3.5 Solving Nonlinear Systems of Equations

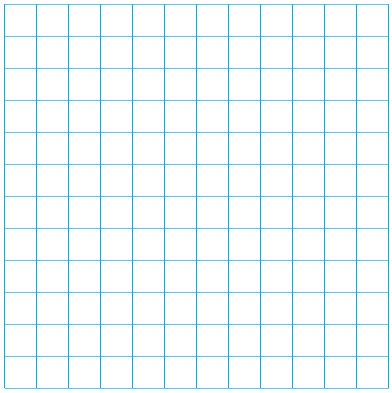
Learning Target Solve nonlinear systems graphically and algebraically.

Success Criteria
- I can describe what a nonlinear system of equations is.
- I can solve nonlinear systems using graphing, substitution, or elimination.
- I can solve quadratic equations by graphing each side of the equation.

EXPLORE IT! Solving Systems of Equations

Work with a partner.

a. Graph the equation $x^2 + y^2 = 4$. Make several observations about the graph.



b. How many intersection points can the graphs of a line and a circle have? Use graphs to support your answers. What do the intersection points represent?

Math Practice

Look for Structure
How did you determine the ordered pairs used to graph the equation $x^2 + y^2 = 4$?

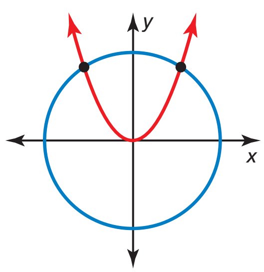
c. Consider the system below. Can you use a graph to solve the system? Explain.

$$x^2 + y^2 = 4$$
$$y = -\frac{1}{2}x + 1$$

d. MP **CHOOSE TOOLS** Find the points of intersection of the graphs of the equations in part (c). Explain your method.

e. Write the equation of a line that intersects the graph of $x^2 + y^2 = 4$ at only one point. Explain how you found your answer.

f. Think of all the ways that a parabola can intersect the graph of a circle. How many points of intersection are possible? Use graphs to support your answers.



g. Use your answers in part (f) to write several equations of parabolas that have different numbers of intersection points with the graph of $x^2 + y^2 = 4$. Then compare your results with your classmates.

Solutions of Nonlinear Systems

Vocabulary

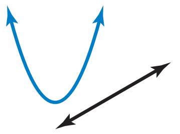
system of nonlinear equations, *p. 128*

Previously, you solved systems of *linear* equations by graphing, substitution, and elimination. You can also use these methods to solve systems of *nonlinear* equations. A **system of nonlinear equations** is a system in which at least one of the equations is nonlinear. For instance, the nonlinear system shown consists of a quadratic equation and a linear equation.

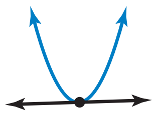
$y = x^2 + 2x - 4$ — Equation 1 is nonlinear.

$y = 2x + 5$ — Equation 2 is linear.

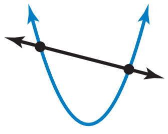
When a nonlinear system consists of a linear equation and a quadratic equation, the graphs can intersect in zero, one, or two points. So, the system can have zero, one, or two real solutions, as shown.



No real solution

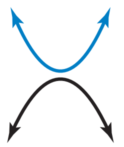


One real solution

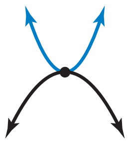


Two real solutions

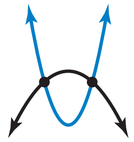
When a nonlinear system consists of two parabolas that open up or open down, the graphs can intersect in zero, one, or two points. So, the system can have zero, one, or two real solutions, as shown.



No real solution



One real solution



Two real solutions

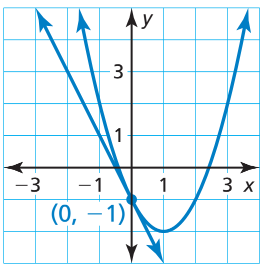
EXAMPLE 1 Solving a Nonlinear System by Graphing

Solve the system by graphing.

$y = x^2 - 2x - 1$ — Equation 1

$y = -2x - 1$ — Equation 2

SOLUTION



Step 1 Graph each equation.

Step 2 Estimate the point of intersection. The parabola and the line appear to intersect at the point $(0, -1)$.

Step 3 Check that $(0, -1)$ is a solution of each equation.

Equation 1

$y = x^2 - 2x - 1$

$-1 \stackrel{?}{=} (0)^2 - 2(0) - 1$

$-1 = -1$ ✓

Equation 2

$y = -2x - 1$

$-1 \stackrel{?}{=} -2(0) - 1$

$-1 = -1$ ✓

▶ The solution is $(0, -1)$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the system by graphing.

1. $y = x^2 - 4x - 2$
$y = x - 2$

2. $y = \frac{1}{2}x^2 - 2x + 4$
$x + y = 3$

3. $y = -3x + 8$
$y = -x^2 + 3x - 1$

EXAMPLE 2 Solving a Nonlinear System by Substitution

WATCH

Solve the system by substitution.

$x^2 + x - y = -1$ Equation 1

$x + y = 4$ Equation 2

SOLUTION

Step 1 Solve for y in Equation 2.

$y = -x + 4$ Solve for y in Equation 2.

Step 2 Substitute $-x + 4$ for y in Equation 1 and solve for x.

$x^2 + x - y = -1$ Equation 1

$x^2 + x - (-x + 4) = -1$ Substitute $-x + 4$ for y.

$x^2 + 2x - 4 = -1$ Simplify.

$x^2 + 2x - 3 = 0$ Write in standard form.

$(x + 3)(x - 1) = 0$ Factor.

$x + 3 = 0$ *or* $x - 1 = 0$ Zero-Product Property

$x = -3$ *or* $x = 1$ Solve for x.

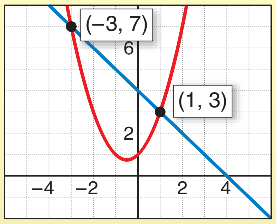
Step 3 Substitute -3 and 1 for x in $y = -x + 4$ and solve for y.

$y = -x + 4 = -(-3) + 4 = 7$ Substitute -3 for x.

$y = -x + 4 = -1 + 4 = 3$ Substitute 1 for x.

▶ So, the solutions are $(-3, 7)$ and $(1, 3)$.

Check



EXAMPLE 3 Solving a Nonlinear System by Elimination

WATCH

Solve the system by elimination.

$2x^2 - 5x - y = -2$ Equation 1

$x^2 + 2x + y = 0$ Equation 2

SOLUTION

The coefficients of the y-terms are opposites. So, add the equations to eliminate the y-terms and obtain a quadratic equation in x.

$$\begin{array}{r} 2x^2 - 5x - y = -2 \\ x^2 + 2x + y = 0 \\ \hline 3x^2 - 3x \quad = -2 \end{array}$$

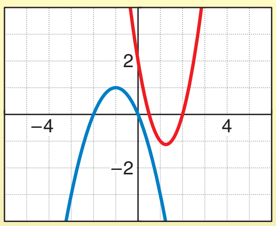
Add the equations.

$3x^2 - 3x + 2 = 0$ Write in standard form.

$x = \dfrac{3 \pm \sqrt{-15}}{6}$ Use the Quadratic Formula.

▶ Because the discriminant is negative, the equation $3x^2 - 3x + 2 = 0$ has no real solution. So, the original system has no real solution.

Check



SELF-ASSESSMENT

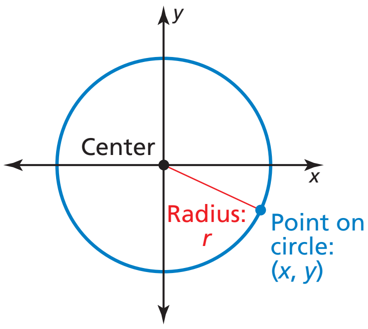
1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the system using any method. Explain your choice of method.

4. $y = -x^2 + 4$
$y = -4x + 8$

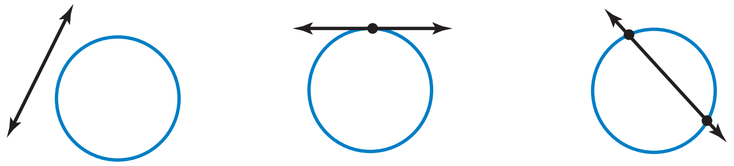
5. $x^2 + 3x + y = 0$
$2x + y = 5$

6. $2x^2 + 4x - y = -2$
$x^2 + y = 2$



An equation of the form $x^2 + y^2 = r^2$ is the standard form of a circle with center (0, 0) and radius r. When a nonlinear system consists of the equation of a circle and a linear equation, the graphs can intersect in zero, one, or two points. So, the system can have zero, one, or two real solutions, as shown.

GO DIGITAL



No real solution | One real solution | Two real solutions

EXAMPLE 4 Solving a Nonlinear System Involving a Circle

Solve the system by substitution.

$x^2 + y^2 = 10$ Equation 1

$y = -3x + 10$ Equation 2

WATCH

SOLUTION

Step 1 Equation 2 is already solved for y. So, substitute $-3x + 10$ for y in Equation 1 and solve for x.

$x^2 + y^2 = 10$ Equation 1

$x^2 + (-3x + 10)^2 = 10$ Substitute $-3x + 10$ for y.

$x^2 + 9x^2 - 60x + 100 = 10$ Expand the power.

$10x^2 - 60x + 90 = 0$ Write in standard form.

$x^2 - 6x + 9 = 0$ Divide each side by 10.

$(x - 3)^2 = 0$ Perfect square trinomial pattern

$x = 3$ Zero-Product Property

COMMON ERROR

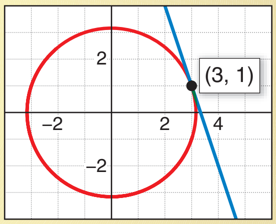
You can also substitute $x = 3$ in Equation 1 to find y. This yields two *apparent* solutions, (3, 1) and (3, −1). However, (3, −1) is *not* a solution because it does not satisfy Equation 2. You can also see (3, −1) is not a solution from the graph.

Step 2 Find the y-coordinate of the solution by substituting $x = 3$ in Equation 2.

$y = -3(3) + 10 = 1$

▶ So, the solution is (3, 1).

Check Use technology to check your answer.



SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the system.

7. $y = 4x + 17$
$x^2 + y^2 = 17$

8. $x^2 + y^2 = 4$
$y = x + 4$

9. $x^2 + y^2 = 1$
$y = \frac{1}{2}x + \frac{1}{2}$

10. WHICH ONE DOESN'T BELONG? Which system does *not* belong with the other three? Explain your reasoning.

$y = 3x + 4$ $y = x^2 + 1$	$y = 2x - 1$ $y = -3x + 6$	$y = 3x^2 + 4x + 1$ $y = -5x^2 - 3x + 1$	$x^2 + y^2 = 4$ $y = -x + 1$

STUDY TIP

You can think of this as solving the system of equations

$y = f(x)$

$y = g(x)$

by graphing.

Solving Equations by Graphing

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KEY IDEA

Solving Equations by Graphing

Step 1 To solve the equation $f(x) = g(x)$, first write functions to represent each side of the equation, $y = f(x)$ and $y = g(x)$.

Step 2 Graph the functions $y = f(x)$ and $y = g(x)$. The x-value of an intersection point of the graphs of the functions is a solution of the equation $f(x) = g(x)$.

EXAMPLE 5 Solving Quadratic Equations by Graphing

WATCH

Solve each equation by graphing.

a. $3x^2 + 5x - 1 = -x^2 + 2x + 1$ **b.** $-(x - 1.5)^2 + 2.25 = 2x(x + 1.5)$

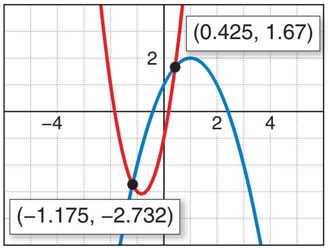
SOLUTION

a. Step 1 Write functions to represent each side of the original equation.

Equation	*Functions*
$3x^2 + 5x - 1 = -x^2 + 2x + 1$	$y = 3x^2 + 5x - 1$ $y = -x^2 + 2x + 1$

Step 2 Use technology to graph the functions and find the x-coordinates of the intersection points of the graphs.

The points of intersection are about $(-1.175, -2.732)$ and $(0.425, 1.67)$.

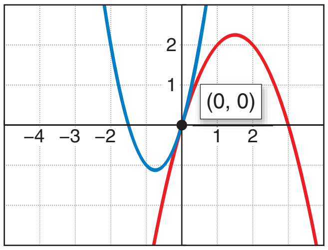


► So, the solutions of the equation are $x \approx -1.175$ and $x \approx 0.425$.

b. Step 1 Write functions to represent each side of the original equation.

Equation	*Functions*
$-(x - 1.5)^2 + 2.25 = 2x(x + 1.5)$	$y = -(x - 1.5)^2 + 2.25$ $y = 2x(x + 1.5)$

Step 2 Use technology to graph the functions and find the x-coordinate of the intersection point of the graphs. The graphs intersect at $(0, 0)$.



► So, the solution of the equation is $x = 0$.

ANOTHER WAY

In Example 5(a), you can also find the solutions by writing the given equation as $4x^2 + 3x - 2 = 0$ and using the Quadratic Formula.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the equation by graphing.

11. $x^2 - 6x + 15 = -(x - 3)^2 + 6$ **12.** $(x + 4)(x - 1) = -x^2 + 3x + 4$

13. WRITING Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$.

3.5 Practice WITH CalcChat® AND CalcView®

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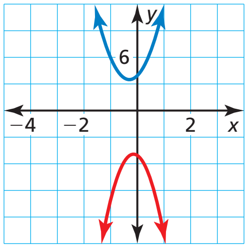
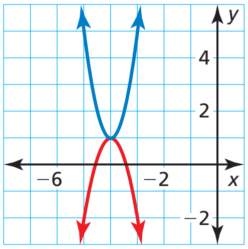
In Exercises 1–4, use the graph to solve the system.

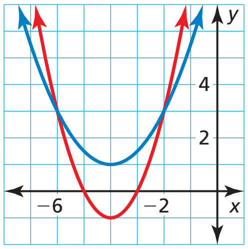
1. $y = -7x^2 - 2x - 5$
$y = 5x^2 + 3x + 4$



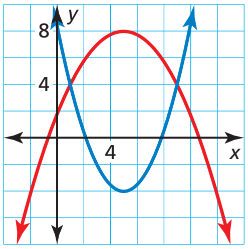
2. $y = -3x^2 - 24x - 47$
$y = 4(x + 4)^2 + 1$



3. $y = x^2 + 8x + 15$
$y = \frac{1}{2}x^2 + 4x + 9$



4. $y = -0.25(x - 5)^2 + 8$
$y = 0.5x^2 - 5x + 8.5$



In Exercises 5–12, solve the system by graphing.
Example 1

5. $y = x + 2$
$y = 0.5(x + 2)^2$

6. $y = (x - 3)^2 + 5$
$y = 5$

7. $y = \frac{1}{3}x + 2$
$y = -3x^2 - 5x - 4$

8. $y = -3x^2 - 30x - 71$
$y = -3x - 17$

9. $y = x^2 + 8x + 18$
$y = -2x^2 - 16x - 30$

10. $y = -2x^2 - 9$
$y = -x^2 - 1$

11. $y = (x - 2)^2$
$y = -x^2 + 4x - 2$

12. $y = \frac{1}{2}(x + 2)^2$
$y = -\frac{1}{2}x^2 + 2$

In Exercises 13–22, solve the system by substitution.
Examples 2 and 4

13. $y = x + 5$
$y = x^2 - x + 2$

14. $x^2 + y^2 = 49$
$y = 7 - x$

15. $x^2 + y^2 = 20$
$y = 2x - 10$

16. $y = -2x - 5$
$-3x^2 + 4x - y = 8$

17. $2x^2 + 4x - y = -3$
$-2x + y = -4$

18. $2x - 3 = y + 5x^2$
$y = -3x - 3$

19. $y = x^2 - 1$
$-7 = -x^2 - y$

20. $y + 16x - 22 = 4x^2$
$4x^2 - 24x + 26 + y = 0$

21. $x^2 + y^2 = 7$
$x + 3y = 21$

22. $x^2 + y^2 = 5$
$-x + y = -1$

In Exercises 23–30, solve the system by elimination.
Example 3

23. $2x^2 - 3x - y = -5$
$-x + y = 5$

24. $-3x^2 + 2x - 5 = y$
$-x + 2 = -y$

25. $-3x^2 + y = -18x + 29$
$-3x^2 - y = 18x - 25$

26. $y = -x^2 - 6x - 10$
$y = 3x^2 + 18x + 22$

27. $y + 2x = -14$
$-x^2 - y - 6x = 11$

28. $y = x^2 + 4x + 7$
$-y = 4x + 7$

29. $y = -3x^2 - 30x - 76$
$y = 2x^2 + 20x + 44$

30. $-10x^2 + y = -80x + 155$
$5x^2 + y = 40x - 85$

31. **ERROR ANALYSIS** Describe and correct the error in using elimination to solve for one of the variables in the system.

✗

$y = 2x^2 - 26$
$\underline{-y = -x - 10}$
$0 = x^2 - 36$
$36 = x^2$
$\pm 6 = x$

32. **MP NUMBER SENSE** The table shows the inputs and outputs of two quadratic functions. Identify the solution(s) of the system. Explain your reasoning.

x	y_1	y_2
−3	29	−11
−1	9	9
1	−3	21
3	−7	25
7	9	9
11	57	−39

In Exercises 33–38, solve the system using any method. Explain your choice of method.

33. $y = x^2 - 1$
$-y = 2x^2 + 1$

34. $y = -4x^2 - 16x - 13$
$-3x^2 + y + 12x = 17$

35. $-2x + 10 + y = \frac{1}{3}x^2$
$y = 10$

36. $y = 0.5x^2 - 10$
$y = -x^2 + 14$

37. $y = -3(x - 4)^2 + 6$
$(x - 4)^2 + 2 - y = 0$

38. $-x^2 + y^2 = 100$
$y = -x + 14$

In Exercises 39–44, solve the equation by graphing.
Example 5

39. $x^2 + 2x = -\frac{1}{2}x^2 + 2x$

40. $2x^2 - 12x - 16 = -6x^2 + 60x - 144$

41. $(x + 2)(x - 2) = -x^2 + 6x - 7$

42. $-2x^2 - 16x - 25 = 6x^2 + 48x + 95$

43. $(x - 2)^2 - 3 = (x + 3)(-x + 9) - 38$

44. $(-x + 4)(x + 8) - 42 = (x + 3)(x + 1) - 1$

45. COLLEGE PREP Which ordered pairs are solutions of the nonlinear system?

$y = \frac{1}{2}x^2 - 5x + \frac{21}{2}$
$y = -\frac{1}{2}x + \frac{13}{2}$

Ⓐ (1, 6) Ⓑ (3, 0)
Ⓒ (8, 2.5) Ⓓ (7, 0)

46. MP REASONING A nonlinear system contains the equations of a constant function and a quadratic function. The system has one solution. Describe the relationship between the graphs.

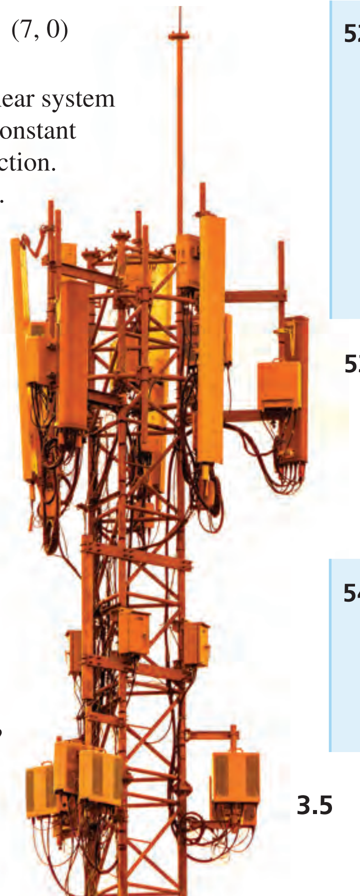
47. MODELING REAL LIFE The range (in miles) of a broadcast signal from a radio tower is bounded by a circle given by the equation

$x^2 + y^2 = 1620.$

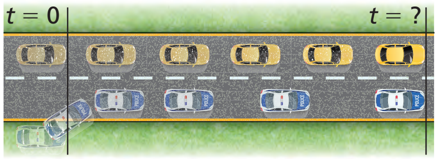
A straight highway can be modeled by the equation

$y = -\frac{1}{3}x + 30.$

For what length of the highway are cars able to receive the broadcast signal?



48. MODELING REAL LIFE A car passes a parked police car and continues at a constant speed r. The police car begins accelerating at a constant rate when it is passed. The diagram indicates the distance d (in miles) the police car travels as a function of time t (in minutes) after being passed. How long does it take the police car to catch up to the other car?



$r = 0.8$ mi/min
$d = 2.5t^2$

49. COMPARING METHODS Describe two different ways you can solve $-2x^2 + 12x - 17 = 2x^2 - 16x + 31$. Which way do you prefer? Explain your reasoning.

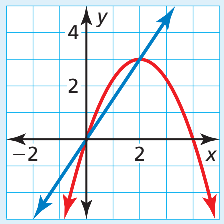
50. OPEN-ENDED Find a value for m so the system has (a) no real solution, (b) one real solution, and (c) two real solutions. Justify each answer using a graph.

$3y = -x^2 + 8x - 7$
$y = mx + 3$

51. MAKING AN ARGUMENT You and a friend solve the system shown and determine that $x = 3$ and $x = -3$. You use Equation 1 to obtain the solutions (3, 3), (3, −3), (−3, 3), and (−3, −3). Your friend uses Equation 2 to obtain the solutions (3, 3) and (−3, −3). Who is correct? Explain your reasoning.

$x^2 + y^2 = 18$ Equation 1
$x - y = 0$ Equation 2

52. HOW DO YOU SEE IT?
The graph of a nonlinear system is shown. Estimate the solution(s). Then describe a transformation of the graph of the linear function that results in a system with no real solution.



53. ANALYZING RELATIONSHIPS The graph of a line that passes through the origin intersects the graph of a circle with its center at the origin. When you know one of the points of intersection, explain how you can find the other point of intersection without performing any calculations.

54. THOUGHT PROVOKING
Write a nonlinear system that has two different solutions with the same y-coordinate and does not include an equation of a constant function. Justify your answer by solving the system.

55. WRITING Describe the possible numbers of real solutions of a system that contains (a) one quadratic equation and one equation of a circle, and (b) two distinct equations of circles. Sketch graphs to justify your answers.

56. MP REASONING Each system shown includes the equation of a circle with center (0, 0) and radius 1 and an equation of a line with a y-intercept of -1.

System A	System B	System C
$x^2 + y^2 = 1$	$x^2 + y^2 = 1$	$x^2 + y^2 = 1$
$y = 3x - 1$	$y = 4x - 1$	$y = 5x - 1$

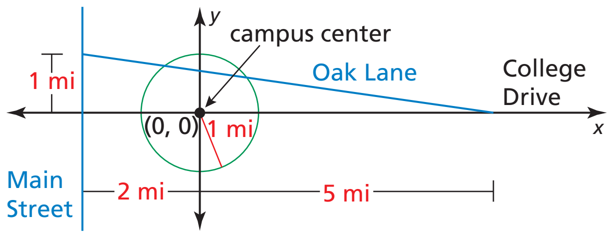
a. Without solving, find one solution that all three systems have in common. Explain your reasoning.

b. Find the other solution of each system. What do you notice about the numerators and denominator of each solution?

57. CRITICAL THINKING Solve the system shown.

$$x^2 + y^2 = 4$$
$$2y = x^2 - 2x + 4$$
$$y = -x + 2$$

58. DIG DEEPER To be eligible for a parking pass on a college campus, a student must live at least 1 mile from the campus center. For what length of Oak Lane are students *not* eligible for a parking pass?



GO DIGITAL

REVIEW & REFRESH

WATCH

In Exercises 59–62, solve the inequality. Graph the solution.

59. $4x - 4 > 8$

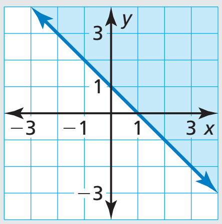
60. $-x + 7 \le 4 - 2x$

61. $-3(x - 4) \ge 24$

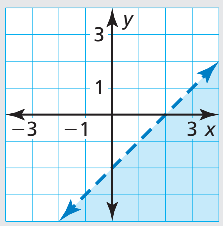
62. $3x - 2 < \frac{3}{2}(2x - 1)$

In Exercises 63 and 64, write an inequality that represents the graph.

63.



64.



In Exercises 65 and 66, solve the system using any method. Explain your choice of method.

65. $2x^2 + 4x + y = -6$
$x^2 + 2x - y = 3$

66. $x^2 + y^2 = 25$
$y = x - 1$

67. Graph $y = \begin{cases} -\frac{1}{2}x + 6, & \text{if } x \le 2 \\ 3x - 5, & \text{if } x > 2 \end{cases}$. Find the domain and range.

68. MODELING REAL LIFE You kick a soccer ball from an initial height of 2 feet. The ball has an initial vertical velocity of 45 feet per second. Does the ball reach a height of 35 feet? Explain your reasoning.

In Exercises 69 and 70, describe the transformation of $f(x) = x^2$ represented by g. Then graph each function.

69. $g(x) = (x - 5)^2 - 3$

70. $g(x) = \frac{1}{4}(x + 2)^2$

71. MP NUMBER SENSE For what values of b can you complete the square for $x^2 + bx$ by adding 81?

72. Use technology to find an equation of the line of best fit for the data. Identify and interpret the correlation coefficient.

x	0	5	10	12	16
y	18	15	9	7	2

In Exercises 73 and 74, find the square root of the number.

73. $\sqrt{-144}$

74. $\sqrt{-52}$

75. Approximate when the function is positive, negative, increasing, or decreasing.