3.6 Quadratic Inequalities

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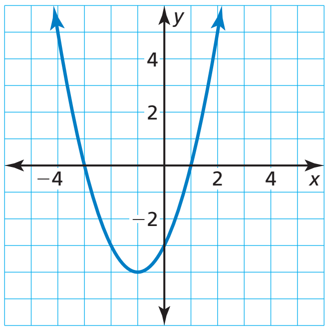
Learning Target Graph quadratic inequalities in two variables and solve quadratic inequalities in one variable.

Success Criteria

- I can describe the graph of a quadratic inequality.
- I can graph quadratic inequalities.
- I can graph systems of quadratic inequalities.
- I can solve quadratic inequalities algebraically and graphically.

EXPLORE IT! Solving Quadratic Inequalities

Work with a partner. The figure shows the graph of $f(x) = x^2 + 2x - 3$.



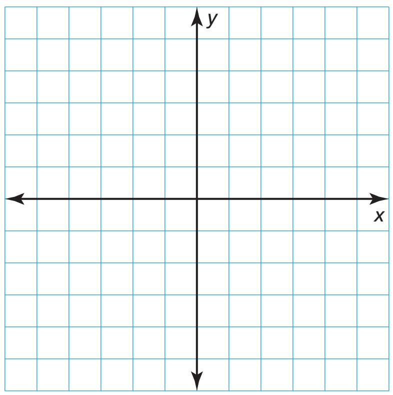
Math Practice

Consider Similar Problems
How is graphing a quadratic inequality in two variables similar to graphing a linear inequality in two variables?

a. Explain how you can use the graph to solve the inequality $0 > x^2 + 2x - 3$. Then graph the solutions of the inequality.

b. Explain how the inequality $y > x^2 + 2x - 3$ is different from the inequality in part (a).

c. Explain how you can use the graph above to represent the solutions of $y > x^2 + 2x - 3$. Then graph the inequality.



d. Repeat parts (a)–(c) by replacing > with ≤.

e. Compare the graphs of the solutions of quadratic inequalities in one variable to the graphs of the solutions of quadratic inequalities in two variables.

Graphing Quadratic Inequalities in Two Variables

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Vocabulary

quadratic inequality in two variables, p. 136
system of quadratic inequalities, p. 137
quadratic inequality in one variable, p. 138

A **quadratic inequality in two variables**, x and y, can be written in one of the following forms, where a, b, and c are real numbers and $a \neq 0$.

$$y < ax^2 + bx + c \qquad y > ax^2 + bx + c$$

$$y \leq ax^2 + bx + c \qquad y \geq ax^2 + bx + c$$

The graph of any such inequality consists of all solutions (x, y) of the inequality.

Previously, you graphed linear inequalities in two variables. You can use a similar procedure to graph quadratic inequalities in two variables.

KEY IDEA

Graphing a Quadratic Inequality in Two Variables

Step 1 Graph the parabola with the equation $y = ax^2 + bx + c$. Make the parabola *dashed* for inequalities with $<$ or $>$ and *solid* for inequalities with $\leq$ or $\geq$.

Step 2 Test a point (x, y) that does not lie on the parabola to determine whether the point is a solution of the inequality.

Step 3 When the test point is a solution, shade the region of the plane that contains the point. When the test point is not a solution, shade the region that does not contain the point.

EXAMPLE 1 Graphing a Quadratic Inequality in Two Variables

Graph $y < -x^2 - 2x - 1$.

WATCH

SOLUTION

Step 1 Graph $y = -x^2 - 2x - 1$. Because the inequality symbol is $<$, make the parabola dashed.

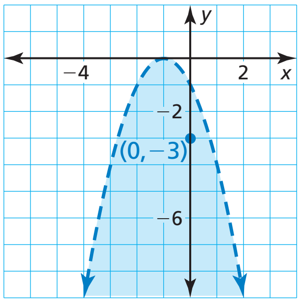
Step 2 Test a point that does not lie on the parabola, such as $(0, -3)$.

$y < -x^2 - 2x - 1$ — Write the inequality.

$-3 \overset{?}{<} -0^2 - 2(0) - 1$ — Substitute.

$-3 < -1$ ✓ — Simplify.

Step 3 Because $(0, -3)$ is a solution, shade the region inside the parabola that contains $(0, -3)$.



Math Practice

Look for Structure
Why is it convenient to choose a test point that is on the y-axis?

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

1. **WRITING** Explain how you can tell whether an ordered pair is a solution of a quadratic inequality.

Graph the inequality.

2. $y \geq x^2 + 2x - 8$
3. $y \leq 2x^2 - x - 1$
4. $y > -x^2 + 2x + 4$

5. **WRITING** When determining which region to shade in the graph of a quadratic inequality in two variables, why is it important to test a point that does not lie on the parabola?

EXAMPLE 2 Modeling Real Life

WATCH INFO

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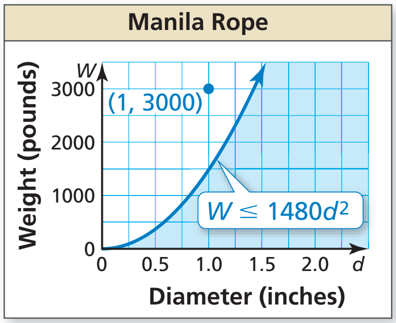
A manila rope used for rappelling down a cliff can safely support a weight W (in pounds) provided $W \leq 1480d^2$, where d is the diameter (in inches) of the rope. Graph the inequality and interpret the solution.

SOLUTION



Graph $W = 1480d^2$ for nonnegative values of d. Because the inequality symbol is $\leq$, make the parabola solid. Test a point that does not lie on the parabola, such as (1, 3000).

$$W \leq 1480d^2$$

$$3000 \overset{?}{\leq} 1480(1)^2$$

$$3000 \leq 1480$$

▶ Because (1, 3000) is not a solution, shade the region outside the parabola. The shaded region represents weights that can be supported by ropes with various diameters.

A **system of quadratic inequalities** is a set of two or more quadratic inequalities in the same variables. Graphing a system of quadratic inequalities is similar to graphing a system of linear inequalities. First graph each inequality in the same coordinate plane. Then identify the region in the coordinate plane common to all of the graphs. This region is called the *graph of the system*.

EXAMPLE 3 Graphing a System of Quadratic Inequalities

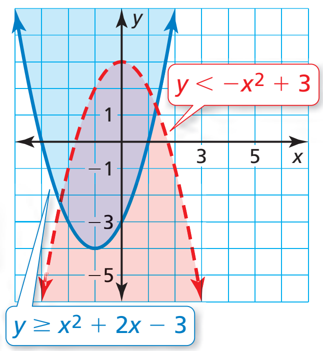
WATCH

Graph the system of quadratic inequalities.

$y < -x^2 + 3$ Inequality 1

$y \geq x^2 + 2x - 3$ Inequality 2

SOLUTION



Step 1 Graph $y < -x^2 + 3$. The graph is the red region inside (but not including) the parabola $y = -x^2 + 3$.

Step 2 Graph $y \geq x^2 + 2x - 3$. The graph is the blue region inside and including the parabola $y = x^2 + 2x - 3$.

Step 3 Identify the purple region where the two graphs overlap. This region is the graph of the system.

Check

Check that a point in the solution region, such as (0, 0), is a solution of the system.

$$y < -x^2 + 3$$

$$0 \overset{?}{<} -0^2 + 3$$

$$0 < 3 \checkmark$$

$$y \geq x^2 + 2x - 3$$

$$0 \overset{?}{\geq} 0^2 + 2(0) - 3$$

$$0 \geq -3 \checkmark$$

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

6. MP **REASONING** Can you use the graph in Example 2 to determine if a manila rope with a $\frac{3}{8}$-inch diameter can safely support a 200-pound person? Explain your reasoning. If not, show how you can solve the problem.

Graph the system of quadratic inequalities.

7. $y \leq -x^2$
$y > x^2 - 3$

8. $y \geq x^2 + 2x - 1$
$y \geq 2x^2 + 4x - 1$

9. $y > x^2 + 1$
$y < -x^2 + x - 1$

Solving Quadratic Inequalities in One Variable

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A **quadratic inequality in one variable**, x, can be written in one of the following forms, where a, b, and c are real numbers and $a \neq 0$.

$ax^2 + bx + c < 0 \qquad ax^2 + bx + c > 0 \qquad ax^2 + bx + c \leq 0 \qquad ax^2 + bx + c \geq 0$

You can solve quadratic inequalities using algebraic methods or graphs.

EXAMPLE 4 Solving a Quadratic Inequality Algebraically

WATCH

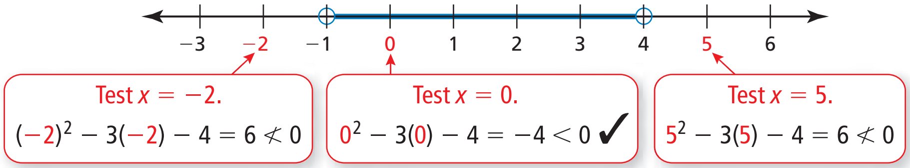
Solve $x^2 - 3x - 4 < 0$ algebraically.

SOLUTION

First, write and solve the equation obtained by replacing $<$ with $=$.

$x^2 - 3x - 4 = 0$	Write the related equation.
$(x - 4)(x + 1) = 0$	Factor.
$x = 4 \quad \text{or} \quad x = -1$	Zero-Product Property

The numbers -1 and 4 are the *critical values* of the original inequality. Plot -1 and 4 on a number line, using open dots because the values do not satisfy the inequality. The critical values partition the number line into three intervals. Test an x-value in each interval to determine whether it satisfies the inequality.



▶ So, the solution is $-1 < x < 4$.

Another way to solve $ax^2 + bx + c < 0$ is to first graph the related function $y = ax^2 + bx + c$. Then, because the inequality symbol is $<$, identify the x-values for which the graph lies *below* the x-axis. You can use a similar procedure to solve quadratic inequalities that involve $\leq$, $>$, or $\geq$.

EXAMPLE 5 Solving a Quadratic Inequality by Graphing

WATCH

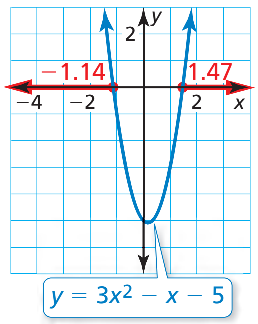
Solve $3x^2 - x - 5 \geq 0$ by graphing.

SOLUTION

The solution consists of the x-values for which the graph of $y = 3x^2 - x - 5$ lies on or above the x-axis. Find the x-intercepts of the graph by letting $y = 0$ and using the Quadratic Formula to solve $0 = 3x^2 - x - 5$ for x.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-5)}}{2(3)} \qquad a = 3, b = -1, c = -5$$

$$x = \frac{1 \pm \sqrt{61}}{6} \qquad \text{Simplify.}$$



The solutions are $x \approx -1.14$ and $x \approx 1.47$. Sketch a parabola that opens up and has -1.14 and 1.47 as x-intercepts. The graph lies on or above the x-axis to the left of (and including) $x = -1.14$ and to the right of (and including) $x = 1.47$.

▶ The solution of the inequality is approximately $x \leq -1.14$ *or* $x \geq 1.47$.

EXAMPLE 6 Modeling Real Life

WATCH

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An archaeologist is roping off a rectangular region of land to dig for artifacts. The region must have a perimeter of 440 feet and an area of at least 8000 square feet. Describe the possible lengths of the archaeological region.

SOLUTION

1. **Understand the Problem** You are given the perimeter and the minimum area of a rectangular region. You are asked to determine the possible lengths of the region.
2. **Make a Plan** Use the perimeter and area formulas to write a quadratic inequality describing the possible lengths of the region. Then solve the inequality.
3. **Solve and Check** Let ℓ represent the length (in feet) and let w represent the width (in feet) of the region.

Perimeter = 440 Area ≥ 8000

$2\ell + 2w = 440$ $\ell w \geq 8000$

Solve the perimeter equation for w to obtain $w = 220 - \ell$. Substitute this into the area inequality to obtain a quadratic inequality in one variable.

$\ell w \geq 8000$ Write the area inequality.

$\ell(220 - \ell) \geq 8000$ Substitute $220 - \ell$ for w.

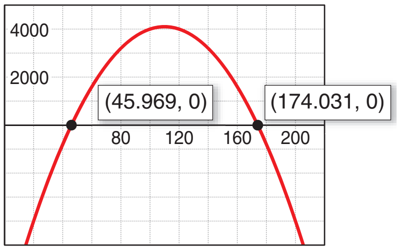
$220\ell - \ell^2 \geq 8000$ Distributive Property

$-\ell^2 + 220\ell - 8000 \geq 0$ Write in standard form.

ANOTHER WAY

You can graph each side of $220\ell - \ell^2 = 8000$ and use the intersection points to determine when $220\ell - \ell^2$ is greater than or equal to 8000.

Use technology to find the ℓ-intercepts of $y = -\ell^2 + 220\ell - 8000$.



The ℓ-intercepts are $\ell \approx 45.969$ and $\ell \approx 174.031$. The solution consists of the ℓ-values for which the graph lies on or above the ℓ-axis. The graph lies on or above the ℓ-axis when $45.969 \leq \ell \leq 174.031$.

▶ So, the approximate lengths of the region are at least 46 feet and at most 174 feet.

Check Choose a length in the solution region, such as $\ell = 100$, and find the width. Then check that the dimensions satisfy the original area inequality.

$2\ell + 2w = 440$ $\ell w \geq 8000$

$2(100) + 2w = 440$ $100(120) \overset{?}{\geq} 8000$

$w = 120$ $12{,}000 \geq 8000$ ✓

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the inequality using any method. Explain your choice of method.

10. $2x^2 + 3x \leq 2$ **11.** $-3x^2 - 4x + 1 < 0$ **12.** $4x^2 + 3 > -13x$ **13.** $x^2 + 6x - 8 < 0$

14. WHAT IF? In Example 6, the area must be at least 8500 square feet. Describe the possible lengths of the region.

3.6 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–4, match the inequality with its graph. Explain your reasoning.

1. $y \leq x^2 + 4x + 3$ **2.** $y > -x^2 + 4x - 3$

3. $y < x^2 - 4x + 3$ **4.** $y \geq x^2 + 4x + 3$

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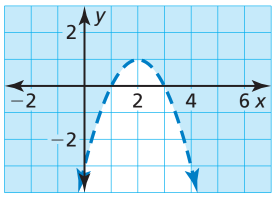

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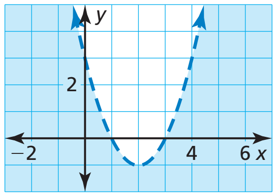

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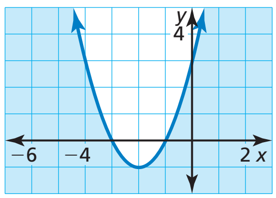

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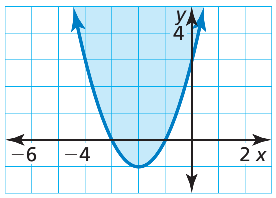

In Exercises 5–12, graph the inequality. *Example 1*

5. $y < -x^2$ **6.** $y \geq 4x^2$

7. $y > x^2 - 9$ **8.** $y < x^2 + 5$

9. $y \leq x^2 + 5x$ **10.** $y \geq -2x^2 + 9x - 4$

11. $y > 2(x + 3)^2 - 1$ **12.** $y \leq \left(x - \frac{1}{2}\right)^2 + \frac{5}{2}$

ANALYZING RELATIONSHIPS **In Exercises 13 and 14, use the graph to write an inequality in terms of $f(x)$ so point P is a solution.**

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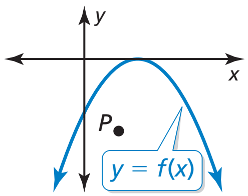
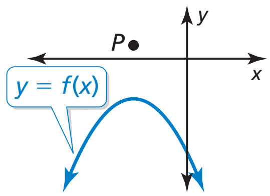


14.


ERROR ANALYSIS **In Exercises 15 and 16, describe and correct the error in graphing $y \geq x^2 + 2$.**

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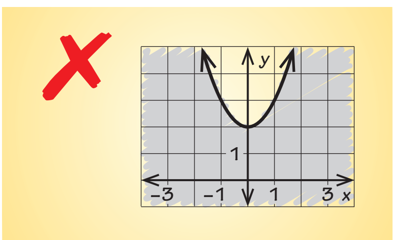

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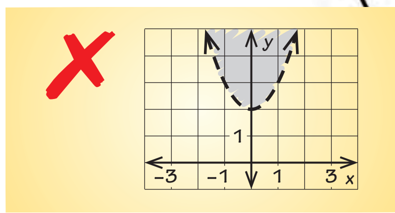
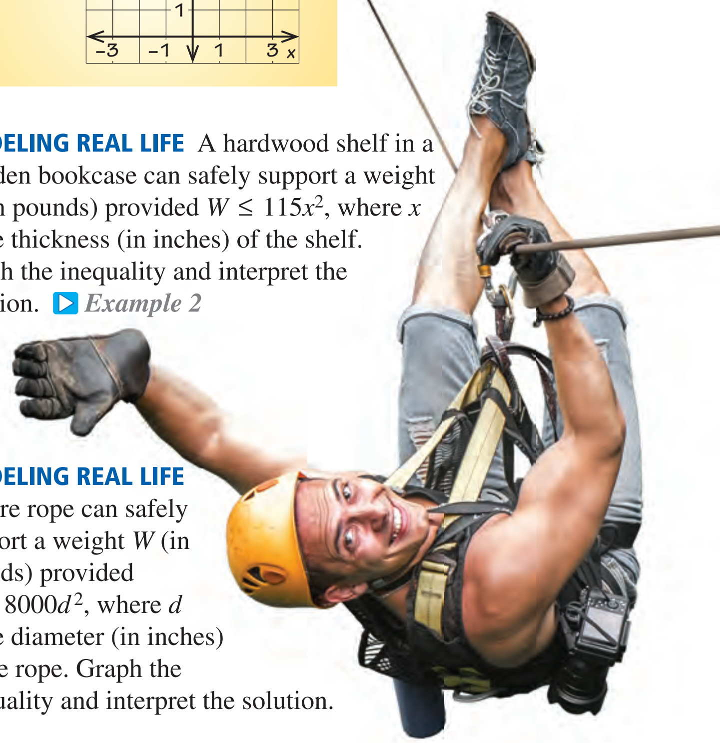


17. MODELING REAL LIFE A hardwood shelf in a wooden bookcase can safely support a weight W (in pounds) provided $W \leq 115x^2$, where x is the thickness (in inches) of the shelf. Graph the inequality and interpret the solution. *Example 2*



18. MODELING REAL LIFE A wire rope can safely support a weight W (in pounds) provided $W \leq 8000d^2$, where d is the diameter (in inches) of the rope. Graph the inequality and interpret the solution.

In Exercises 19–26, graph the system of quadratic inequalities. *Example 3*

19. $y \geq 2x^2$
$y < -x^2 + 1$

20. $y > -5x^2$
$y > 3x^2 - 2$

21. $y \leq -x^2 + 4x - 4$
$y < x^2 + 2x - 8$

22. $y \geq x^2 - 4$
$y \leq -2x^2 + 7x + 4$

23. $y \geq 2x^2 + x - 5$
$y < -x^2 + 5x + 10$

24. $y \geq x^2 - 3x - 6$
$y \geq x^2 + 7x + 6$

25. $x^2 + y < 3x + 2$
$y < -x^2 + 5x + 10$

26. $3x^2 + y \leq -x - 3$
$\frac{1}{2}x^2 + 2x \geq y - 2$

In Exercises 27–34, solve the inequality algebraically. *Example 4*

27. $4x^2 < 25$ **28.** $x^2 + 10x + 9 < 0$

29. $x^2 - 11x \geq -28$ **30.** $3x^2 - 13x > -10$

31. $2x^2 - 5x - 3 \leq 0$ **32.** $4x^2 + 8x - 21 \geq 0$

33. $\frac{1}{2}x^2 - x > 4$ **34.** $-\frac{1}{2}x^2 + 4x \leq 1$

In Exercises 35–42, solve the inequality by graphing. Example 5

35. $x^2 - 3x + 1 < 0$

36. $x^2 - 4x + 2 > 0$

37. $x^2 + 8x > -7$

38. $x^2 + 6x < -3$

39. $3x^2 - 8 \le -2x$

40. $3x^2 + 5x - 3 < 1$

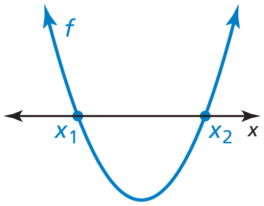
41. $\frac{1}{3}x^2 + 2x \ge 2$

42. $\frac{3}{4}x^2 + 4x \ge 3$

43. **MODELING REAL LIFE** A rectangular fountain display has a perimeter of 400 feet and an area of at least 9100 square feet. Describe the possible widths of the fountain. Example 6

44. **MODELING REAL LIFE** An animal shelter director is planning to build a rectangular playpen. The playpen must have a perimeter of 150 feet and an area of at least 1000 square feet. Describe the possible lengths of the playpen.

45. **MP STRUCTURE** Consider the graph of the function $f(x) = ax^2 + bx + c$.



a. What are the solutions of $ax^2 + bx + c < 0$?

b. What are the solutions of $ax^2 + bx + c > 0$?

c. The graph of g represents a reflection in the x-axis of the graph of f. For which values of x is $g(x)$ positive?

46. **MODELING REAL LIFE** The arch of the Sydney Harbor Bridge in Sydney, Australia, can be modeled by $y = -0.00211x^2 + 1.06x$, where x is the distance (in meters) from the left pylons and y is the height (in meters) of the arch above the water. See photo at the bottom of the page. For what distances x is the arch above the road?

47. **MP PROBLEM SOLVING** The number of teams that have participated in an engineering competition for high-school students over a recent period of time x (in years) can be modeled by $T(x) = 17.155x^2 + 193.68x + 235.81, 0 \le x \le 6$. After how many years is the number of teams greater than 1000? Justify your answer.

48. **MP PROBLEM SOLVING** A study found that a driver's reaction time $A(x)$ to audio stimuli and his or her reaction time $V(x)$ to visual stimuli (both in milliseconds) can be modeled by

$$A(x) = 0.0051x^2 - 0.319x + 15, \quad 16 \le x \le 70$$

$$V(x) = 0.005x^2 - 0.23x + 22, \quad 16 \le x \le 70$$

where x is the age (in years) of the driver.

a. Use technology to solve the inequality $A(x) < V(x)$. Explain how to use the domain to determine a reasonable solution.

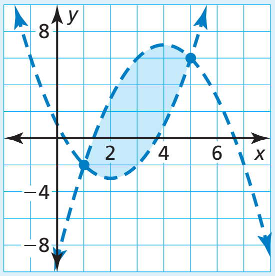
b. Based on your result from part (a), do you think a driver reacts more quickly to a traffic light changing from green to yellow or to the siren of an approaching ambulance? Explain.

49. **MODELING REAL LIFE** The length (in millimeters) of the larvae of a black porgy fish can be modeled by

$$L(x) = 0.0058x^2 + 0.201x + 2.59, \quad 0 \le x \le 44$$

where x is the age (in days) of the larvae. At what ages is a larva's length typically greater than 10 millimeters? Explain how the given domain affects the solution.

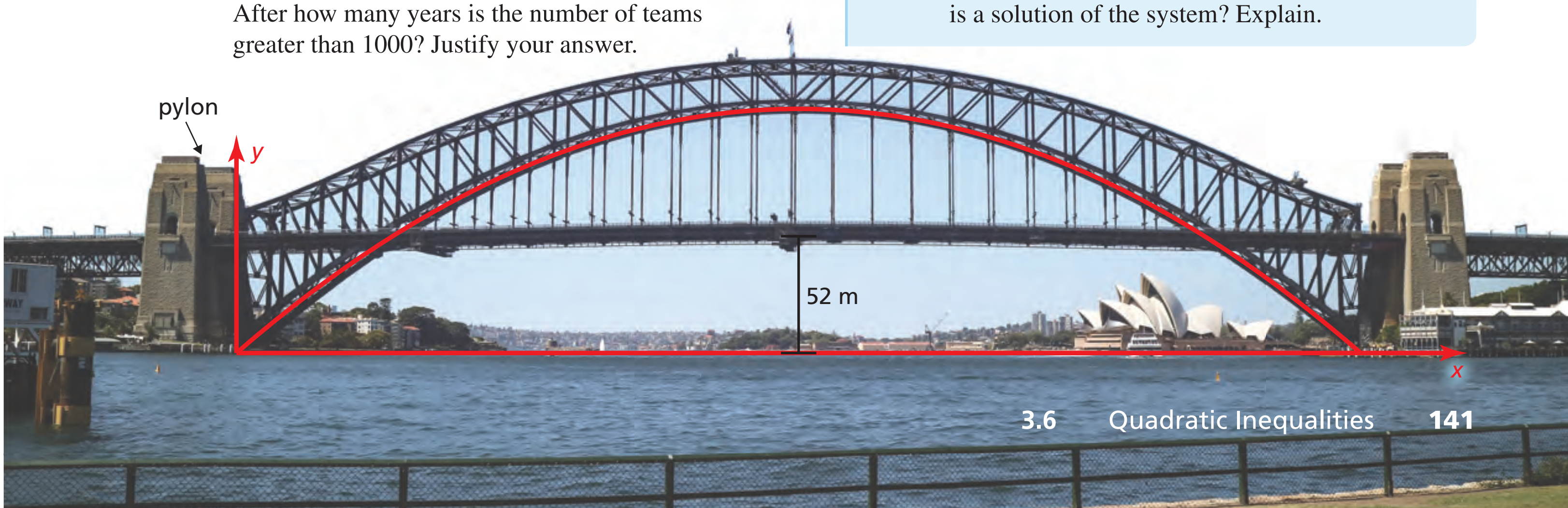
50. **HOW DO YOU SEE IT?**
The graph shows a system of quadratic inequalities.



a. Identify two solutions of the system.

b. Are the points $(1, -2)$ and $(5, 6)$ solutions of the system? Explain.

c. Is it possible to change the inequality symbol(s) so that one, but not both, of the points in part (b) is a solution of the system? Explain.



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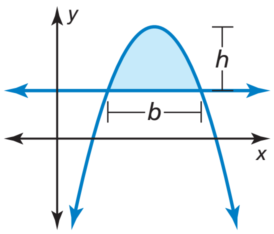
51. **CONNECTING CONCEPTS** The area A of the region bounded by a parabola and a horizontal line can be modeled by $A = \frac{2}{3}bh$, where b and h are as defined in the diagram. Find the area of the region determined by each pair of inequalities.



a. $y \leq -x^2 + 4x$
$y \geq 0$

b. $y \geq x^2 - 4x - 5$
$y \leq 7$

52. **OPEN-ENDED** Design a company logo that is created by the intersection of two quadratic inequalities. Justify your answer.

53. **MP REASONING** A truck that is 11 feet tall and 7 feet wide is traveling under an arch. The arch can be modeled by $y = -0.0625x^2 + 1.25x + 5.75$, where x and y are measured in feet.

a. Will the truck fit under the arch? Explain.

b. What is the maximum width that a truck 11 feet tall can have and still make it under the arch?

c. What is the maximum height that a truck 7 feet wide can have and still make it under the arch?

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54. **THOUGHT PROVOKING**
Consider the system of inequalities below, where a, b, c, and d are real numbers, $a < b$, and $c < d$. Write the solutions of the system, if any, as an inequality.

$$y \leq -ax^2 + c$$
$$y < -bx^2 + d$$

REVIEW & REFRESH

WATCH

In Exercises 55 and 56, graph the function. Label the *x*-intercept(s) and the *y*-intercept.

55. $f(x) = (x + 7)(x - 9)$ 56. $h(x) = -x^2 + 5x - 6$

In Exercises 57 and 58, find the minimum value or maximum value of the function and when the function is increasing and decreasing.

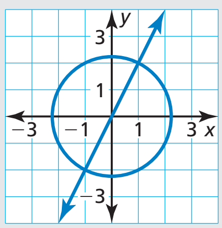
57. $f(x) = -x^2 - 6x - 10$

58. $h(x) = \frac{1}{2}(x + 2)^2 - 1$

In Exercises 59 and 60, graph the inequality.

59. $y \geq \frac{1}{2}x^2 + 3$ 60. $y < -x^2 + 4x - 5$

61. Solve the system of nonlinear equations using the graph.



In Exercises 62 and 63, find the discriminant of the quadratic equation and describe the number and type of solutions of the equation.

62. $2x^2 - x + 7 = 0$ 63. $16x = -x^2 - 10$

In Exercises 64 and 65, solve the equation by completing the square.

64. $x^2 + 12x + 4 = 0$ 65. $4x(x + 6) = -40$

66. Write an equation for the nth term of the geometric sequence. Then find a_8.

n	1	2	3	4
a_n	6	18	54	162

67. **MODELING REAL LIFE** The linear function $y = 50 + 30x$ represents the cost y (in dollars) of renting a picnic pavilion for x hours. The pavilion can be rented for at most 24 hours.

a. Interpret the terms and coefficient in the equation.

b. Find the domain of the function. Is the domain discrete or continuous? Explain.

c. Graph the function using its domain.

In Exercises 68 and 69, solve the system.

68. $3x - y + 2z = 16$
$-2x + 4y + 3z = -2$
$6x + y - z = 0$

69. $2x + 3y + z = 7$
$-x - 5y + 4z = -6$
$6x + 9y + 3z = 14$

70. Write an equation of the line that passes through $(2, -2)$ and is perpendicular to $y = \frac{1}{3}x - 6$.