4.1 Graphing Polynomial Functions

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Learning Target Graph and describe polynomial functions.

Success Criteria
- I can identify and evaluate polynomial functions.
- I can graph polynomial functions.
- I can describe end behavior of polynomial functions.

EXPLORE IT! Graphing Polynomial Functions

$f(x) = -x^2 - 1$	$f(x) = \frac{1}{x}$
$f(x) = 2^x$	$f(x) = -4x^3$
$f(x) = x^3 + 1$	$f(x) = \frac{1}{2}x^2 + x$
$f(x) = -\frac{1}{4}x^4 - x^3$	$f(x) = x^3 + x^2$
$f(x) = \sqrt{x}$	$f(x) = 2x^4 - x$

MP CHOOSE TOOLS **Work with a partner.**

a. Identify each function in the list at the left in which $f(x)$ is a polynomial. Graph each function you identified. For each function,

- describe the end behavior.
- identify the term with the greatest exponent. How does the exponent affect the graph? How does the coefficient of this term affect the graph?

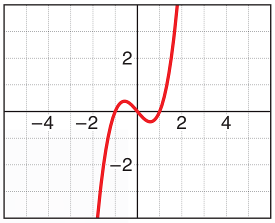
b. Graph $y = x^3$ and $y = x^4$. Compare the graphs. One of these graphs is *cubic* and the other is *quartic*. Which do you think is which? Explain.

c. Identify each function as cubic or quartic. Then match each function with its graph. Explain your reasoning.

i. $f(x) = x^3 - x$ **ii.** $f(x) = -x^3 + x$

iii. $f(x) = -x^4 + 1$ **iv.** $f(x) = x^4 - x^2$

A.



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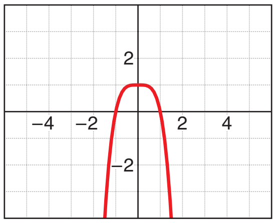

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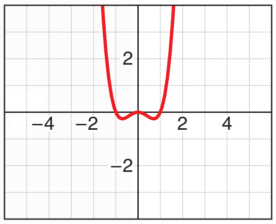
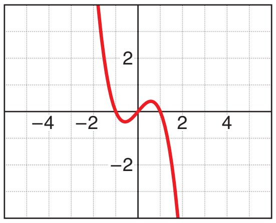


D.



d. What are some characteristics of the graphs of cubic polynomial functions? quartic polynomial functions?

Math Practice

Find Entry Points
How can rewriting the functions help you match the functions with their graphs?

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Polynomial Functions

Vocabulary (AZ VOCAB)

polynomial, *p. 154*
polynomial function, *p. 154*
end behavior, *p. 155*

A monomial is a number, a variable, or the product of a number and one or more variables with whole number exponents. A **polynomial** is a monomial or a sum of monomials. A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers. For this function, a_n is the leading coefficient, n is the degree, and a_0 is the constant term. A polynomial function is in *standard form* when its terms are written in descending order of exponents from left to right.

You are already familiar with some types of polynomial functions, such as linear and quadratic. Here is a summary of common types of polynomial functions.

Common Polynomial Functions

Degree	Type	Standard Form	Example
0	Constant	$f(x) = a_0$	$f(x) = -14$
1	Linear	$f(x) = a_1x + a_0$	$f(x) = 5x - 7$
2	Quadratic	$f(x) = a_2x^2 + a_1x + a_0$	$f(x) = 2x^2 + x - 9$
3	Cubic	$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$	$f(x) = x^3 - x^2 + 3x$
4	Quartic	$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$	$f(x) = x^4 + 2x - 1$

EXAMPLE 1 Identifying Polynomial Functions

WATCH

Determine whether each function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

a. $f(x) = -2x^3 + 5x + 8$

b. $g(x) = -0.8x^3 + \sqrt{2}x^4 - 12$

c. $h(x) = -x^2 + 7x^{-1} + 4x$

d. $k(x) = x^2 + 3^x$

SOLUTION

a. The function is a polynomial function that is already written in standard form. It has degree 3 (cubic) and leading coefficient -2.

b. The function is a polynomial function written as $g(x) = \sqrt{2}x^4 - 0.8x^3 - 12$ in standard form. It has degree 4 (quartic) and leading coefficient $\sqrt{2}$.

c. The function is not a polynomial function because the term $7x^{-1}$ has an exponent that is not a whole number.

d. The function is not a polynomial function because the term 3^x does not have a variable base and an exponent that is a whole number.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Determine whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

1. $f(x) = 7 - 1.6x^2 - 5x$

2. $p(x) = x + 2x^{-2} + 9.5$

3. $q(x) = x^3 - 6x + 3x^4$

4. WHICH ONE DOESN'T BELONG? Which function does *not* belong with the other three? Explain your reasoning.

$f(x) = 7x^5 + 3x^2 - 2x$

$g(x) = 3x^3 - 2x^8 + \frac{3}{4}$

$h(x) = -3x^4 + 5x^{-1} - 3x^2$

$k(x) = \sqrt{3}x + 8x^4$

EXAMPLE 2 Evaluating a Polynomial Function

WATCH

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Evaluate $f(x) = 2x^4 - 8x^2 + 5x - 7$ when $x = 3$.

SOLUTION

$f(x) = 2x^4 - 8x^2 + 5x - 7$ Write the function.

$f(3) = 2(3)^4 - 8(3)^2 + 5(3) - 7$ Substitute 3 for x.

$= 162 - 72 + 15 - 7$ Evaluate powers and multiply.

$= 98$ Simplify.

The **end behavior** of a function is the behavior of the graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$). For a polynomial function, the end behavior is determined by the function's degree and the sign of its leading coefficient.

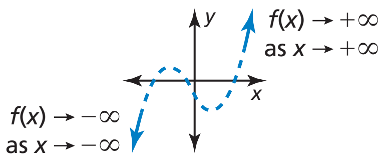
REMEMBER

The expression "$x \to +\infty$" is read as "x approaches positive infinity."

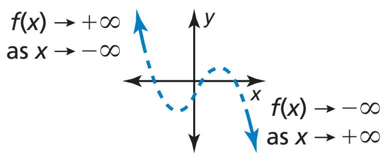
KEY IDEA

End Behavior of Polynomial Functions

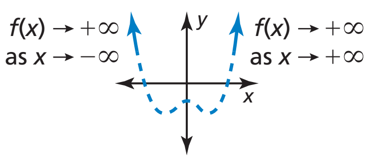
Degree: odd
Leading coefficient: positive



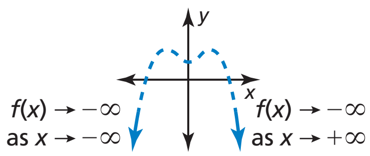
Degree: odd
Leading coefficient: negative



Degree: even
Leading coefficient: positive



Degree: even
Leading coefficient: negative



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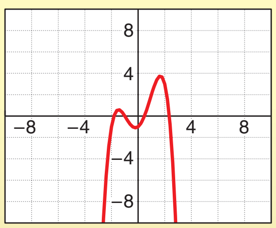

EXAMPLE 3 Describing End Behavior

WATCH

Describe the end behavior of $f(x) = -0.5x^4 + 2.5x^2 + x - 1$.

SOLUTION

The function has degree 4 and leading coefficient -0.5. Because the degree is even and the leading coefficient is negative, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Evaluate the function for the given value of x.

5. $f(x) = -x^3 + 3x^2 + 9; x = 4$

6. $f(x) = 3x^5 - x^4 - 6x + 10; x = -2$

7. **WRITING** Explain what is meant by the end behavior of a polynomial function.

8. Describe the end behavior of $f(x) = 0.25x^3 - x^2 - 1$.

Graphing Polynomial Functions

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To graph a polynomial function, first plot points to determine the shape of the graph's middle portion. Then connect the points with a smooth continuous curve and use what you know about end behavior to sketch the graph.

EXAMPLE 4 Graphing Polynomial Functions

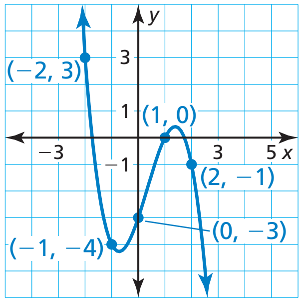
WATCH

Graph (a) $f(x) = -x^3 + x^2 + 3x - 3$ and (b) $f(x) = x^4 - x^3 - 4x^2 + 4$.

SOLUTION

a. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

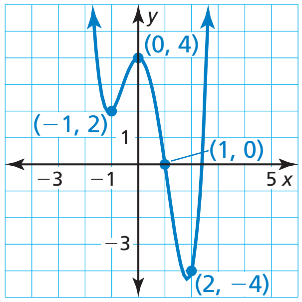
x	−2	−1	0	1	2
$f(x)$	3	−4	−3	0	−1



The degree is odd and the leading coefficient is negative. So, $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$.

b. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

x	−2	−1	0	1	2
$f(x)$	12	2	4	0	−4



The degree is even and the leading coefficient is positive. So, $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

EXAMPLE 5 Sketching a Graph

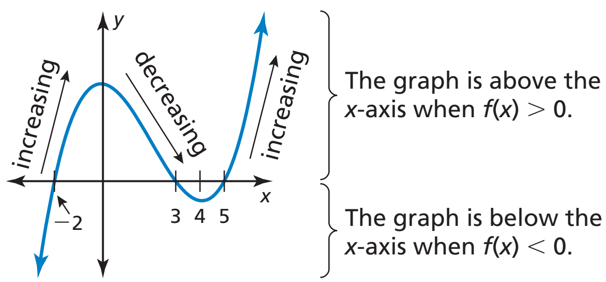
WATCH

Sketch a graph of the polynomial function f with the following characteristics.

- f is increasing when $x < 0$ and $x > 4$.
- f is decreasing when $0 < x < 4$.
- $f(x) > 0$ when $-2 < x < 3$ and $x > 5$.
- $f(x) < 0$ when $x < -2$ and $3 < x < 5$.

Use the graph to describe the degree and leading coefficient of f.

SOLUTION



▶ From the graph, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$. So, the degree is odd and the leading coefficient is positive.

EXAMPLE 6 Modeling Real Life

WATCH

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The African wild dog is one of the most endangered carnivores on Earth. The estimated population of African wild dogs under human care can be modeled by the polynomial function

$$P(t) = 0.368t^3 - 11.45t^2 + 109.5t + 286$$

where t represents the number of years after 2000.

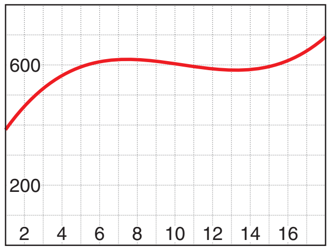
a. Use technology to graph the function for $1 \leq t \leq 18$. Describe the behavior of the graph on this interval.

b. What is the average rate of change in the number of dogs from 2001 to 2018?

c. Do you think this model can be used for years after 2018? Explain your reasoning.

SOLUTION

a. Using technology and the domain $1 \leq t \leq 18$, you obtain the graph shown.



▶ The number of dogs increases from 2001 to 2007, decreases slightly from 2007 to 2013, and increases from 2013 to 2018.

b. The years 2001 and 2018 correspond to $t = 1$ and $t = 18$.

Average rate of change over $1 \leq t \leq 18$:

$$\frac{P(18) - P(1)}{18 - 1} = \frac{693.376 - 384.418}{17} = 18.174$$

▶ The average rate of change from 2001 to 2018 is about 18 dogs per year.

c. Because the degree is odd and the leading coefficient is positive, $P(t) \to -\infty$ as $t \to -\infty$ and $P(t) \to +\infty$ as $t \to +\infty$. The end behavior indicates that the model has unlimited growth as t increases. While the model may be valid for a few years after 2018, over time, unlimited growth is not reasonable.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Graph the polynomial function.

9. $f(x) = x^4 + x^2 - 3$

10. $f(x) = 4 - x^3$

11. $f(x) = x^3 - x^2 + x - 1$

12. Sketch a graph of the polynomial function f with the following characteristics.

- f is decreasing when $x < -1.5$ and $x > 2.5$; f is increasing when $-1.5 < x < 2.5$.
- $f(x) > 0$ when $x < -3$ and $1 < x < 4$; $f(x) < 0$ when $-3 < x < 1$ and $x > 4$.

Use the graph to describe the degree and leading coefficient of f.

13. **WHAT IF?** Repeat Example 6 using the following model for the African wild dog population.

$$P(t) = 0.0012t^4 + 0.321t^3 - 10.86t^2 + 106.8t + 289$$

4.1 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–6, determine whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient. *Example 1*

1. $f(x) = -3x + 5x^3 - 6x^2 + 2$

2. $p(x) = \frac{1}{2}x^2 + 3x - 4x^3 + 6x^4 - 1$

3. $f(x) = 9x^4 + 8x^3 - 6x^{-2} + 2x$

4. $g(x) = \sqrt{3} - 12x + 13x^2$

5. $h(x) = \frac{5}{3}x^2 - \sqrt{7}x^4 + 8x^3 - \frac{1}{2} + x$

6. $h(x) = 3x^4 + 2x - \frac{5}{x} + 9x^3 - 7$

ERROR ANALYSIS In Exercises 7 and 8, describe and correct the error in analyzing the function.

7. $f(x) = 8x^3 - 7x^4 - 9x - 3x^2 + 11$

> ✗ f is a polynomial function.
> The degree is 3 and f is a cubic function.
> The leading coefficient is 8.

8. $f(x) = 2x^4 + 4x - 9\sqrt{x} + 3x^2 - 8$

> ✗ f is a polynomial function.
> The degree is 4 and f is a quartic function.
> The leading coefficient is 2.

In Exercises 9–16, evaluate the function for the given value of *x*. *Example 2*

9. $f(x) = 2x^3 - 5x^2 + 16;\ x = -4$

10. $p(x) = -x^5 + 11x^3 + 7;\ x = 3$

11. $h(x) = -3x^4 + 2x^3 - 12x - 6;\ x = -2$

12. $f(x) = 7x^4 - 10x^2 + 14x - 26;\ x = -7$

13. $g(x) = x^6 - 64x^4 + x^2 - 7x - 51;\ x = 8$

14. $g(x) = -x^3 + 3x^2 + 5x + 1;\ x = -12$

15. $p(x) = 2x^3 + 4x^2 + 6x + 7;\ x = \frac{1}{2}$

16. $h(x) = 5x^3 - 3x^2 + 2x + 4;\ x = -\frac{1}{3}$

17. WRITING Let $f(x) = 13$. State the degree, type, and leading coefficient. Describe the end behavior of the function. Explain your reasoning.

18. MODELING REAL LIFE The weight of an ideal round-cut diamond can be modeled by

$$w = 0.00583d^3 - 0.0125d^2 + 0.022d - 0.01$$

where w is the weight of the diamond (in carats) and d is the diameter (in millimeters). According to the model, what is the weight of a diamond with a diameter of 12 millimeters?

In Exercises 19–22, describe the end behavior of the function. *Example 3*

19. $h(x) = -5x^4 + 7x^3 - 6x^2 + 9x + 2$

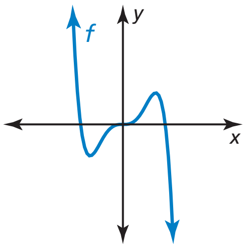
20. $g(x) = 7x^7 + 12x^5 - 6x^3 - 2x - 18$

21. $f(x) = -2x^4 + 12x^8 + 17 + 15x^2$

22. $f(x) = 11 - 18x^2 - 5x^5 - 12x^4 - 2x$

In Exercises 23 and 24, use the graph to describe the degree and leading coefficient of *f*.

23.



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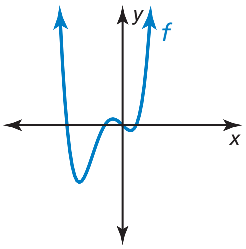


In Exercises 25–32, graph the polynomial function. *Example 4*

25. $p(x) = 3 - x^4$

26. $g(x) = x^3 + x + 3$

27. $f(x) = 4x - 9 - x^3$

28. $p(x) = x^5 - 3x^3 + 2$

29. $h(x) = x^4 - 2x^3 + 3x$

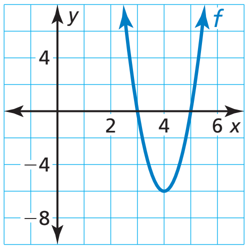
30. $h(x) = 5 + 3x^2 - x^4$

31. $g(x) = x^5 - 3x^4 + 2x - 4$

32. $p(x) = x^6 - 2x^5 - 2x^3 + x + 5$

ANALYZING RELATIONSHIPS **In Exercises 33 and 34, describe the *x*-values for which (a) *f* is increasing, (b) *f* is decreasing, (c) $f(x) > 0$, and (d) $f(x) < 0$.**

33.



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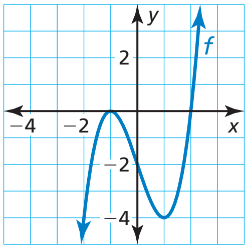


In Exercises 35–38, sketch a graph of the polynomial function *f* with the given characteristics. Use the graph to describe the degree and leading coefficient of the function *f*. *Example 5*

35.
- f is increasing when $x > 0.5$; f is decreasing when $x < 0.5$.
- $f(x) > 0$ when $x < -2$ and $x > 3$; $f(x) < 0$ when $-2 < x < 3$.

36.
- f is increasing when $-2 < x < 3$; f is decreasing when $x < -2$ and $x > 3$.
- $f(x) > 0$ when $x < -4$ and $1 < x < 5$; $f(x) < 0$ when $-4 < x < 1$ and $x > 5$.

37.
- f is increasing when $-2 < x < 0$ and $x > 2$; f is decreasing when $x < -2$ and $0 < x < 2$.
- $f(x) > 0$ when $x < -3$, $-1 < x < 1$, and $x > 3$; $f(x) < 0$ when $-3 < x < -1$ and $1 < x < 3$.

38.
- f is increasing when $x < -1$ and $x > 1$; f is decreasing when $-1 < x < 1$.
- $f(x) > 0$ when $-1.5 < x < 0$ and $x > 1.5$; $f(x) < 0$ when $x < -1.5$ and $0 < x < 1.5$.

39. **MODELING REAL LIFE** The estimated population of Sumatran tigers can be modeled by the function

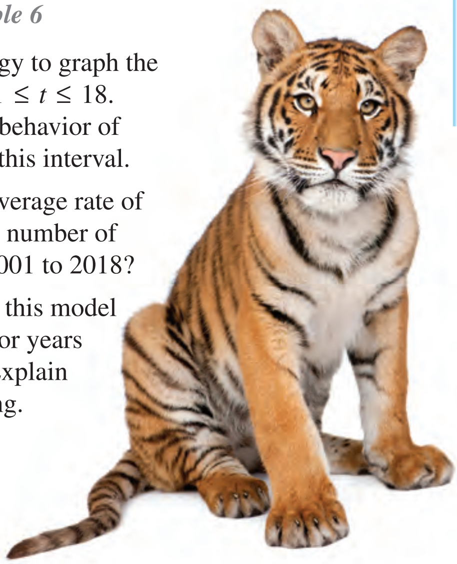
$$P(t) = -0.077t^3 + 2.11t^2 - 7.1t + 166$$

where t is the number of years after 2000. *Example 6*

a. Use technology to graph the function for $1 \le t \le 18$. Describe the behavior of the graph on this interval.

b. What is the average rate of change in the number of tigers from 2001 to 2018?

c. Do you think this model can be used for years after 2018? Explain your reasoning.



40. **MODELING REAL LIFE** The number of drive-in movie theaters in the United States from 1995 to 2018 can be modeled by the function

$$d(t) = -0.086t^3 + 3.71t^2 - 53.7t + 643$$

where t is the number of years after 1995.

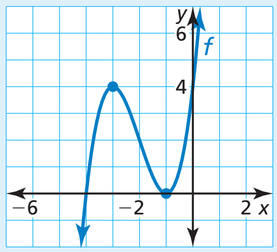
a. Use technology to graph the function for $1 \le t \le 23$. Describe the behavior of the graph on this interval.

b. Find and interpret the average rates of change in the number of drive-in movie theaters from 1996 to 2006 and from 2006 to 2018.

c. Do you think this model can be used for years before 1995 or after 2018? Explain.

41. MP **USING TOOLS** Your friend uses technology to graph $f(x) = (x - 1)(x - 2)(x + 12)$ in the viewing window $-10 \le x \le 10$, $-10 \le y \le 10$, and says the graph is a parabola. Is your friend correct? Explain.

42. **HOW DO YOU SEE IT?**
The graph of a polynomial function is shown.



a. State the degree and leading coefficient of f.

b. Describe the intervals for which the function is increasing and decreasing.

c. What is the constant term of the polynomial function? Explain.

43. **ABSTRACT REASONING** The end behavior of a polynomial function f is described by $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. Describe the end behavior of $g(x) = -f(x)$. Justify your answer.

44. **THOUGHT PROVOKING**
The end behavior of a polynomial function f is described by $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$. Is it possible that f has no zeros? one zero? more than one zero? Explain your reasoning.

45. **ANALYZING RELATIONSHIPS** Use technology to graph $f(x) = x^2$, $g(x) = x^4$, and $h(x) = x^6$ in the same coordinate plane.

a. What do you notice about the functions and their points of intersection? Does the pattern continue for greater, even powers of x? Explain.

b. Is there a similar pattern for functions with odd powers of x? Explain.

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46. **DRAWING CONCLUSIONS** The graph of a function is symmetric about the y-axis if for each point (a, b) on the graph, $(-a, b)$ is also a point on the graph. It is symmetric about the origin if for each point (a, b) on the graph, $(-a, -b)$ is also a point on the graph.

a. Use technology to graph the function $y = x^n$ when $n = 1, 2, 3, 4, 5,$ and 6. In each case, identify the symmetry of the graph.

b. Predict the symmetry of the graphs of $y = x^{10}$ and $y = x^{11}$. Explain your reasoning. Then confirm your predictions by graphing the functions.

47. **MAKING AN ARGUMENT** Can you use the table to determine whether the polynomial function f has an even degree or an odd degree? Explain.

x	−5	−1	0	1	5
$f(x)$	92.5	58.9	55	58.9	92.5

48. **DIG DEEPER** A cubic polynomial function f has a leading coefficient of 2 and a constant term of −5. When $f(1) = 0$ and $f(2) = 3$, what is $f(-5)$? Explain your reasoning.

49. **COLLEGE PREP** The function f represented by the table is a polynomial function. Which statement is true?

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x	2	5	8	11	13
$f(x)$	−75	39	−22	−47	−9

Ⓐ f has a maximum value of 39.
Ⓑ $f(x) = 0$ for at least one value of x.
Ⓒ f is increasing when $2 < x < 5$.
Ⓓ $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

50. **CRITICAL THINKING** The weight y (in pounds) of a rainbow trout can be modeled by $y = 0.000304x^3$, where x is the length (in inches) of the trout.

a. Write a function that relates the weight y and length x of a rainbow trout when y is measured in kilograms and x is measured in centimeters.

b. Graph the original function and the function from part (a) in the same coordinate plane. What type of transformation can you apply to the original function to produce the graph from part (a)?

REVIEW & REFRESH

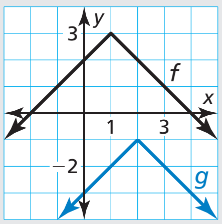
WATCH

In Exercises 51 and 52, simplify the expression.

51. $-wk + 3kz - 2kw + 9zk - kw$

52. $a^2(m - 7a^3) - m(a^2 - 10)$

53. Write a function g whose graph represents the indicated transformation of the graph of $f(x) = -|x - 1| + 3$.



In Exercises 54 and 55, describe the transformation of $f(x) = x^2$ represented by g. Then graph each function.

54. $g(x) = (x + 5)^2$

55. $g(x) = -\frac{3}{2}x^2$

In Exercises 56 and 57, solve the system using any method. Explain your choice of method.

56. $y = x^2 + x - 6$
$y = 2x^2 + x - 10$

57. $x^2 - 7x = y - 6$
$y = -8$

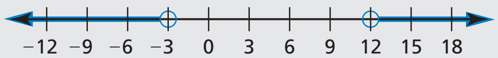
58. **MODELING REAL LIFE** You toss a penny into a park fountain. The penny leaves your hand 4 feet above the ground and has an initial vertical velocity of 25 feet per second. Does the penny reach a height of 10 feet? 15 feet? Explain your reasoning.

In Exercises 59 and 60, solve the inequality using any method. Explain your choice of method.

59. $2x^2 - 7x - 4 \leq 0$

60. $5x + 1 > 3x^2$

61. Write an inequality that represents the graph.



In Exercises 62 and 63, graph the polynomial function.

62. $g(x) = -x^4 + 3x^2 - 5$

63. $h(x) = 2x^2 - 7x + \frac{1}{2}x^3 + 4$

64. **MP STRUCTURE** Determine whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

$$f(x) = 5x^3x + \frac{5}{2}x^3 - 9x^4 + \sqrt{2}x^2 + 4x - 1 - x^{-5}x^5 - 4$$