Integrating Technology

Some graphing utilities can be used to draw the graph of the inverse of a function without the user having to find the inverse function. For instance, Figure 4.11 shows the graph of $f(x) = 0.1x^3 - 4$. The graphs of f and f^{-1} are both shown in Figure 4.12, along with the graph of $y = x$. Note that the graph of f^{-1} is the reflection of the graph of f with respect to the graph of $y = x$. The display shown in Figure 4.12 was produced on a TI-83/TI-83 Plus/TI-84 Plus graphing calculator by using the DrawInv command, which is in the DRAW menu.

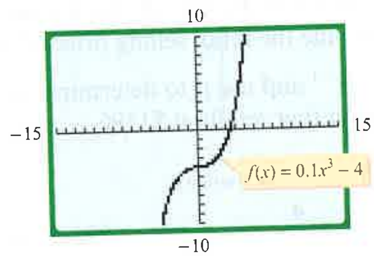


Figure 4.11

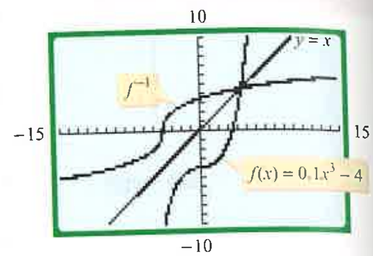


Figure 4.12

Answer graphs to Exercises 13–20 are on page AA15.

EXERCISE SET 4.1

Concept Check

1. What is a one-to-one function?
A function f is a one-to-one function if and only if $f(a) = f(b)$ implies $a = b$.

2. What is the horizontal line test?
If every horizontal line intersects the graph of a function at most once, then the graph is the graph of a one-to-one function.

3. How is the graph of the inverse of a function related to the graph of the function?
The graph of the inverse is a reflection of the graph of the function about the line $y = x$.

4. What are the steps for finding the inverse of a one-to-one function that is defined by an equation?
1. Substitute y for $f(x)$. 2. Interchange x and y. 3. Solve, if possible, for y in terms of x. 4. Substitute $f^{-1}(x)$ for y.

In Exercises 5 to 8, assume that the given function has an inverse function.

5. Given $f(3) = 7$, find $f^{-1}(7)$. 3

6. Given $g(-3) = 5$, find $g^{-1}(5)$. −3

7. Given $h^{-1}(-3) = -4$, find $h(-4)$. −3

8. Given $f^{-1}(7) = 0$, find $f(0)$. 7

9. If 3 is in the domain of f^{-1}, find $f[f^{-1}(3)]$. 3

10. If f is a one-to-one function and $f(0) = 5$, $f(1) = 2$, and $f(2) = 7$, find the following.

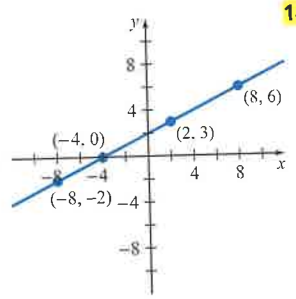
a. $f^{-1}(5)$ 0 **b.** $f^{-1}(2)$ 1

11. The domain of the inverse function f^{-1} is the ______ of f. range

12. The range of the inverse function f^{-1} is the ______ of f. domain

In Exercises 13 to 20, draw the graph of the inverse relation. Is the inverse relation a function?

13.



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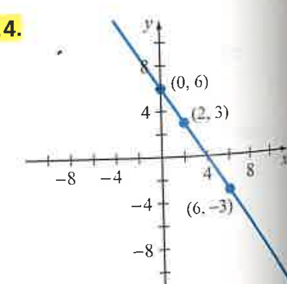

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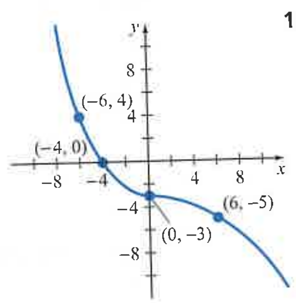
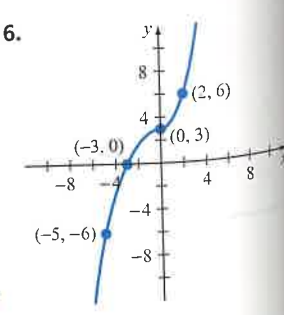
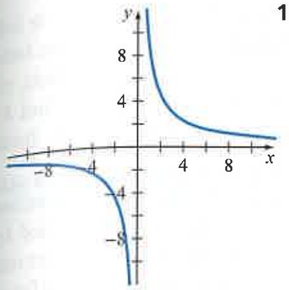


16.



Indicates Try It Exercises

17.



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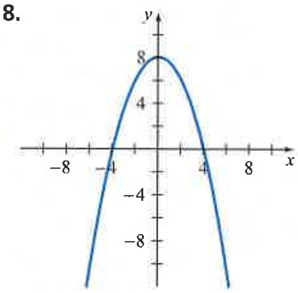
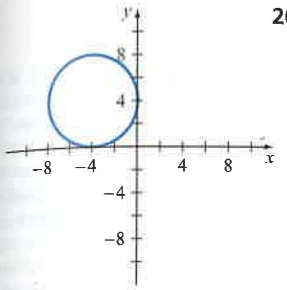


19.



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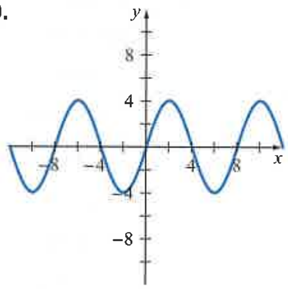


In Exercises 21 to 30, use composition of functions to determine whether *f* and *g* are inverses of one another.

21. $f(x) = 2x + 7;\ f^{-1}(x) = \frac{1}{2}x - \frac{7}{2}$ Yes

22. $f(x) = \frac{3}{4}x - 5;\ f^{-1}(x) = \frac{4}{3}x + \frac{20}{3}$ Yes

23. $f(x) = 4x - 1;\ g(x) = \frac{1}{4}x + \frac{1}{4}$ Yes

24. $f(x) = \frac{1}{2}x - \frac{3}{2};\ g(x) = 2x + 3$ Yes

25. $f(x) = -\frac{1}{2}x - \frac{1}{2};\ g(x) = -2x + 1$ No

26. $f(x) = 3x + 2;\ g(x) = \frac{1}{3}x - \frac{2}{3}$ Yes

27. $f(x) = \frac{5}{x - 3};\ g(x) = \frac{5}{x} + 3$ Yes

28. $f(x) = \frac{2x}{x - 1};\ g(x) = \frac{x}{x - 2}$ Yes

29. $f(x) = x^3 + 2;\ g(x) = \sqrt[3]{x - 2}$ Yes

30. $f(x) = (x + 5)^3;\ g(x) = \sqrt[3]{x} - 5$ Yes

In Exercises 31 to 34, find the inverse of the function. If the function does not have an inverse function, write "no inverse function."

31. $\{(-3, 1), (-2, 2), (1, 5), (4, -7)\}$
$\{(1, -3), (2, -2), (5, 1), (-7, 4)\}$

32. $\{(-5, 4), (-2, 3), (0, 1), (3, 2), (7, 11)\}$
$\{(4, -5), (3, -2), (1, 0), (2, 3), (11, 7)\}$

33. $\{(0, 1), (1, 2), (2, 4), (3, 8), (4, 16)\}$
$\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4)\}$

34. $\{(1, 0), (10, 1), (100, 2), (1000, 3), (10{,}000, 4)\}$
$\{(0, 1), (1, 10), (2, 100), (3, 1000), (4, 10{,}000)\}$

In Exercises 35 to 52, find $f^{-1}(x)$. State any restrictions on the domain of $f^{-1}(x)$.

35. $f(x) = 2x + 4$ $f^{-1}(x) = \frac{1}{2}x - 2$

36. $f(x) = 4x - 8$ $f^{-1}(x) = \frac{1}{4}x + 2$

37. $f(x) = 3x - 7$ $f^{-1}(x) = \frac{1}{3}x + \frac{7}{3}$

38. $f(x) = -3x - 8$ $f^{-1}(x) = -\frac{1}{3}x - \frac{8}{3}$

39. $f(x) = -3x + 11$ $f^{-1}(x) = -\frac{1}{3}x + \frac{11}{3}$

40. $f(x) = -\frac{1}{2}x - \frac{3}{4}$ $f^{-1}(x) = -2x - \frac{3}{2}$

41. $f(x) = \frac{2x}{x - 1},\ x \neq 1$ $f^{-1}(x) = \frac{x}{x - 2},\ x \neq 2$

42. $f(x) = \frac{x}{x - 2},\ x \neq 2$ $f^{-1}(x) = \frac{2x}{x - 1},\ x \neq 1$

43. $f(x) = \frac{x - 1}{x + 1},\ x \neq -1$ $f^{-1}(x) = \frac{x + 1}{1 - x},\ x \neq 1$

44. $f(x) = \frac{2x - 1}{x + 3},\ x \neq -3$ $f^{-1}(x) = \frac{3x + 1}{2 - x},\ x \neq 2$

45. $f(x) = x^2 + 1,\ x \geq 0$ $f^{-1}(x) = \sqrt{x - 1},\ x \geq 1$

46. $f(x) = x^2 - 4,\ x \geq 0$ $f^{-1}(x) = \sqrt{x + 4},\ x \geq -4$

47. $f(x) = \sqrt{x - 2},\ x \geq 2$ $f^{-1}(x) = x^2 + 2,\ x \geq 0$

48. $f(x) = \sqrt{4 - x},\ x \leq 4$ $f^{-1}(x) = -x^2 + 4,\ x \geq 0$

49. $f(x) = x^2 + 4x,\ x \geq -2$ $f^{-1}(x) = \sqrt{x + 4} - 2,\ x \geq -4$

50. $f(x) = x^2 - 6x,\ x \leq 3$ $f^{-1}(x) = -\sqrt{x + 9} + 3,\ x \geq -9$

51. $f(x) = x^2 + 4x - 1,\ x \leq -2$
$f^{-1}(x) = -\sqrt{x + 5} - 2,\ x \geq -5$

52. $f(x) = x^2 - 6x + 1,\ x \geq 3$
$f^{-1}(x) = \sqrt{x + 8} + 3,\ x \geq -8$

53. **Geometry** The volume of a cube is given by $V(x) = x^3$, where x is the measure of the length of a side of the cube. Find $V^{-1}(x)$ and explain what it represents.
$V^{-1}(x) = \sqrt[3]{x}$. V^{-1} is the length of a side of a cube that has a volume of x cubic units.

54. **Unit Conversion** The function $f(x) = 12x$ converts feet, x, into inches. Find $f^{-1}(x)$ and explain what it represents.
$f^{-1}(x) = \frac{x}{12}$. f^{-1} converts x inches into feet.

55. **Fahrenheit to Celsius** The function

$$f(x) = \frac{5}{9}(x - 32)$$

is used to convert x degrees Fahrenheit to an equivalent Celsius temperature. Find f^{-1} and explain how it is used. $f^{-1}(x) = \frac{9}{5}x + 32$; $f^{-1}(x)$ is used to convert x degrees Celsius to an equivalent Fahrenheit temperature.

56. **Retail Sales** A clothing merchant uses the function

$$S(x) = \frac{3}{2}x + 18$$

to determine the retail selling price S, in dollars, of a winter coat for which she has paid a wholesale price of x dollars.

a. The merchant paid a wholesale price of \$96 for a winter coat. Use S to determine the retail selling price she will charge for this coat. \$162

b. Find S^{-1} and use it to determine the merchant's wholesale price for a coat that retails at \$399. $S^{-1}(x) = \frac{2}{3}x - 12$, \$254

57. **Fashion** The function $s(x) = 2x + 24$ can be used to convert a U.S. women's shoe size into an Italian women's shoe size. Determine the function $s^{-1}(x)$ that can be used to convert an Italian women's shoe size to its equivalent U.S. shoe size. $s^{-1}(x) = \frac{1}{2}x - 12$

58. **Fashion** The function $K(x) = 1.3x - 4.7$ converts a men's shoe size in the United States to the equivalent shoe size in the United Kingdom. Determine the function $K^{-1}(x)$ that can be used to convert a U.K. men's shoe size to its equivalent U.S. shoe size. $K^{-1}(x) = \frac{x + 4.7}{1.3}$

59. **Catering** A catering service uses the function

$$c(x) = \frac{300 + 12x}{x}$$

to determine the amount, in dollars, it charges per person for a sit-down dinner, where x is the number of people in attendance.

a. Find $c(30)$ and explain what it represents. $c(30) = \$22$; the company charges \$22 per person to cater a dinner for 30 people.

b. Find c^{-1}. $c^{-1}(x) = \frac{300}{x - 12}$

c. Use c^{-1} to determine how many people attended a dinner for which the cost per person was \$15.00. 100 people

60. **Landscaping** A landscaping company uses the function

$$c(x) = \frac{600 + 140x}{x}$$

to determine the amount, in dollars, it charges per tree to deliver and plant x palm trees.

a. Find $c(5)$ and explain what it represents. $c(5) = \$260$; the company charges \$260 per tree to deliver and plant 5 trees.

b. Find c^{-1}. $c^{-1}(x) = \frac{600}{x - 140}$

c. Use c^{-1} to determine how many palm trees were delivered and planted if the cost per tree was \$160. 30 palm trees

61. **Compensation** The monthly earnings $E(s)$, in dollars, of a software sales executive are given by the function $E(s) = 0.05s + 2500$, where s is the value, in dollars, of the software sold by the executive during the month. Find $E^{-1}(s)$ and explain how the executive could use this function. $E^{-1}(s) = 20s - 50{,}000$. The executive can use the inverse function to determine the value of the software that must be sold to achieve a given monthly income.

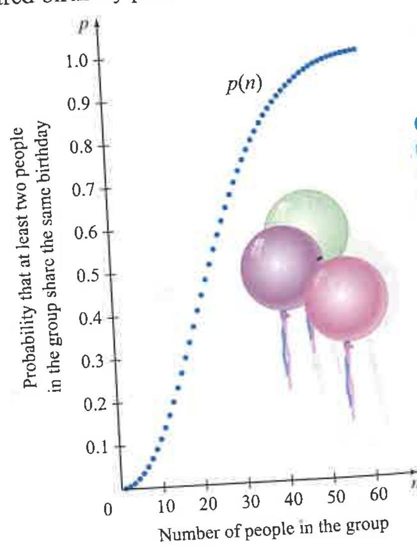
62. **Grading** A professor uses the function defined by the following table to determine the grade a student receives on a test. Does this grading function have an inverse function? Explain your answer.

Grading Scale

Score	Grade
90–100	A
80–89	B
70–79	C
60–69	D
0–59	F

No. The function does not have an inverse function because it is not a one-to-one function.

63. **The Birthday Problem** A famous problem called the *birthday problem* goes like this: Suppose there is a randomly selected group of n people in a room. What is the probability that at least two of the people have a birthday on the same day of the year? It may surprise you that for a group of 23 people, the probability that at least two of the people share a birthday is about 50.7%. The following graph can be used to estimate shared birthday probabilities for $1 \le n \le 60$.



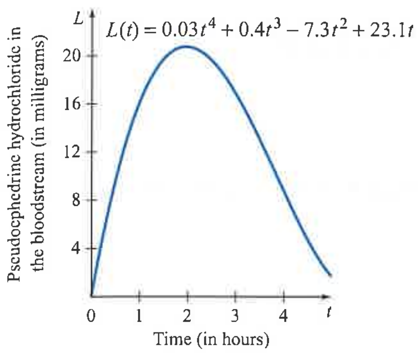
63. c. $p^{-1}(0.223)$ represents the number of people required to be in the group for a 22.3% probability that at least two of the people will share a birthday.

a. Use the graph of p to estimate $p(10)$ and $p(30)$. $p(10) \approx 0.12 = 12\%$; $p(30) \approx 0.71 = 71\%$

b. Consider the function p with $1 \le n \le 60$, as shown in the graph. Explain how you can tell that p has an inverse that is a function. The graph of p, for $1 \le n \le 60$, is an increasing function. Thus p has an inverse that is a function.

c. Write a sentence that explains the meaning of $p^{-1}(0.223)$ in the context of this application. Answer is (above) right of figure.

64. **Medication Level** The function L shown in the following graph models the level of pseudoephedrine hydrochloride, in milligrams, in the bloodstream of a patient t hours after 30 milligrams of the medication have been administered.



a. Use the graph of L to estimate two different values of t for which the pseudoephedrine hydrochloride levels are the same. Answers will vary.

b. Does L have an inverse that is a function? Explain. No. L is not a one-to-one function.

65. Cryptology Cryptology is the study of making and breaking secret codes. Secret codes are often used to send messages over the Internet. By devising a code that is difficult to break, the sender hopes to prevent the messages from being read by an unauthorized person.

In practice, complicated one-to-one functions and their inverses are used to encode and decode messages. The following procedure uses the simple function $f(x) = 2x - 1$ to illustrate the basic concepts that are involved.

Assign to each letter of the alphabet, and a blank space, a two-digit numerical value, as shown below.

A	10	H	17	O	24	V	31
B	11	I	18	P	25	W	32
C	12	J	19	Q	26	X	33
D	13	K	20	R	27	Y	34
E	14	L	21	S	28	Z	35
F	15	M	22	T	29		36
G	16	N	23	U	30		

Note: A blank space is represented by the numerical value 36.

Using these numerical values, the message MEET YOU AT NOON would be represented by

22 14 14 29 36 34 24 30 36 10 29 36 23 24 24 23

Let $f(x) = 2x - 1$ define a coding function. The above message can be encoded by finding $f(22)$, $f(14)$, $f(14)$, $f(29)$, $f(36)$, $f(34)$, $f(24)$, . . ., $f(23)$, which yields

43 27 27 57 71 67 47 59 71 19 57 71 45 47 47 45

The inverse of f, which is

$$f^{-1}(x) = \frac{x+1}{2}$$

is used by the receiver of the message to decode the message. For instance,

$$f^{-1}(43) = \frac{43+1}{2} = 22$$

which represents M, and

$$f^{-1}(27) = \frac{27+1}{2} = 14$$

which represents E.

a. Use the preceding coding procedure to encode the message DO YOUR HOMEWORK
25 47 71 67 47 59 53 71 33 47 43 27 63 47 53 39

b. Use $f^{-1}(x)$ to decode the message

49 33 47 45 27 71 33 47 43 27 PHONE HOME

c. Explain why it is important to use a one-to-one function to encode a message. Answers will vary.

66. Cryptography A friend is using the letter–number correspondence in Exercise 65 and the coding function $g(x) = 2x + 3$. Your friend sends you the coded message

59 31 39 73 31 75 61 37 31 75 29 23 71

Use $g^{-1}(x)$ to decode this message. SEIZE THE DAY

In Exercises 67 to 70, answer the question without finding the equation of the linear function.

67. Suppose that f is a linear function, $f(2) = 7$, and $f(5) = 12$. If $f(4) = c$, then is c less than 7, between 7 and 12, or greater than 12? Explain your answer. Because the function is increasing and 4 is between 2 and 5, c must be between 7 and 12.

68. Suppose that f is a linear function, $f(1) = 13$, and $f(4) = 9$. If $f(3) = c$, then is c less than 9, between 9 and 13, or greater than 13? Explain your answer. Because the function is decreasing and 3 is between 1 and 4, c must be between 9 and 13.

69. Suppose that f is a linear function, $f(2) = 3$, and $f(5) = 9$. Between which two numbers is $f^{-1}(6)$? Between 2 and 5

70. Suppose that f is a linear function, $f(5) = -1$, and $f(9) = -3$. Between which two numbers is $f^{-1}(-2)$? Between 5 and 9

Only one-to-one functions have inverses that are functions. In Exercises 71 to 78, determine whether the given function is a one-to-one function.

71. $f(x) = x^2 + 1$ No

72. $v(t) = \sqrt{16 + t}$ Yes

73. $F(x) = |x| + x$ No

74. $T(x) = |x^2 - 6|, \quad x \ge 0$ No

75. $g(x) = x^3 - 2x$ No

76. $k(x) = \sqrt{x}$ Yes

77. $j(x) = x^3$ Yes

78. $n(x) = \dfrac{1}{x}$ Yes

79. Use a graph of $f(x) = -x + 3$ to explain why f is its own inverse. The reflection of f across the line given by $y = x$ yields f. Thus f is its own inverse.

80. Use a graph of $f(x) = \sqrt{16 - x^2}$, with $0 \le x \le 4$, to explain why f is its own inverse. The reflection of f across the line given by $y = x$ yields f. Thus f is its own inverse.

Enrichment Exercises

81. Consider the linear function $f(x) = mx + b$, $m \ne 0$. The graph of f has a slope of m and a y-intercept of $(0, b)$. What are the slope and y-intercept of the graph of f^{-1}?
Slope: $\frac{1}{m}$; y-intercept: $\left(0, -\frac{b}{m}\right)$

82. Find the inverse of $f(x) = ax^2 + bx + c$, $a \ne 0$, $x \ge -\dfrac{b}{2a}$.
$f^{-1}(x) = \dfrac{-b + \sqrt{b^2 + 4ax - 4ac}}{2a}$, $a \ne 0$, $x \ge \dfrac{4ac - b^2}{4a}$

SECTION 4.2

Exponential Functions and Their Applications

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A24.

PS1. Evaluate: 2^3 [P.2] 8

PS2. Evaluate: 3^{-4} [P.2] $\frac{1}{81}$

PS3. Evaluate: $\dfrac{2^2 + 2^{-2}}{2}$ [P.2/P.5] $\frac{17}{8}$

PS4. Evaluate: $\dfrac{3^2 - 3^{-2}}{2}$ [P.2/P.5] $\frac{40}{9}$

PS5. Evaluate $f(x) = 10^x$ for $x = -1, 0, 1,$ and 2. [P.2] $\frac{1}{10}$, 1, 10, and 100

PS6. Evaluate $f(x) = \left(\dfrac{1}{2}\right)^x$ for $x = -1, 0, 1,$ and 2. [P.2] 2, 1, $\frac{1}{2}$, and $\frac{1}{4}$

Exponential Functions

When a parking facility opened in 1980, it charged \$1 for all-day parking. Since then, it has doubled its daily parking fee every 8 years as shown in Table 4.1.

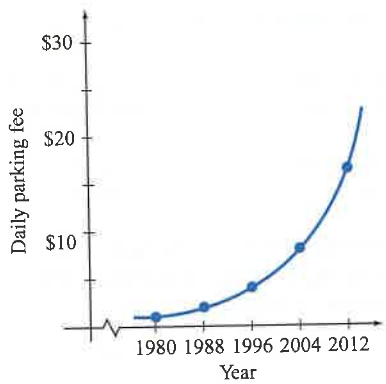


Figure 4.13

Table 4.1

Year	1980	1988	1996	2004	2012
Daily parking fee	\$1	\$2	\$4	\$8	\$16

In Figure 4.13, we have plotted the data in Table 4.1 and modeled the upward trend in the parking fee by a smooth curve. This model is based on an *exponential function*, which is one of the major topics of this chapter.

The effectiveness of a drug that is used for sedation during a surgical procedure depends on the concentration of the drug in the patient. Through natural body chemistry, the amount of this drug in the body decreases over time. The graph in Figure 4.14 models this decrease. This model is another example of an exponential model.

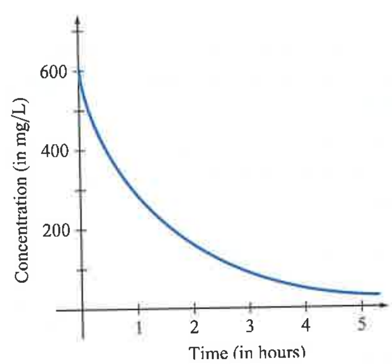


Figure 4.14

Definition of an Exponential Function

The **exponential function with base b** is defined by

$$f(x) = b^x$$

where $b > 0$, $b \neq 1$, and x is a real number.

The base b of $f(x) = b^x$ is required to be positive. If the base were a negative number, the value of the function would be a complex number for some values of x. For