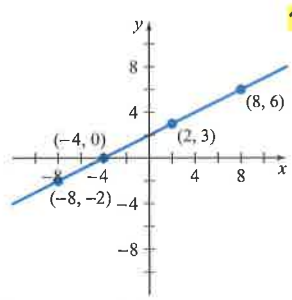
a. $f^{-1}(5)$ **b.** $f^{-1}(2)$

11. The domain of the inverse function f^{-1} is the ________ of f.

12. The range of the inverse function f^{-1} is the ________ of f.

In Exercises 13 to 20, draw the graph of the inverse relation. Is the inverse relation a function?

13.



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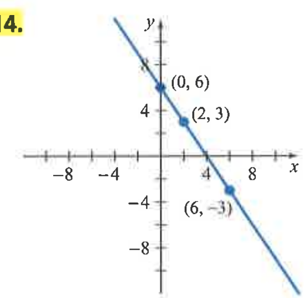

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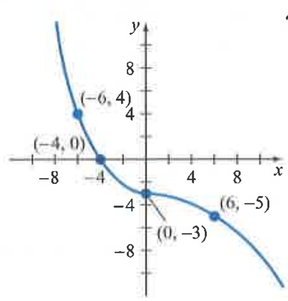

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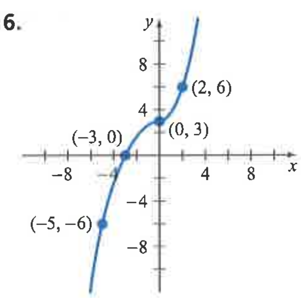
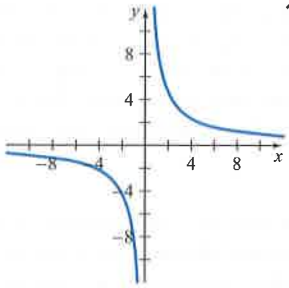


Indicates Try It Exercises

17.



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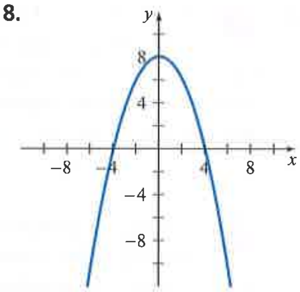

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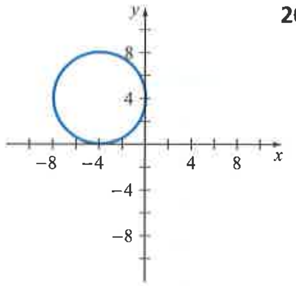

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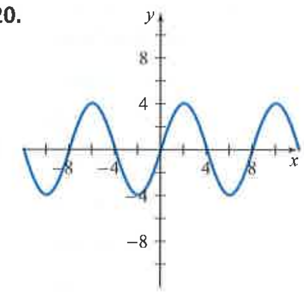


In Exercises 21 to 30, use composition of functions to determine whether *f* and *g* are inverses of one another.

21. $f(x) = 2x + 7;\ f^{-1}(x) = \frac{1}{2}x - \frac{7}{2}$

22. $f(x) = \frac{3}{4}x - 5;\ f^{-1}(x) = \frac{4}{3}x + \frac{20}{3}$

23. $f(x) = 4x - 1;\ g(x) = \frac{1}{4}x + \frac{1}{4}$

24. $f(x) = \frac{1}{2}x - \frac{3}{2};\ g(x) = 2x + 3$

25. $f(x) = -\frac{1}{2}x - \frac{1}{2};\ g(x) = -2x + 1$

26. $f(x) = 3x + 2;\ g(x) = \frac{1}{3}x - \frac{2}{3}$

27. $f(x) = \frac{5}{x-3};\ g(x) = \frac{5}{x} + 3$

28. $f(x) = \frac{2x}{x-1};\ g(x) = \frac{x}{x-2}$

29. $f(x) = x^3 + 2;\ g(x) = \sqrt[3]{x-2}$

30. $f(x) = (x+5)^3;\ g(x) = \sqrt[3]{x} - 5$

In Exercises 31 to 34, find the inverse of the function. If the function does not have an inverse function, write "no inverse function."

31. $\{(-3,1), (-2,2), (1,5), (4,-7)\}$

32. $\{(-5,4), (-2,3), (0,1), (3,2), (7,11)\}$

33. $\{(0,1), (1,2), (2,4), (3,8), (4,16)\}$

34. $\{(1,0), (10,1), (100,2), (1000,3), (10{,}000,4)\}$

In Exercises 35 to 52, find $f^{-1}(x)$. State any restrictions on the domain of $f^{-1}(x)$.

35. $f(x) = 2x + 4$

36. $f(x) = 4x - 8$

37. $f(x) = 3x - 7$

38. $f(x) = -3x - 8$

39. $f(x) = -3x + 11$

40. $f(x) = -\frac{1}{2}x - \frac{3}{4}$

41. $f(x) = \frac{2x}{x-1},\quad x \neq 1$

42. $f(x) = \frac{x}{x-2},\quad x \neq 2$

43. $f(x) = \frac{x-1}{x+1},\quad x \neq -1$

44. $f(x) = \frac{2x-1}{x+3},\quad x \neq -3$

45. $f(x) = x^2 + 1,\quad x \geq 0$

46. $f(x) = x^2 - 4,\quad x \geq 0$

47. $f(x) = \sqrt{x-2},\quad x \geq 2$

48. $f(x) = \sqrt{4-x},\quad x \leq 4$

49. $f(x) = x^2 + 4x,\quad x \geq -2$

50. $f(x) = x^2 - 6x,\quad x \leq 3$

51. $f(x) = x^2 + 4x - 1,\quad x \leq -2$

52. $f(x) = x^2 - 6x + 1,\quad x \geq 3$

53. **Geometry** The volume of a cube is given by $V(x) = x^3$, where x is the measure of the length of a side of the cube. Find $V^{-1}(x)$ and explain what it represents.

54. **Unit Conversion** The function $f(x) = 12x$ converts feet, x, into inches. Find $f^{-1}(x)$ and explain what it represents.

55. **Fahrenheit to Celsius** The function

$$f(x) = \frac{5}{9}(x - 32)$$

is used to convert x degrees Fahrenheit to an equivalent Celsius temperature. Find f^{-1} and explain how it is used.

56. **Retail Sales** A clothing merchant uses the function

$$S(x) = \frac{3}{2}x + 18$$

to determine the retail selling price S, in dollars, of a winter coat for which she has paid a wholesale price of x dollars.

a. The merchant paid a wholesale price of \$96 for a winter coat. Use S to determine the retail selling price she will charge for this coat.

b. Find S^{-1} and use it to determine the merchant's wholesale price for a coat that retails at \$399.

57. **Fashion** The function $s(x) = 2x + 24$ can be used to convert a U.S. women's shoe size into an Italian women's shoe size. Determine the function $s^{-1}(x)$ that can be used to convert an Italian women's shoe size to its equivalent U.S. shoe size.

58. **Fashion** The function $K(x) = 1.3x - 4.7$ converts a men's shoe size in the United States to the equivalent shoe size in the United Kingdom. Determine the function $K^{-1}(x)$ that can be used to convert a U.K. men's shoe size to its equivalent U.S. shoe size.

59. **Catering** A catering service uses the function

$$c(x) = \frac{300 + 12x}{x}$$

to determine the amount, in dollars, it charges per person for a sit-down dinner, where x is the number of people in attendance.

a. Find $c(30)$ and explain what it represents.

b. Find c^{-1}.

c. Use c^{-1} to determine how many people attended a dinner for which the cost per person was \$15.00.

60. **Landscaping** A landscaping company uses the function

$$c(x) = \frac{600 + 140x}{x}$$

to determine the amount, in dollars, it charges per tree to deliver and plant x palm trees.

a. Find $c(5)$ and explain what it represents.

b. Find c^{-1}.

c. Use c^{-1} to determine how many palm trees were delivered and planted if the cost per tree was \$160.

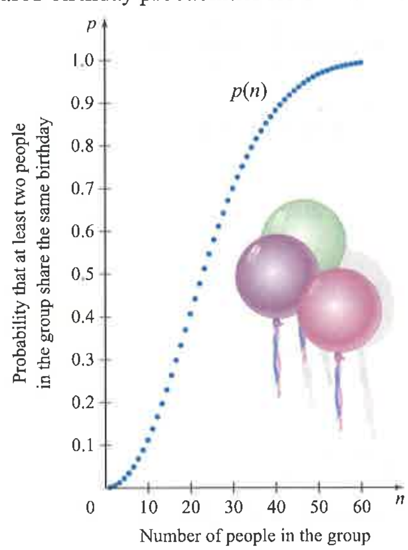
61. **Compensation** The monthly earnings $E(s)$, in dollars, of a software sales executive are given by the function $E(s) = 0.05s + 2500$, where s is the value, in dollars, of the software sold by the executive during the month. Find $E^{-1}(s)$ and explain how the executive could use this function.

62. **Grading** A professor uses the function defined by the following table to determine the grade a student receives on a test. Does this grading function have an inverse function? Explain your answer.

Grading Scale

Score	Grade
90–100	A
80–89	B
70–79	C
60–69	D
0–59	F

63. **The Birthday Problem** A famous problem called the *birthday problem* goes like this: Suppose there is a randomly selected group of n people in a room. What is the probability that at least two of the people have a birthday on the same day of the year? It may surprise you that for a group of 23 people, the probability that at least two of the people share a birthday is about 50.7%. The following graph can be used to estimate shared birthday probabilities for $1 \le n \le 60$.

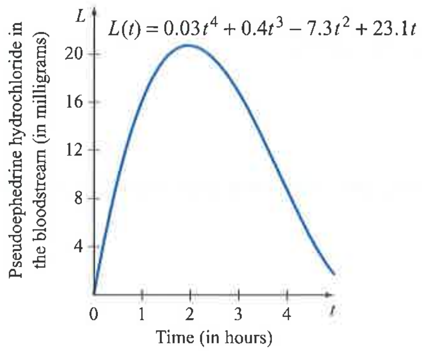


a. Use the graph of p to estimate $p(10)$ and $p(30)$.

b. Consider the function p with $1 \le n \le 60$, as shown in the graph. Explain how you can tell that p has an inverse that is a function.

c. Write a sentence that explains the meaning of $p^{-1}(0.223)$ in the context of this application.

64. **Medication Level** The function L shown in the following graph models the level of pseudoephedrine hydrochloride, in milligrams, in the bloodstream of a patient t hours after 30 milligrams of the medication have been administered.



a. Use the graph of L to estimate two different values of t for which the pseudoephedrine hydrochloride levels are the same.

b. Does L have an inverse that is a function? Explain.

65. **Cryptology** Cryptology is the study of making and breaking secret codes. Secret codes are often used to send messages over the Internet. By devising a code that is difficult to break, the sender hopes to prevent the messages from being read by an unauthorized person.

In practice, complicated one-to-one functions and their inverses are used to encode and decode messages. The following procedure uses the simple function $f(x) = 2x - 1$ to illustrate the basic concepts that are involved.

Assign to each letter of the alphabet, and a blank space, a two-digit numerical value, as shown below.

A	10	H	17	O	24	V	31
B	11	I	18	P	25	W	32
C	12	J	19	Q	26	X	33
D	13	K	20	R	27	Y	34
E	14	L	21	S	28	Z	35
F	15	M	22	T	29		36
G	16	N	23	U	30		

Note: A blank space is represented by the numerical value 36.

Using these numerical values, the message MEET YOU AT NOON would be represented by

22 14 14 29 36 34 24 30 36 10 29 36 23 24 24 23

Let $f(x) = 2x - 1$ define a coding function. The above message can be encoded by finding $f(22)$, $f(14)$, $f(14)$, $f(29)$, $f(36)$, $f(34)$, $f(24)$, . . . , $f(23)$, which yields

43 27 27 57 71 67 47 59 71 19 57 71 45 47 47 45

The inverse of f, which is

$$f^{-1}(x) = \frac{x+1}{2}$$

is used by the receiver of the message to decode the message. For instance,

$$f^{-1}(43) = \frac{43+1}{2} = 22$$

which represents M, and

$$f^{-1}(27) = \frac{27+1}{2} = 14$$

which represents E.

a. Use the preceding coding procedure to encode the message DO YOUR HOMEWORK.

b. Use $f^{-1}(x)$ to decode the message

49 33 47 45 27 71 33 47 43 27

c. Explain why it is important to use a one-to-one function to encode a message.

66. **Cryptography** A friend is using the letter–number correspondence in Exercise 65 and the coding function $g(x) = 2x + 3$. Your friend sends you the coded message

59 31 39 73 31 75 61 37 31 75 29 23 71

Use $g^{-1}(x)$ to decode this message.

In Exercises 67 to 70, answer the question without finding the equation of the linear function.

67. Suppose that f is a linear function, $f(2) = 7$, and $f(5) = 12$. If $f(4) = c$, then is c less than 7, between 7 and 12, or greater than 12? Explain your answer.

68. Suppose that f is a linear function, $f(1) = 13$, and $f(4) = 9$. If $f(3) = c$, then is c less than 9, between 9 and 13, or greater than 13? Explain your answer.

69. Suppose that f is a linear function, $f(2) = 3$, and $f(5) = 9$. Between which two numbers is $f^{-1}(6)$?

70. Suppose that f is a linear function, $f(5) = -1$, and $f(9) = -3$. Between which two numbers is $f^{-1}(-2)$?

Only one-to-one functions have inverses that are functions. In Exercises 71 to 78, determine whether the given function is a one-to-one function.

71. $f(x) = x^2 + 1$

72. $v(t) = \sqrt{16 + t}$

73. $F(x) = |x| + x$

74. $T(x) = |x^2 - 6|, \quad x \geq 0$

75. $g(x) = x^3 - 2x$

76. $k(x) = \sqrt{x}$

77. $j(x) = x^3$

78. $n(x) = \dfrac{1}{x}$

79. Use a graph of $f(x) = -x + 3$ to explain why f is its own inverse.

80. Use a graph of $f(x) = \sqrt{16 - x^2}$, with $0 \leq x \leq 4$, to explain why f is its own inverse.

Enrichment Exercises

81. Consider the linear function $f(x) = mx + b$, $m \neq 0$. The graph of f has a slope of m and a y-intercept of $(0, b)$. What are the slope and y-intercept of the graph of f^{-1}?

82. Find the inverse of $f(x) = ax^2 + bx + c$, $a \neq 0$, $x \geq -\dfrac{b}{2a}$.

SECTION 4.2

Exponential Functions
Graphs of Exponential Functions
Natural Exponential Function

Exponential Functions and Their Applications

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A24.

PS1. Evaluate: 2^3 [P.2]

PS2. Evaluate: 3^{-4} [P.2]

PS3. Evaluate: $\dfrac{2^2 + 2^{-2}}{2}$ [P.2/P.5]

PS4. Evaluate: $\dfrac{3^2 - 3^{-2}}{2}$ [P.2/P.5]

PS5. Evaluate $f(x) = 10^x$ for $x = -1, 0, 1,$ and 2. [P.2]

PS6. Evaluate $f(x) = \left(\dfrac{1}{2}\right)^x$ for $x = -1, 0, 1,$ and 2. [P.2]

Exponential Functions

When a parking facility opened in 1980, it charged $1 for all-day parking. Since then, it has doubled its daily parking fee every 8 years as shown in Table 4.1.

Table 4.1

Year	1980	1988	1996	2004	2012
Daily parking fee	$1	$2	$4	$8	$16

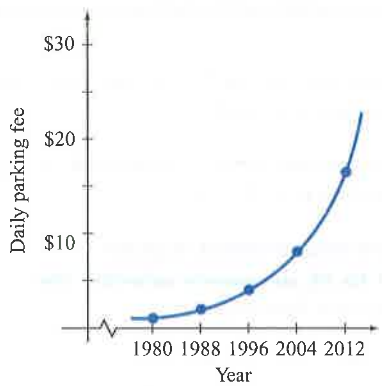


Figure 4.13

In Figure 4.13, we have plotted the data in Table 4.1 and modeled the upward trend in the parking fee by a smooth curve. This model is based on an *exponential function*, which is one of the major topics of this chapter.

The effectiveness of a drug that is used for sedation during a surgical procedure depends on the concentration of the drug in the patient. Through natural body chemistry, the amount of this drug in the body decreases over time. The graph in Figure 4.14 models this decrease. This model is another example of an exponential model.

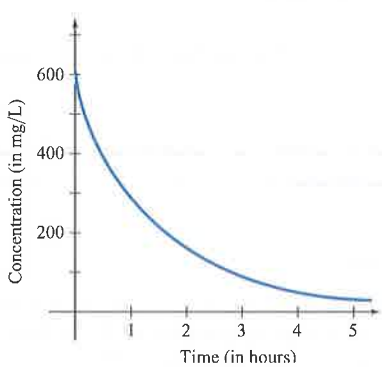


Figure 4.14

Definition of an Exponential Function

The **exponential function with base b** is defined by

$$f(x) = b^x$$

where $b > 0$, $b \neq 1$, and x is a real number.

The base b of $f(x) = b^x$ is required to be positive. If the base were a negative number, the value of the function would be a complex number for some values of x. For