b. Cardano then showed that a solution of the reduced cubic is given by

$$\sqrt[3]{\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}-\sqrt[3]{-\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}$$

Use Cardano's procedure to solve the equation $x^3 - 6x^2 + 20x - 33 = 0$.

71. Polynomial Function Give an example of a polynomial function that has the given properties, or explain why no such polynomial function exists.

a. A polynomial function of degree 3 that has one rational zero and two irrational zeros

b. A polynomial function of degree 4 that has four irrational zeros

c. A polynomial function of degree 3, with real coefficients, that has no real zeros

d. A polynomial function of degree 4, with real coefficients, that has no real zeros

SECTION 3.5

Graphs of Rational Functions and Their Applications

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A20.

PS1. Simplify: $\dfrac{x^2 - 9}{x^2 - 2x - 15}$ [P.5]

PS2. Evaluate $\dfrac{x + 4}{x^2 - 2x - 5}$ for $x = -1$. [P.1]

PS3. Evaluate $\dfrac{2x^2 + 4x - 5}{x + 6}$ for $x = -3$. [P.1]

PS4. For what values of x does the denominator of $\dfrac{x^2 - x - 5}{2x^3 + x^2 - 15x}$ equal zero? [1.4]

PS5. Determine the degree of the numerator and the degree of the denominator of $\dfrac{x^3 + 3x^2 - 5}{x^2 - 4}$. [P.3]

PS6. Write $\dfrac{x^3 + 2x^2 - x - 11}{x^2 - 2x}$ in $Q(x) + \dfrac{R(x)}{x^2 - 2x}$ form. [3.1]

Vertical and Horizontal Asymptotes

If $P(x)$ and $Q(x)$ are polynomials, then the function F given by

$$F(x) = \frac{P(x)}{Q(x)}$$

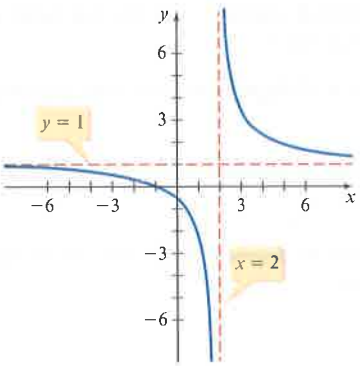
is called a **rational function.** The domain of F is the set of all real numbers except those for which $Q(x) = 0$. For example, let

$$F(x) = \frac{x^2 - x - 5}{2x^3 + x^2 - 15x}$$

Setting the denominator equal to 0, we have

$$2x^3 + x^2 - 15x = 0$$
$$x(2x - 5)(x + 3) = 0$$

The denominator is 0 for $x = 0$, $x = \frac{5}{2}$, and $x = -3$. Thus the domain of F is the set of all real numbers except 0, $\frac{5}{2}$, and -3.



$G(x) = \dfrac{x+1}{x-2}$

Figure 3.23

The graph of $G(x) = \dfrac{x+1}{x-2}$ is given in Figure 3.23. The graph shows that G has the following properties:

- The graph has an x-intercept at $(-1, 0)$ and a y-intercept at $\left(0, -\dfrac{1}{2}\right)$.
- The graph does not exist at $x = 2$.

Note the behavior of the graph as x takes on values that are close to 2 but *less than* 2. Mathematically, we say that "x approaches 2 from the left."

x	1.9	1.95	1.99	1.995	1.999
$G(x)$	−29	−59	−299	−599	−2999

From this table and the graph, it appears that as x approaches 2 from the left, the functional values $G(x)$ decrease without bound.

- In this case, we say that "$G(x)$ approaches negative infinity."

Now observe the behavior of the graph as x takes on values that are close to 2 but *greater than* 2. Mathematically, we say that "x approaches 2 from the right."

x	2.1	2.05	2.01	2.005	2.001
$G(x)$	31	61	301	601	3001

From this table and the graph, it appears that as x approaches 2 from the right, the functional values $G(x)$ increase without bound.

- In this case, we say that "$G(x)$ approaches positive infinity."

Now consider the values of $G(x)$ as x *increases* without bound. The following table gives values of $G(x)$ for selected values of x.

x	1000	5000	10,000	50,000	100,000
$G(x)$	1.00301	1.00060	1.00030	1.00006	1.00003

- As x increases without bound, the values of $G(x)$ become closer to 1.

Now let the values of x *decrease* without bound. The following table gives the values of $G(x)$ for selected values of x.

x	−1000	−5000	−10,000	−50,000	−100,000
$G(x)$	0.997006	0.999400	0.999700	0.999940	0.999970

- As x decreases without bound, the values of $G(x)$ become closer to 1.

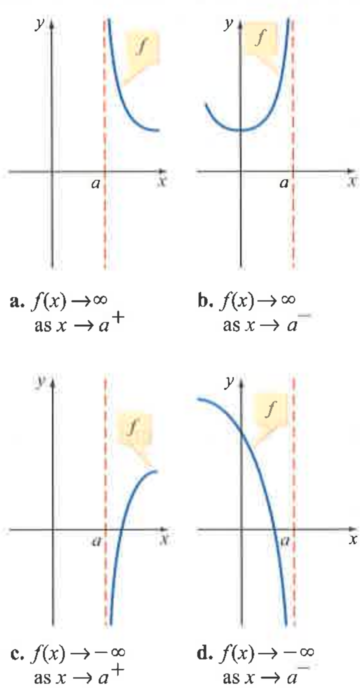


Figure 3.24

When we are discussing functional values that increase or decrease without bound, it is convenient to use mathematical notation. The notation

$$f(x) \to \infty \quad \text{as} \quad x \to a^+$$

means that the functional values $f(x)$ increase without bound as x approaches a from the right. Recall that the symbol ∞ does not represent a real number but is used merely to describe the concept of a variable taking on larger and larger values without bound. See Figure 3.24a.

The notation

$$f(x) \to \infty \quad \text{as} \quad x \to a^-$$

means that the function values $f(x)$ increase without bound as x approaches a from the left. See Figure 3.24b.

Unless otherwise noted, all content on this page is © Cengage Learning

The notation

$$f(x) \to -\infty \quad \text{as} \quad x \to a^+$$

means that the functional values $f(x)$ decrease without bound as x approaches a from the right. See Figure 3.24c.

The notation

$$f(x) \to -\infty \quad \text{as} \quad x \to a^-$$

means that the functional values $f(x)$ decrease without bound as x approaches a from the left. See Figure 3.24d.

Each graph in Figure 3.24 approaches a vertical line through $(a, 0)$ as $x \to a^+$ or as $x \to a^-$. The line is said to be a *vertical asymptote* of the graph.

Definition of a Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function F provided

$$F(x) \to \infty \quad \text{or} \quad F(x) \to -\infty$$

as x approaches a from either the left or right.

In Figure 3.23 on page 308, the line $x = 2$ is a vertical asymptote of the graph of G. Note that the graph of G in Figure 3.23 also approaches the horizontal line $y = 1$ as $x \to \infty$ and as $x \to -\infty$. The line $y = 1$ is a *horizontal asymptote* of the graph of G.

Definition of a Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function F provided

$$F(x) \to b \quad \text{as} \quad x \to \infty \quad \text{or} \quad x \to -\infty$$

Figure 3.25 illustrates some of the ways in which the graph of a rational function may approach its horizontal asymptote. It is common practice to display the asymptotes of the graph of a rational function by using dashed lines. Although a rational function may have several vertical asymptotes, it can have at most one horizontal asymptote. The graph may intersect its horizontal asymptote.

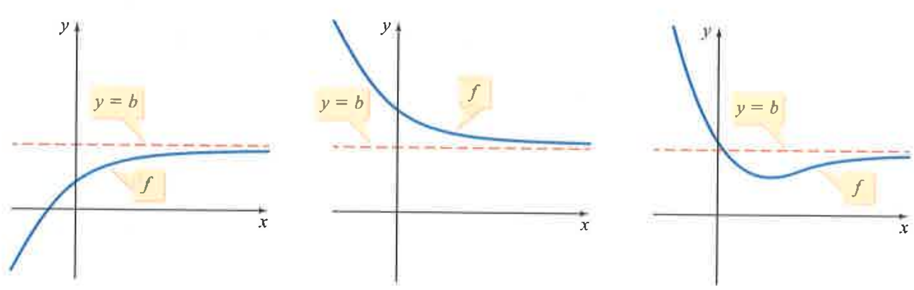


Figure 3.25

$f(x) \to b$ as $x \to \infty$

Geometrically, a line is an asymptote of a curve if the distance between the line and a point $P(x, y)$ on the curve approaches zero as the distance between the origin and the point P increases without bound.

Vertical asymptotes of the graph of a rational function can be found by using the following theorem.

Unless otherwise noted, all content on this page is © Cengage Learning.

Theorem on Vertical Asymptotes

If the real number a is a zero of the denominator $Q(x)$, then the graph of $F(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ *have no common factors,* has the vertical asymptote $x = a$.

EXAMPLE 1 Find the Vertical Asymptotes of a Rational Function

Find the vertical asymptotes of each rational function.

a. $f(x) = \dfrac{x^3}{x^2 + 1}$

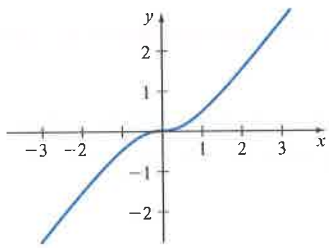
b. $g(x) = \dfrac{x}{x^2 - x - 6}$

Solution

a. To find the vertical asymptotes, determine the real zeros of the denominator. The denominator $x^2 + 1$ has no real zeros, so the graph of f has no vertical asymptotes. See Figure 3.26.

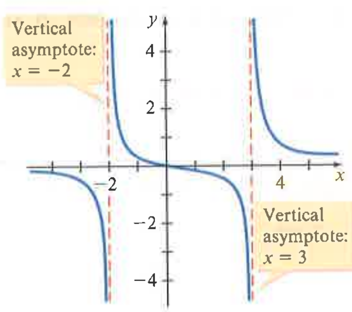
b. The denominator $x^2 - x - 6 = (x - 3)(x + 2)$ has zeros of 3 and -2. The numerator has no common factors with the denominator, so $x = 3$ and $x = -2$ are both vertical asymptotes of the graph of g, as shown in Figure 3.27.

▸ Try Exercise 14, page 320



$f(x) = \dfrac{x^3}{x^2 + 1}$

Figure 3.26



$g(x) = \dfrac{x}{x^2 - x - 6}$

Figure 3.27

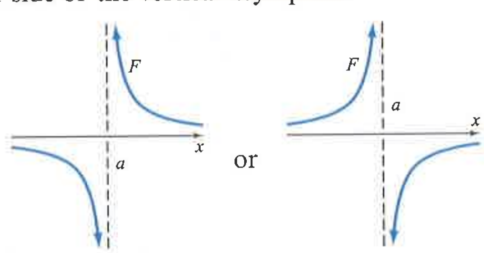
Question • Can a graph of a rational function cross its vertical asymptote?

If the denominator of a rational function is written as a product of linear factors, then the following theorem can be used to determine the manner in which the graph of the rational function approaches its vertical asymptotes.

Behavior Near a Vertical Asymptote Theorem

Let F be a function defined by a rational expression *in simplest form.* If $(x - a)^n$ is the largest power of $(x - a)$ that is a factor of the denominator of F, then the graph of F will have a vertical asymptote at $x = a$.

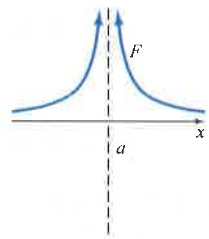
1. If n is *odd*, F will approach ∞ on one side of the vertical asymptote and $-\infty$ on the other side of the vertical asymptote.



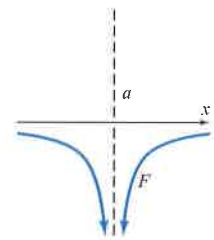
Answer • No. If $x = a$ is a vertical asymptote of a rational function R, then $R(a)$ is undefined.

Unless otherwise noted, all content on this page is © Cengage Learning.

2. If n is *even*, F will approach ∞ on both sides of the vertical asymptote or F will approach $-\infty$ on both sides of the vertical asymptote.



or

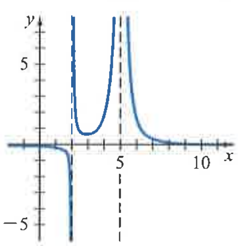


Here is a specific example of how we can use the Behavior Near a Vertical Asymptote Theorem. Consider the rational function

$$F(x) = \frac{x}{(x-2)(x-5)^2}$$

The exponent of $(x - 2)$ is 1. Because 1 is an odd number, the graph of F will approach ∞ on one side of the vertical asymptote $x = 2$ and it will approach $-\infty$ on the other side. The exponent of $(x - 5)$ is 2. Because 2 is an even number, the graph of F will approach ∞ on both sides of the vertical asymptote $x = 5$ or it will approach $-\infty$ on both sides of $x = 5$.

We can examine the factors of F to determine the *exact* manner in which F approaches its vertical asymptotes. For instance, as $x \to 2^+$, the factor $(x - 2)$ approaches 0 through *positive* values and the factor $(x - 5)^2$ approaches $(-3)^2 = 9$ through *positive* values. Thus, as $x \to 2^+$, the denominator of F approaches $0 \cdot 9 = 0$ through positive values and the numerator of F approaches 2 through positive values. From this analysis, we see that as $x \to 2^+$ the quotient of x and $(x - 2)(x - 5)^2$ will be positive and that it will become larger and larger as x gets closer and closer to 2. That is, as $x \to 2^+$, $F \to \infty$. We could use a similar analysis to determine the behavior of F as $x \to 2^-$; however, we have already determined that F approaches ∞ on one side of $x = 2$ and $-\infty$ on the other side. Thus we can conclude that as $x \to 2^-$, $F \to -\infty$. See Figure 3.28.



$F(x) = \dfrac{x}{(x-2)(x-5)^2}$

Figure 3.28

As $x \to 2^-$, $F \to -\infty$.
As $x \to 2^+$, $F \to \infty$.
As $x \to 5^-$, $F \to \infty$.
As $x \to 5^+$, $F \to \infty$.

As $x \to 5^+$, the factor $(x - 2)$ approaches 3 through positive values and the factor $(x - 5)^2$ approaches $(0)^2 = 0$ through positive values. Thus, as $x \to 5^+$, the denominator of F approaches $3 \cdot 0 = 0$ through positive values and the numerator of F approaches 5 through positive values. From this analysis, we see that as $x \to 5^+$ the quotient of x and $(x - 2)(x - 5)^2$ will be positive and that it will become larger and larger as x gets closer and closer to 5. That is, as $x \to 5^+$, $F \to \infty$. We have already determined that F approaches ∞ on both sides of $x = 5$ or F approaches $-\infty$ on both sides of $x = 5$. Thus we can conclude that as $x \to 5^-$, $F \to \infty$. See Figure 3.28.

The following theorem indicates that a horizontal asymptote can be determined by examining the leading terms of the numerator and the denominator of a rational function.

Theorem on Horizontal Asymptotes

Let $$F(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

be a rational function with numerator of degree n and denominator of degree m.

1. If $n < m$, then the x-axis, which is the line given by $y = 0$, is the horizontal asymptote of the graph of F.

(continued)

Unless otherwise noted, all content on this page is © Cengage Learning.

2. If $n = m$, then the line given by $y = a_n/b_m$ is the horizontal asymptote of the graph of F.
3. If $n > m$, then the graph of F has no horizontal asymptote.

EXAMPLE 2 Find the Horizontal Asymptote of a Rational Function

Find the horizontal asymptote of each rational function.

a. $f(x) = \dfrac{2x+3}{x^2+1}$ **b.** $g(x) = \dfrac{4x^2+1}{3x^2}$ **c.** $h(x) = \dfrac{x^3+1}{x-2}$

Solution

a. The degree of the numerator $2x + 3$ is less than the degree of the denominator $x^2 + 1$. By the Theorem on Horizontal Asymptotes, the x-axis is the horizontal asymptote of f. See the graph of f in Figure 3.29.

b. The numerator $4x^2 + 1$ and the denominator $3x^2$ of g are both of degree 2.

By the Theorem on Horizontal Asymptotes, the line $y = \dfrac{4}{3}$ is the horizontal asymptote of g. See the graph of g in Figure 3.30.

c. The degree of the numerator $x^3 + 1$ is larger than the degree of the denominator $x - 2$, so by the Theorem on Horizontal Asymptotes, the graph of h has no horizontal asymptotes.

▶ Try Exercise 24, page 320

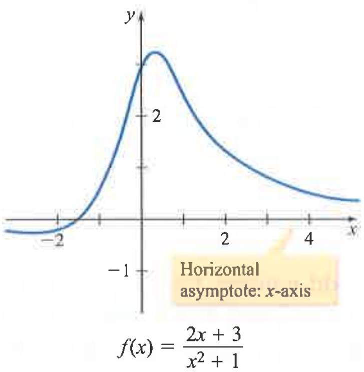


Figure 3.29

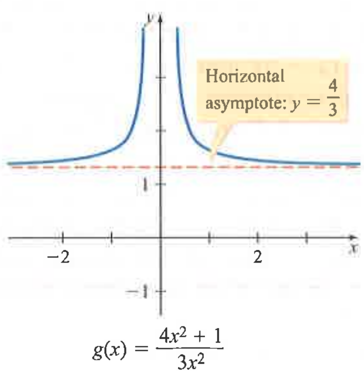


Figure 3.30

The proof of the Theorem on Horizontal Asymptotes uses the technique employed in the following verification. To verify that

$$y = \frac{5x^2+4}{3x^2+8x+7}$$

has a horizontal asymptote of $y = \dfrac{5}{3}$, divide the numerator and the denominator by the largest power of the variable x (x^2 in this case).

$$y = \frac{\dfrac{5x^2+4}{x^2}}{\dfrac{3x^2+8x+7}{x^2}} = \frac{5 + \dfrac{4}{x^2}}{3 + \dfrac{8}{x} + \dfrac{7}{x^2}}, \quad x \neq 0$$

As x increases without bound or decreases without bound, the fractions $\dfrac{4}{x^2}$, $\dfrac{8}{x}$, and $\dfrac{7}{x^2}$ approach zero. Thus

$$y \to \frac{5+0}{3+0+0} = \frac{5}{3} \quad \text{as} \quad x \to \pm\infty$$

Hence the line given by $y = \dfrac{5}{3}$ is a horizontal asymptote of the graph.

Sign Property of Rational Functions

The zeros and vertical asymptotes of a rational function F divide the x-axis into intervals. In each interval, $F(x)$ is positive for all x in the interval or $F(x)$ is negative for all x in the interval. For example, consider the rational function

Unless otherwise noted, all content on this page is © Cengage Learning.

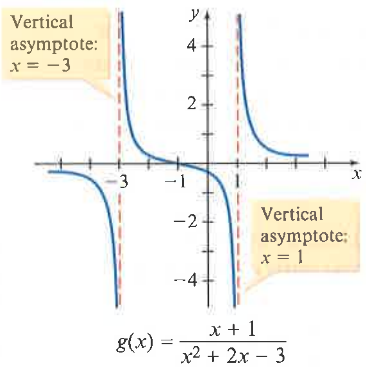


Figure 3.31

$$g(x) = \frac{x + 1}{x^2 + 2x - 3}$$

which has vertical asymptotes of $x = -3$ and $x = 1$ and a zero of -1. These three numbers divide the x-axis into the four intervals $(-\infty, -3)$, $(-3, -1)$, $(-1, 1)$, and $(1, \infty)$. Note in Figure 3.31 that the graph of g is negative for all x such that $x < -3$, positive for all x such that $-3 < x < -1$, negative for all x such that $-1 < x < 1$, and positive for all x such that $x > 1$.

General Graphing Procedure

If $F(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials that have no common factors, then the following general procedure offers useful guidelines for graphing F.

General Procedure for Graphing Rational Functions That Have No Common Factors

1. ***Asymptotes*** Find the real zeros of the denominator $Q(x)$. For each zero a, draw the dashed line $x = a$. Each line is a vertical asymptote of the graph of F. Also graph any horizontal asymptotes.
2. ***Intercepts*** Find the real zeros of the numerator $P(x)$. For each real zero c, plot the point $(c, 0)$. Each such point is an x-intercept of the graph of F. For each x-intercept, use the even and odd powers of $(x - c)$ to determine whether the graph crosses the x-axis at the intercept or intersects but does not cross the x-axis. Also evaluate $F(0)$. Plot $(0, F(0))$, the y-intercept of the graph of F.
3. ***Symmetry*** Use the tests for symmetry to determine whether the graph of the function has symmetry with respect to the y-axis or symmetry with respect to the origin.
4. ***Additional points*** Plot some points that lie in the intervals between and beyond the vertical asymptotes and the x-intercepts.
5. ***Behavior near asymptotes*** If $x = a$ is a vertical asymptote, determine whether $F(x) \to \infty$ or $F(x) \to -\infty$ as $x \to a^-$ and as $x \to a^+$.
6. ***Sketch the graph*** Use all the information obtained in steps 1 through 5 to sketch the graph of F.

EXAMPLE 3 Graph a Rational Function

Sketch a graph of $f(x) = \dfrac{2x^2 - 18}{x^2 + 3}$.

Solution

1. ***Asymptotes*** The denominator $x^2 + 3$ has no real zeros, so the graph of f has no vertical asymptotes. The numerator and denominator both are of degree 2. The leading coefficients are 2 and 1, respectively. By the Theorem on Horizontal Asymptotes, the graph of f has a horizontal asymptote of $y = \dfrac{2}{1} = 2$.

(continued)

Unless otherwise noted, all content on this page is © Cengage Learning.

2. ***Intercepts*** The zeros of the numerator occur when $2x^2 - 18 = 0$ or, solving for x, when $x = -3$ and $x = 3$. Therefore, the x-intercepts are $(-3, 0)$ and $(3, 0)$. The factored numerator is $2(x + 3)(x - 3)$. Each linear factor has an exponent of 1, an odd number. Thus the graph crosses the x-axis at its x-intercepts. To find the y-intercept, evaluate f when $x = 0$. This gives $y = -6$. Therefore, the y-intercept is $(0, -6)$.

3. ***Symmetry*** Below we show that $f(-x) = f(x)$, which means that f is an even function and therefore its graph is symmetric with respect to the y-axis.

$$f(-x) = \frac{2(-x)^2 - 18}{(-x)^2 + 3} = \frac{2x^2 - 18}{x^2 + 3} = f(x)$$

4. ***Additional points*** The intervals determined by the x-intercepts are $x < -3$, $-3 < x < 3$, and $x > 3$. Generally, it is necessary to determine points in all intervals. However, because f is an even function, its graph is symmetric with respect to the y-axis. The following table lists a few points for $x > 0$. Symmetry can be used to locate corresponding points for $x < 0$.

x	1	2	6
$f(x)$	-4	$-\frac{10}{7} \approx -1.43$	$\frac{18}{13} \approx 1.38$

5. ***Behavior near asymptotes*** As x increases or decreases without bound, $f(x)$ approaches the horizontal asymptote $y = 2$.

   To determine whether the graph of f intersects the horizontal asymptote at any point, solve the equation $f(x) = 2$. There are no solutions of $f(x) = 2$ because

$$\frac{2x^2 - 18}{x^2 + 3} = 2 \quad \text{implies} \quad 2x^2 - 18 = 2x^2 + 6 \quad \text{implies} \quad -18 = 6$$

   This is not possible. Thus the graph of f does not intersect the horizontal asymptote but approaches it from below as x increases or decreases without bound.

6. ***Sketch the graph*** Use the summary in Table 3.3 to sketch the graph. The completed graph is shown in Figure 3.32.

Table 3.3

Vertical asymptote	None
Horizontal asymptote	$y = 2$
x-intercepts	Crosses at $(-3, 0)$ and $(3, 0)$
y-intercept	$(0, -6)$
Additional points	$(1, -4)$, $(2, -1.43)$, $(6, 1.38)$

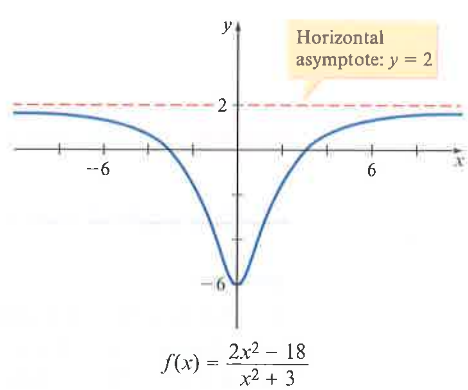


Figure 3.32

▶ Try Exercise 34, page 321

Unless otherwise noted, all content on this page is © Cengage Learning.

EXAMPLE 4 Graph a Rational Function

Sketch a graph of $h(x) = \dfrac{x^2 + 1}{x^2 + x - 2}$.

Solution

1. ***Asymptotes*** The denominator $x^2 + x - 2 = (x + 2)(x - 1)$ has zeros -2 and 1; because there are no common factors of the numerator and the denominator, the lines $x = -2$ and $x = 1$ are vertical asymptotes.

   The numerator and denominator both are of degree 2. The leading coefficients of the numerator and denominator are both 1. Thus h has the horizontal asymptote $y = \dfrac{1}{1} = 1$.

2. ***Intercepts*** The numerator $x^2 + 1$ has no real zeros, so the graph of h has no x-intercepts. Because $h(0) = -0.5$, h has the y-intercept $(0, -0.5)$.

3. ***Symmetry*** By applying the tests for symmetry, we can determine that the graph of h is not symmetric with respect to the origin or with respect to the y-axis.

4. ***Additional points*** The intervals determined by the vertical asymptotes are $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$. Plot a few points from each interval.

x	-5	-3	-1	0.5	2	3	4
$h(x)$	$\frac{13}{9}$	$\frac{5}{2}$	-1	-1	$\frac{5}{4}$	1	$\frac{17}{18}$

   The graph of h will intersect the horizontal asymptote $y = 1$ exactly once. This can be determined by solving the equation $h(x) = 1$.

$$\frac{x^2 + 1}{x^2 + x - 2} = 1$$

$$x^2 + 1 = x^2 + x - 2 \qquad \text{• Multiply both sides by } x^2 + x - 2.$$

$$1 = x - 2$$

$$3 = x$$

   The only solution is $x = 3$. Therefore, the graph of h intersects the horizontal asymptote at $(3, 1)$.

5. ***Behavior near asymptotes*** The manner in which h approaches its vertical asymptote $x = -2$ as $x \to -2^-$ can be determined by examining the numerator and the factors of the denominators as $x \to -2^-$. For instance, as $x \to -2^-$, the numerator $x^2 + 1$ approaches $(-2)^2 + 1 = 5$ through *positive* values. As $x \to -2^-$, the $(x + 2)$ factor in the denominator approaches 0 through *negative* values and the $(x - 1)$ factor in the denominator approaches -3 through *negative* values. Thus, as $x \to -2^-$, the denominator of h approaches $0 \cdot (-3) = 0$ through *positive* values. From this analysis, we see that as $x \to -2^-$, the quotient of the numerator and the denominator will be positive, and it will become larger and larger as $x \to -2^-$. That is,

$$h(x) \to \infty \text{ as } x \to -2^-$$

   We could use a similiar analysis to determine the behavior of h as $x \to -2^+$. However, the Behavior Near a Vertical Asymptote Theorem

(continued)

Unless otherwise noted, all content on this page is © Cengage Learning

indicates that h approaches ∞ on one side of the vertical asymptote $x = -2$, and h approaches $-\infty$ on the other side. Thus we know that

$$h(x) \to -\infty \text{ as } x \to -2^+$$

A similar analysis can be used to show that

$$h(x) \to -\infty \text{ as } x \to 1^-$$
$$h(x) \to \infty \text{ as } x \to 1^+$$

6. ***Sketch the graph*** Use the summary in Table 3.4 to sketch the graph. See Figure 3.33.

Table 3.4

Vertical asymptote	$x = -2, x = 1$
Horizontal asymptote	$y = 1$
x-intercepts	None
y-intercept	$(0, -0.5)$
Additional points	$(-5, 1.4), (-3, 2.5), (-1, -1), (0.5, -1), (2, 1.25), (3, 1), (4, 0.94)$

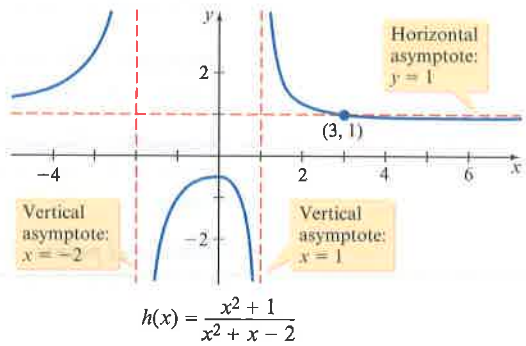


Figure 3.33

▶ **Try Exercise 50, page 321**

Slant Asymptotes

Some rational functions have an asymptote that is neither vertical nor horizontal but slanted.

Definition of a Slant Asymptote

The line given by $y = mx + b$, $m \neq 0$, is a **slant asymptote** of the graph of a function F provided $F(x) \to mx + b$ as $x \to \infty$ or $x \to -\infty$.

The following theorem can be used to determine which rational functions have a slant asymptote.

Theorem on Slant Asymptotes

The rational function given by $F(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ have no common factors, has a slant asymptote if the degree of $P(x)$ is 1 greater than the degree of $Q(x)$.

To find the slant asymptote, divide $P(x)$ by $Q(x)$ and write $F(x)$ in the form

$$F(x) = \frac{P(x)}{Q(x)} = (mx + b) + \frac{r(x)}{Q(x)}$$

Unless otherwise noted, all content on this page is © Cengage Learning.

where the degree of $r(x)$ is less than the degree of $Q(x)$. Because

$$\frac{r(x)}{Q(x)} \to 0 \quad \text{as} \quad x \to \pm\infty$$

we know that $F(x) \to mx + b$ as $x \to \pm\infty$.

The line represented by $y = mx + b$ is the slant asymptote of the graph of F.

EXAMPLE 5 Find the Slant Asymptote of a Rational Function

Find the slant asymptote of $f(x) = \dfrac{2x^3 + 5x^2 + 1}{x^2 + x + 3}$.

Solution

Because the degree of the numerator $2x^3 + 5x^2 + 1$ is exactly 1 larger than the degree of the denominator $x^2 + x + 3$ and f is in simplest form, f has a slant asymptote. To find the asymptote, divide $2x^3 + 5x^2 + 1$ by $x^2 + x + 3$.

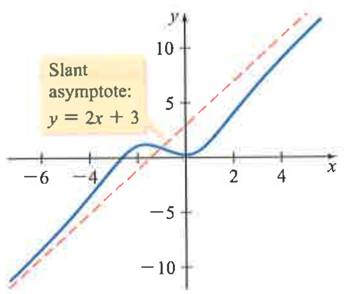
$$\begin{array}{r} 2x + 3 \\ x^2 + x + 3\overline{)\,2x^3 + 5x^2 + 0x + 1} \\ \underline{2x^3 + 2x^2 + 6x} \\ 3x^2 - 6x + 1 \\ \underline{3x^2 + 3x + 9} \\ -9x - 8 \end{array}$$

Therefore,

$$f(x) = \frac{2x^3 + 5x^2 + 1}{x^2 + x + 3} = 2x + 3 + \frac{-9x - 8}{x^2 + x + 3}$$

and the line given by $y = 2x + 3$ is the slant asymptote for the graph of f. Figure 3.34 shows the graph of f and its slant asymptote.

▶ **Try Exercise 56, page 321**



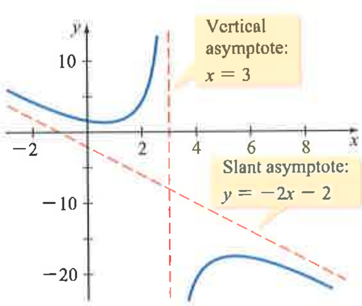
$f(x) = \dfrac{2x^3 + 5x^2 + 1}{x^2 + x + 3}$

Figure 3.34

The function f in Example 5 does not have a vertical asymptote because the denominator $x^2 + x + 3$ does not have any real zeros. However, the function

$$g(x) = \frac{2x^2 - 4x + 5}{3 - x}$$

has both a slant asymptote and a vertical asymptote. The vertical asymptote is $x = 3$, and the slant asymptote is $y = -2x - 2$. Figure 3.35 shows the graph of g and its asymptotes.



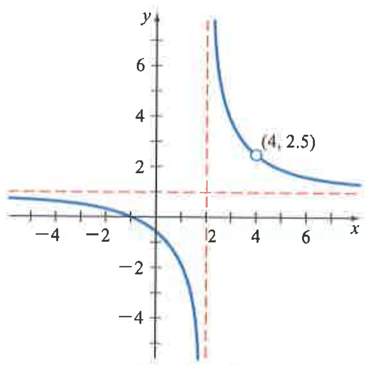
$g(x) = \dfrac{2x^2 - 4x + 5}{3 - x}$

Figure 3.35

Graphing Rational Functions That Have a Common Factor

If a rational function has a numerator and denominator that have a common factor, then you should reduce the rational function to simplest form before you apply the general procedure for sketching the graph of a rational function.

Unless otherwise noted, all content on this page is © Cengage Learning.



$f(x) = \frac{x^2 - 3x - 4}{x^2 - 6x + 8}$

Figure 3.36

EXAMPLE 6 Graph a Rational Function That Has a Common Factor

Sketch the graph of $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 6x + 8}$.

Solution

Factor the numerator and denominator to obtain

$$f(x) = \frac{x^2 - 3x - 4}{x^2 - 6x + 8} = \frac{(x+1)(x-4)}{(x-2)(x-4)}, \quad x \neq 2, x \neq 4$$

Thus for all x values other than $x = 4$, the graph of f is the same as the graph of

$$G(x) = \frac{x+1}{x-2}$$

Figure 3.23 on page 308 shows a graph of G. The graph of f will be the same as this graph, except that it will have an open circle at (4, 2.5) to indicate that it is undefined at $x = 4$. See the graph of f in Figure 3.36. The height of the open circle was found by evaluating the resulting reduced rational function $G(x) = \dfrac{x+1}{x-2}$ at $x = 4$.

▶ **Try Exercise 78, page 321**

Question • Does $F(x) = \dfrac{x^2 - x - 6}{x^2 - 9}$ have a vertical asymptote at $x = 3$?

Applications of Rational Functions

EXAMPLE 7 Determine the Average Speed for a Trip

Jordan averages 30 miles per hour during 12 miles of city driving. For the remainder of the trip, she drives on a highway at a constant rate of 60 miles per hour. Her average speed for the entire trip is given by

$$s(x) = \frac{12 + x}{\frac{2}{5} + \frac{1}{60}x}$$

where x is the number of miles she drives on the highway.

a. How far will she need to drive on the highway to bring her average speed for the entire trip up to 50 miles per hour?

b. Determine the horizontal asymptote of the graph of s, and explain the meaning of the horizontal asymptote in the context of this application.

Answer • No. $F(x) = \dfrac{x^2 - x - 6}{x^2 - 9} = \dfrac{(x-3)(x+2)}{(x-3)(x+3)} = \dfrac{x+2}{x+3}, x \neq 3$. As $x \to 3$, $F(x) \to \dfrac{5}{6}$.

Unless otherwise noted, all content on this page is © Cengage Learning.

Algebraic Solution

a.

$$\frac{12 + x}{\frac{2}{5} + \frac{1}{60}x} = 50 \qquad \text{• Set } s(x) \text{ equal to 50.}$$

$$12 + x = 50\left(\frac{2}{5} + \frac{1}{60}x\right) \qquad \text{• Multiply each side by } \left(\frac{2}{5} + \frac{1}{60}x\right).$$

$$12 + x = 20 + \frac{5}{6}x \qquad \text{• Simplify.}$$

$$x - \frac{5}{6}x = 20 - 12 \qquad \text{• Solve for } x.$$

$$\frac{1}{6}x = 8$$

$$x = 48$$

Jordan needs to drive 48 miles at 60 miles per hour to bring her average speed up to 50 miles per hour. See Figure 3.37.

b. The numerator and denominator of s are both of degree 1. The leading coefficient of the numerator is 1, and the leading coefficient of the denominator is 3/60. Thus the graph of s has a horizontal asymptote of

$$y = \frac{1}{\left(\frac{1}{60}\right)} = 60$$

The horizontal asymptote of the graph of s is the line $y = 60$. As Jordan continues to drive at 60 miles per hour, her average speed for the entire trip will approach 60 miles per hour. See Figure 3.38.

Visualize the Solution

a. The following graph shows that $s(x) = 50$ when $x = 48$.

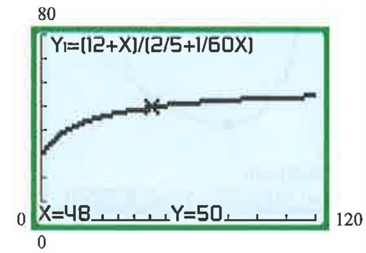


Figure 3.37

b. The graphs of s and $y = 60$ are shown in the same window.

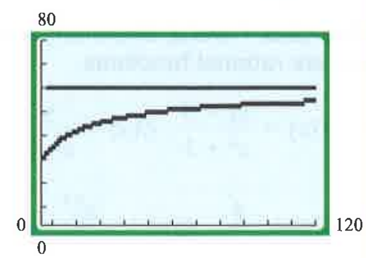


Figure 3.38

▶ **Try Exercise 84, page 322**

EXAMPLE 8 Solve an Application

A cylindrical soft drink can is to be constructed so that it will have a volume of 21.6 cubic inches. See Figure 3.39.

a. Write the total surface area A of the can as a function of r, where r is the radius of the can in inches.

b. Estimate the value of r (to the nearest tenth of an inch) that produces the minimum surface area.

Figure 3.39

Solution

a. The formula for the volume of a cylinder is $V = \pi r^2 h$, where r is the radius and h is the height. Because we are given that the volume is 21.6 cubic inches, we have

$$21.6 = \pi r^2 h$$

$$\frac{21.6}{\pi r^2} = h \qquad \text{• Solve for } h.$$

(continued)

Integrating Technology

A Web applet is available to explore the relationship between the radius of a cylinder with a given volume and the surface area of the cylinder. This applet, CYLINDER, can be found at www.cengagebrain.com

Unless otherwise noted, all content on this page is © Cengage Learning.

$y = \dfrac{2\pi x^3 + 43.2}{x}$

Figure 3.40

The surface area of the cylinder is given by

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 2\pi r^2 + 2\pi r\left(\frac{21.6}{\pi r^2}\right) \qquad \text{• Substitute for } h.$$

$$A = 2\pi r^2 + \frac{2(21.6)}{r} \qquad \text{• Simplify.}$$

$$A = \frac{2\pi r^3 + 43.2}{r} \qquad (1)$$

b. Use Equation (1) with $y = A$ and $x = r$ and a graphing utility to determine that A is a minimum when $r \approx 1.5$ inches. See Figure 3.40.

▶ **Try Exercise 88, page 322**

EXERCISE SET 3.5

Concept Check

In Exercises 1 and 2, determine which of the given functions are rational functions.

1. $F(x) = \dfrac{3x - 5}{x^2 + 3}$ $\quad G(x) = \dfrac{\sqrt{x}}{x^3 - 2}$ $\quad H(x) = \dfrac{1}{2x - 3}$

2. $J(x) = \dfrac{4}{|x|}$ $\quad K(x) = \dfrac{x^{0.5}}{x^2 - 3}$ $\quad L(x) = \dfrac{2x}{3x^3 - x^2 - 1}$

In Exercises 3 to 12, determine the domain of the rational function.

3. $F(x) = \dfrac{1}{x}$

4. $F(x) = \dfrac{2}{x - 3}$

5. $F(x) = \dfrac{x^2 - 3}{x^2 + 1}$

6. $F(x) = \dfrac{x^3 + 4}{x^2 - 25}$

7. $F(x) = \dfrac{2x - 1}{2x^2 - 15x + 18}$

8. $F(x) = \dfrac{3x - 2}{4x^2 - 27x + 18}$

9. $F(x) = \dfrac{2x^2}{x^3 - 4x^2 - 12x}$

10. $F(x) = \dfrac{3x^2}{x^2 - 5}$

11. $F(x) = \dfrac{x^2 + 1}{x^3 - 8}$

12. $F(x) = \dfrac{x^3 + x + 5}{x^4 - 8x^2 - 9}$

In Exercises 13 to 22, find all vertical asymptotes of each rational function.

13. $F(x) = \dfrac{2x - 1}{x^2 + 3x}$

14. $F(x) = \dfrac{3x^2 + 5}{x^2 - 4}$

15. $F(x) = \dfrac{x^2 + 11}{6x^2 - 5x - 4}$

16. $F(x) = \dfrac{3x - 5}{x^3 - 8}$

17. $F(x) = \dfrac{5x^2 - 3}{4x^3 - 25x^2 + 6x}$

18. $F(x) = \dfrac{5x}{x^4 - 81}$

19. $F(x) = \dfrac{3x - 5}{x^2 - 2}$

20. $F(x) = \dfrac{4x^2 + 1}{2x^3 - 3x^2 - 20x}$

21. $F(x) = \dfrac{2x}{3x^2 + 5}$

22. $F(x) = \dfrac{4x^3 + 1}{5x^2}$

In Exercises 23 to 32, find the horizontal asymptote of each rational function.

23. $F(x) = \dfrac{4x^2 + 1}{x^2 + x + 1}$

24. $F(x) = \dfrac{3x^3 - 27x^2 + 5x - 11}{x^5 - 2x^3 + 7}$

25. $F(x) = \dfrac{15{,}000x^3 + 500x - 2000}{700 + 500x^3}$

26. $F(x) = 6000\left(1 - \dfrac{25}{(x + 5)^2}\right)$

27. $F(x) = \dfrac{4x^2 - 11x + 6}{4 - x + \frac{1}{3}x^2}$

28. $F(x) = \dfrac{(2x - 3)(3x + 4)}{(1 - 2x)(3 - 5x)}$

Indicates Try It Exercises

Unless otherwise noted, all content on this page is © Cengage Learning.

29. $F(x) = \dfrac{2x^2 + x + 7}{\frac{1}{3}x^2 + 5x - 4}$

30. $F(x) = \dfrac{3x^3 - 5x - 1}{2x^2 - 3x + 11}$

31. $F(x) = \dfrac{-3x^2 + 8}{x^3 + x^2 - 5x + 2}$

32. $F(x) = \dfrac{1}{x}$

In Exercises 33 to 54, determine the vertical and horizontal asymptotes and sketch the graph of the rational function *F*. Label all intercepts and asymptotes.

33. $F(x) = \dfrac{1}{x + 4}$

34. $F(x) = \dfrac{1}{x - 2}$

35. $F(x) = \dfrac{-4}{x - 3}$

36. $F(x) = \dfrac{-3}{x + 2}$

37. $F(x) = \dfrac{4}{x}$

38. $F(x) = \dfrac{-4}{x}$

39. $F(x) = \dfrac{x}{x + 4}$

40. $F(x) = \dfrac{x}{x - 2}$

41. $F(x) = \dfrac{x + 4}{2 - x}$

42. $F(x) = \dfrac{x + 3}{1 - x}$

43. $F(x) = \dfrac{1}{x^2 - 9}$

44. $F(x) = \dfrac{-2}{x^2 - 4}$

45. $F(x) = \dfrac{1}{x^2 + 2x - 3}$

46. $F(x) = \dfrac{1}{x^2 - 2x - 8}$

47. $F(x) = \dfrac{x^2}{x^2 + 4x + 4}$

48. $F(x) = \dfrac{2x^2}{x^2 - 1}$

49. $F(x) = \dfrac{10}{x^2 + 2}$

50. $F(x) = \dfrac{x^2}{x^2 - 6x + 9}$

51. $F(x) = \dfrac{2x^2 - 2}{x^2 - 9}$

52. $F(x) = \dfrac{6x^2 - 5}{2x^2 + 6}$

53. $F(x) = \dfrac{x^2 + x + 4}{x^2 + 2x - 1}$

54. $F(x) = \dfrac{2x^2 - 14}{x^2 - 6x + 5}$

In Exercises 55 to 64, find the slant asymptote of each rational function.

55. $F(x) = \dfrac{3x^2 + 5x - 1}{x + 4}$

56. $F(x) = \dfrac{x^3 - 2x^2 + 3x + 4}{x^2 - 3x + 5}$

57. $F(x) = \dfrac{x^3 - 1}{x^2}$

58. $F(x) = \dfrac{4000 + 20x + 0.0001x^2}{x}$

59. $F(x) = \dfrac{-4x^2 + 15x + 18}{x - 5}$

60. $F(x) = \dfrac{-x^4 - 2x^3 - 3x^2 + 4x - 1}{x^3 - 1}$

61. $F(x) = \dfrac{x^5 - 2}{x^4 + 1}$

62. $F(x) = \dfrac{6x^3 - 2x^2 + 7x - 11}{2x^2 + x + 1}$

63. $F(x) = \dfrac{\frac{1}{3}x^2 - 1}{2x + 3}$

64. $F(x) = \dfrac{3 - 2x - 5x^2}{6 + x}$

In Exercises 65 to 74, determine the vertical and slant asymptotes and sketch the graph of the rational function *F*.

65. $F(x) = \dfrac{x^2 - 4}{x}$

66. $F(x) = \dfrac{x^2 + 10}{2x}$

67. $F(x) = \dfrac{x^2 - 3x - 4}{x + 3}$

68. $F(x) = \dfrac{x^2 - 4x - 5}{2x + 5}$

69. $F(x) = \dfrac{2x^2 + 5x + 3}{x - 4}$

70. $F(x) = \dfrac{4x^2 - 9}{x + 3}$

71. $F(x) = \dfrac{x^2 - x}{x + 2}$

72. $F(x) = \dfrac{x^2 + x}{x - 1}$

73. $F(x) = \dfrac{x^3 + 1}{x^2 - 4}$

74. $F(x) = \dfrac{x^3 - 1}{3x^2}$

In Exercises 75 to 82, sketch the graph of the rational function *F*.

75. $F(x) = \dfrac{x^2 + x}{x + 1}$

76. $F(x) = \dfrac{x^2 - 3x}{x - 3}$

77. $F(x) = \dfrac{2x^3 + 4x^2}{2x + 4}$

78. $F(x) = \dfrac{x^2 - x - 12}{x^2 - 2x - 8}$

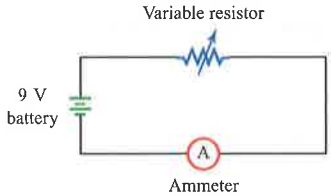
79. $F(x) = \dfrac{-2x^3 + 6x}{2x^2 - 6x}$

80. $F(x) = \dfrac{x^3 + 3x^2}{x(x + 3)(x - 1)}$

81. $F(x) = \dfrac{x^2 - 3x - 10}{x^2 + 4x + 4}$

82. $F(x) = \dfrac{2x^2 + x - 3}{x^2 - 2x + 1}$

83. **Electrical Current** A variable resistor, an ammeter, and a 9-volt battery are connected as shown in the following diagram.



The internal resistance of the ammeter is 4.5 ohms. The current I, in amperes, through the ammeter is given by

$$I(x) = \frac{9}{x + 4.5}$$

where x is the resistance, in ohms, provided by the variable resistor.

a. Find the current through the ammeter when the variable resistor has a resistance of 3 ohms.

b. Determine the resistance of the variable resistor when the current through the ammeter is 0.24 ampere.

c. Determine the horizontal asymptote of the graph of I, and explain the meaning of the horizontal asymptote in the context of this application.

84. **Average Speed** During the first 30 miles of city driving, you average 40 miles per hour. For the remainder of the trip, you drive on a highway at a constant rate of 70 miles per hour. Your average speed for the entire trip is given by

$$r(x) = \frac{30 + x}{\frac{3}{4} + \frac{1}{70}x}$$

where x is the number of miles you have driven on the highway.

a. How far will you need to drive on the highway to bring your average speed up to 60 miles per hour?

b. Determine the horizontal asymptote of the graph of r, and write a sentence that explains the meaning of the horizontal asymptote in the context of this application.

85. **Average Cost of Golf Balls** The cost, in dollars, of producing x golf balls is given by

$$C(x) = 0.43x + 76{,}000$$

The average cost per golf ball is given by

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{0.43x + 76{,}000}{x}$$

a. Find the average cost per golf ball of producing 1000, 10,000, and 100,000 golf balls.

b. What is the equation of the horizontal asymptote of the graph of $\overline{C}$? Explain the significance of the horizontal asymptote as it relates to this application.

86. **Average Cost of Blu-ray Players** The cost, in dollars, of producing x Blu-ray players is given by

$$C(x) = 0.001x^2 + 54x + 175{,}000$$

The average cost per Blu-ray player is given by

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{0.001x^2 + 54x + 175{,}000}{x}$$

a. Find the average cost per Blu-ray player of producing 1000, 10,000, and 100,000 Blu-ray players.

b. What is the minimum average cost per Blu-ray player? How many Blu-ray players should be produced to minimize the average cost per Blu-ray player?

87. **Desalinization** The cost C, in dollars, to remove $p\%$ of the salt in a tank of seawater is given by

$$C(p) = \frac{2000p}{100 - p}, \quad 0 \le p < 100$$

a. Find the cost of removing 40% of the salt.

b. Find the cost of removing 80% of the salt.

c. Sketch the graph of C.

88. **Production Costs** The cost, in dollars, of producing x cell phones is given by

$$C(x) = 0.0006x^2 + 9x + 401{,}000$$

The average cost per cell phone is

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{0.0006x^2 + 9x + 401{,}000}{x}$$

a. Find the average cost per cell phone when 1000, 10,000, and 100,000 phones are produced.

b. What is the minimum average cost per cell phone? How many cell phones should be produced to minimize the average cost per phone?

89. **Average Wait Time** Rational functions can be used to model the expected average time a customer will wait in line, dependent on conditions such as the customer arrival rate, the average time it takes to serve a customer, and the number of service lanes that are open. For instance, the rational function

$$W(x) = \frac{7.5}{x(x - 15)}, \text{ where } 15 < x \le 40$$

models the average expected wait time, in hours, for customers of a convenience store, under the following conditions.

- The store has one cashier.
- The customers arrive at a rate of 15 people per hour.
- The cashier can serve x customers per hour, where $x > 15$.

a. Use W to determine the average expected wait time for a cashier who can serve 25 customers per hour. State your answer in hours and then convert that result to minutes.

b. How many customers must the cashier serve per hour, to reduce the average expected wait time to 0.0125 hour (45 seconds)? (*Hint:* Replace $W(x)$ with 0.0125 and solve the resulting equation for x.) Round your answer to the nearest tenth.

Unless otherwise noted, all content on this page is © Cengage Learning.

c. Explain, in the context of this application, why W decreases as x increases.

90. **Medication Model** The rational function

$$M(t) = \frac{0.5t + 400}{0.04t^2 + 1}$$

models the number of milligrams of medication in the bloodstream of a patient t hours after 400 milligrams of the medication have been injected into the patient's bloodstream.

a. Find $M(5)$ and $M(10)$. Round to the nearest milligram.

b. What will M approach as $t \to \infty$?

91. **Minimizing Surface Area** A cylindrical soft drink can is to be made so that it will have a volume of 354 milliliters. If r is the radius of the can in centimeters, then the total surface area A in square centimeters of the can is given by the rational function

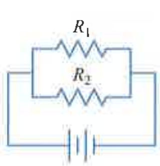
$$A(r) = \frac{2\pi r^3 + 708}{r}$$

a. Graph A and use the graph to estimate (to the nearest tenth of a centimeter) the value of r that produces the minimum value of A.

b. Does the graph of A have a slant asymptote?

c. Explain the meaning of the following statement as it applies to the graph of A.
As $r \to \infty, A \to 2\pi r^2$.

92. **Resistors in Parallel** The electronic circuit below shows two resistors connected in parallel.



One resistor has a resistance of R_1 ohms, and the other has a resistance of R_2 ohms. The total resistance for the circuit, measured in ohms, is given by the formula

$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$

Assume that R_1 has a fixed resistance of 10 ohms.

a. Compute R_T for $R_2 = 2$ ohms and for $R_2 = 20$ ohms.

b. Find R_2 when $R_T = 6$ ohms.

c. What happens to R_T as $R_2 \to \infty$?

93. Determine the point at which the graph of

$$F(x) = \frac{2x^2 + 3x + 4}{x^2 + 4x + 7}$$

intersects its horizontal asymptote.

Enrichment Exercises

94. **Parabolic Asymptotes** It can be shown that the rational function $F(x) = R(x)/S(x)$, where $R(x)$ and $S(x)$ have no common factors, has a parabolic asymptote provided the degree of $R(x)$ is 2 *greater than* the degree of $S(x)$. For instance, the rational function

$$F(x) = \frac{x^3 + 2}{x + 1}$$

has a parabolic asymptote given by $y = x^2 - x + 1$.

a. Use a graphing utility to graph F and the parabola given by $y = x^2 - x + 1$ in the same viewing window. Does the parabola appear to be an asymptote for the graph of F? Explain.

b. Write a paragraph that explains how to determine the equation of the parabolic asymptote for a rational function $F(x) = R(x)/S(x)$, where $R(x)$ and $S(x)$ have no common factors and the degree of $R(x)$ is 2 greater than the degree of $S(x)$.

c. What is the equation of the parabolic asymptote for the rational function $G(x) = \dfrac{x^4 + x^2 + 2}{x^2 - 1}$? Use a graphing utility to graph G and the parabolic asymptote in the same viewing window. Does the parabola appear to be an asymptote for the graph of G?

d. Create a rational function that has $y = x^2 + x + 2$ as its parabolic asymptote. Explain the procedure you used to create your rational function.

In Exercises 95 and 96, create a rational function whose graph has the given characteristics.

95. Is symmetric to the y-axis, has vertical asymptotes at $x = 3$ and $x = -3$, has a horizontal asymptote at $y = 2$, and passes through the origin

96. Has a vertical asymptote at $x = 5$, has $y = x - 3$ as a slant asymptote, and intersects the x-axis at $(4, 0)$

Unless otherwise noted, all content on this page is © Cengage Learning.

Scan the following QR code to access WolframAlpha on a mobile device.

www.wolframalpha.com

Exploring Concepts with Technology

Zeros, Cubic and Quartic Regressions, and WolframAlpha

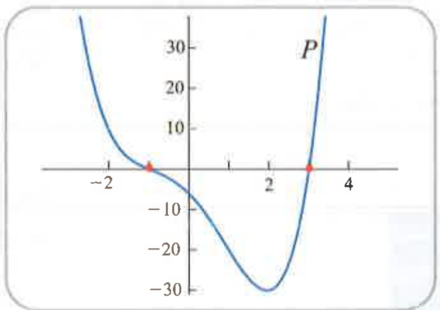
WolframAlpha can be used to find the zeros of polynomial functions and to determine the cubic and quartic regression functions for a data set.

Find Zeros of Polynomial Functions

Recall that a zero of a function is a domain value x for which the value of the function is 0. The idea behind finding the zeros of a polynomial function P is to find the solutions of the equation $P(x) = 0$. For instance, suppose you wish to find the zeros of $P(x) = x^4 - 5x^2 - 10x - 6 = 0$. Click in WolframAlpha's input field and enter the equation shown below.

```
x^4 - 5x^2 - 10x - 6 = 0
```

Now click on the equal sign icon to view a graph of P and a list of its zeros.



Real zeros of P:
$x = -1, x = 3$

Complex zeros of P:
$x = -1 - i, x = -1 + i$

WolframAlpha attempts to provide exact solutions. However, there are no formulas similar to the quadratic formula that yield exact solutions for all fifth-degree (or higher) polynomial equations. In these situations WolframAlpha provides approximate decimal solutions. For instance, for the equation

$$x^5 - 3x^2 + 2x - 1 = 0$$

WolframAlpha lists approximate decimal values for the one real solution and the four complex solutions as:

Real solution:
$x \approx 1.2693$

Complex solutions:
$x \approx -0.948152 - 1.25849i$
$x \approx -0.948152 + 1.25849i$
$x \approx 0.3135 - 0.468011i$
$x \approx 0.3135 + 0.468011i$

Find Cubic and Quartic Regression Functions

To find the cubic regression function for the data set

$$\{(1, 4), (3, 2), (7, 5.5), (10, 7.2), (14, 3.7)\}$$

enter the text shown below

```
cubic fit {(1, 4),(3, 2),(7, 5.5),(10, 7.2),(14, 3.7)}
```

Click on the equal sign icon to display

$$P(x) = -0.0245083x^3 + 0.50779x^2 - 2.46101x + 5.82431$$

as the cubic regression function with $R^2 = 0.973261$. A graph of the regression function and a scatter plot of the data is also provided. To determine a quartic regression function for the data, just change "cubic fit" to "quartic fit."

Unless otherwise noted, all content on this page is © Cengage Learning.