SECTION 3.4

Fundamental Theorem of Algebra

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A19.

PS1. What is the conjugate of $3 - 2i$? [P.6]

PS2. What is the conjugate of $2 + i\sqrt{5}$? [P.6]

PS3. Multiply: $(x - 1)(x - 3)(x - 4)$ [P.3]

PS4. Multiply: $[x - (2 + i)][x - (2 - i)]$ [P.3/P.6]

PS5. Solve: $x^2 + 9 = 0$ [1.3]

PS6. Solve: $x^2 - x + 5 = 0$ [1.3]

Fundamental Theorem of Algebra

The German mathematician Carl Friedrich Gauss (1777–1855) was the first to prove that every polynomial function has at least one complex zero. This concept is so basic to the study of algebra that it is called the **Fundamental Theorem of Algebra.** The proof of the Fundamental Theorem is beyond the scope of this text; however, it is important to understand the theorem and its consequences. As you consider each of the following theorems, keep in mind that the terms *complex coefficients* and *complex zeros* include real coefficients and real zeros because the set of real numbers is a subset of the set of complex numbers.

Fundamental Theorem of Algebra

If P is a polynomial function of degree $n \geq 1$ with complex coefficients, then P has at least one complex zero.

Math Matters

Bettmann/CORBIS

Carl Friedrich Gauss (1777–1855) has often been referred to as the Prince of Mathematics. His work covered topics in algebra, calculus, analysis, probability, number theory, non-Euclidean geometry, astronomy, and physics, to name but a few. The following quote by Eric Temple Bell gives credence to the assertion that Gauss was one of the greatest mathematicians of all time. "Archimedes, Newton, and Gauss, these three, are in a class by themselves among the great mathematicians, and it is not for ordinary mortals to attempt to range them in order of merit."*

Men of Mathematics, by E. T. Bell, New York, Simon and Schuster, 1937.

Number of Zeros of a Polynomial Function

Let P be a polynomial function of degree $n \geq 1$ with complex coefficients. The Fundamental Theorem implies that P has a complex zero—say, c_1. The Factor Theorem implies that

$$P(x) = (x - c_1)Q(x)$$

where $Q(x)$ is a polynomial of degree 1 less than the degree of P. Recall that the polynomial $Q(x)$ is called a *reduced polynomial.* Assuming that the degree of $Q(x)$ is 1 or more, the Fundamental Theorem implies that it also must have a zero. A continuation of this reasoning process leads to the following theorem.

Linear Factor Theorem

If P is a polynomial function of degree $n \geq 1$ with leading coefficient $a_n \neq 0$,

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x^1 + a_0$$

then P has exactly n linear factors

$$P(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

The following theorem follows directly from the Linear Factor Theorem.

Number of Zeros of a Polynomial Function Theorem

If P is a polynomial function of degree $n \geq 1$, then P has exactly n complex zeros, provided each zero is counted according to its multiplicity.

The Linear Factor Theorem and the Number of Zeros of a Polynomial Function Theorem are referred to as **existence theorems**. They state that an nth-degree polynomial will have n linear factors and n complex zeros, but they do not provide any information on how to determine the linear factors or the zeros. In Example 1, we use previously developed methods to actually find the linear factors and zeros of some polynomial functions.

EXAMPLE 1 Find the Zeros and Linear Factors of a Polynomial Function

Find all the zeros of each of the following polynomial functions, and write each function as a product of its leading coefficient and its linear factors.

a. $P(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

b. $S(x) = 2x^4 + x^3 + 39x^2 + 136x - 78$

Solution

a. By the Linear Factor Theorem, P will have four linear factors and thus four zeros. The possible rational zeros are $\pm 1, \pm 2, \pm 4, \pm 8$, and ± 16. Use synthetic division to show that 2 is a zero of multiplicity 2.

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 8 & -16 & 16 \\ & & 2 & -4 & 8 & -16 \\ \hline & 1 & -2 & 4 & -8 & 0 \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & -2 & 4 & -8 \\ & & 2 & 0 & 8 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

The final reduced polynomial is $x^2 + 4$. Solve $x^2 + 4 = 0$ to find the remaining zeros.

$$x^2 + 4 = 0$$
$$x^2 = -4$$
$$x = \pm\sqrt{-4}$$
$$x = \pm 2i$$

The four zeros of P are $2, 2, -2i$, and $2i$. The leading coefficient of P is 1. Thus the linear factored form of P is $P(x) = 1(x - 2)(x - 2)(x - (-2i))(x - 2i)$ or, after simplifying,

$$P(x) = (x - 2)^2(x + 2i)(x - 2i)$$

TO REVIEW

Concepts Involving Complex Numbers
See Section P.6.

b. By the Linear Factor Theorem, S will have four linear factors and thus four zeros. The possible rational zeros are

$$\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \pm 3, \pm 6, \pm\frac{13}{2}, \pm 13, \pm 26, \pm\frac{39}{2}, \pm 39, \text{ and } \pm 78$$

Use synthetic division to show that -3 and $\frac{1}{2}$ are zeros of S.

$$\begin{array}{r|rrrrr} -3 & 2 & 1 & 39 & 136 & -78 \\ & & -6 & 15 & -162 & 78 \\ \hline & 2 & -5 & 54 & -26 & 0 \end{array} \qquad \begin{array}{r|rrrr} \frac{1}{2} & 2 & -5 & 54 & -26 \\ & & 1 & -2 & 26 \\ \hline & 2 & -4 & 52 & 0 \end{array}$$

The final reduced polynomial is $2x^2 - 4x + 52$. The remaining zeros can be found by using the quadratic formula to solve

$$2x^2 - 4x + 52 = 0$$

$$x^2 - 2x + 26 = 0 \qquad \text{• Divide each side by 2.}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(26)}}{2}$$

$$= \frac{2 \pm \sqrt{-100}}{2}$$

$$= \frac{2 \pm 10i}{2} = 1 \pm 5i$$

The four zeros are $-3, \frac{1}{2}, 1 + 5i$, and $1 - 5i$. The leading coefficient of S is 2. Thus the linear factored form of S is

$$S(x) = 2[x - (-3)]\left(x - \frac{1}{2}\right)[x - (1 + 5i)][x - (1 - 5i)]$$

or, after simplifying,

$$S(x) = 2(x + 3)\left(x - \frac{1}{2}\right)(x - 1 - 5i)(x - 1 + 5i)$$

Try Exercise 6, page 305

Conjugate Pair Theorem

You may have noticed that the complex zeros of the polynomial function in Example 1 were complex conjugates. The following theorem shows that this is not a coincidence.

TO REVIEW

Complex Conjugates
See page 63.

Conjugate Pair Theorem

If $a + bi$ $(b \neq 0)$ is a complex zero of a polynomial function *with real coefficients*, then the conjugate $a - bi$ is also a complex zero of the polynomial function.

EXAMPLE 2 Use the Conjugate Pair Theorem to Find Zeros

Find all the zeros of $P(x) = x^4 - 4x^3 + 14x^2 - 36x + 45$, given that $2 + i$ is a zero.

Solution

Because the coefficients are real numbers and $2 + i$ is a zero, the Conjugate Pair Theorem implies that $2 - i$ also must be a zero. Using synthetic division with $2 + i$ and $2 - i$, we have

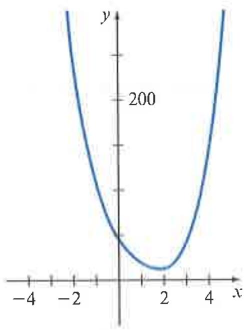
(continued)

$2+i$	1	-4	14	-36	45	
		$2+i$	-5	$18+9i$	-45	
	1	$-2+i$	9	$-18+9i$	0	• The coefficients of the reduced polynomial

$2-i$	1	$-2+i$	9	$-18+9i$	
		$2-i$	0	$18-9i$	
	1	0	9	0	• The coefficients of the next reduced polynomial

The resulting reduced polynomial is $x^2 + 9$, which has $3i$ and $-3i$ as zeros. Therefore, the four zeros of $x^4 - 4x^3 + 14x^2 - 36x + 45$ are $2 + i$, $2 - i$, $3i$, and $-3i$.

▸ Try Exercise 22, page 305



$P(x) = x^4 - 4x^3 + 14x^2 - 36x + 45$

Figure 3.20

A graph of $P(x) = x^4 - 4x^3 + 14x^2 - 36x + 45$ is shown in Figure 3.20. Because the polynomial in Example 2 is a fourth-degree polynomial and because we have verified that P has four nonreal solutions, it comes as no surprise that the graph does not intersect the x-axis.

When performing synthetic division with complex numbers, it is helpful to write the coefficients of the given polynomial as complex coefficients. For instance, -10 can be written as $-10 + 0i$. This technique is illustrated in the next example.

EXAMPLE 3 Apply the Conjugate Pair Theorem

Find all the zeros of $P(x) = x^5 - 10x^4 + 65x^3 - 184x^2 + 274x - 204$, given that $3 - 5i$ is a zero.

Solution

Because the coefficients of P are real numbers and $3 - 5i$ is a zero of P, we know that $3 + 5i$ also must be a zero of P. Start by using synthetic division to show that $3 - 5i$ is a zero of P.

$3-5i$	1	$-10+0i$	$65+0i$	$-184+0i$	$274+0i$	-204
		$3-5i$	$-46+20i$	$157-35i$	$-256+30i$	204
	1	$-7-5i$	$19+20i$	$-27-35i$	$18+30i$	0

Now proceed to show that $3 + 5i$ is a zero of the reduced polynomial that has the coefficients shown in the blue screen above.

$3+5i$	1	$-7-5i$	$19+20i$	$-27-35i$	$18+30i$
		$3+5i$	$-12-20i$	$21+35i$	$-18-30i$
	1	-4	7	-6	0

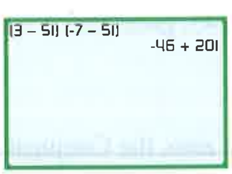
This synthetic division shows that the coefficients of the next reduced polynomial are 1, -4, 7, and -6. Thus the three remaining zeros of P are zeros of

$$x^3 - 4x^2 + 7x - 6$$

Descartes' Rule of Signs indicates that this reduced polynomial has three or one positive zeros and no negative zeros. Using the Rational Zero Theorem, we see that the possible rational zeros of the reduced polynomial are

Integrating Technology

Many graphing calculators can be used to do computations with complex numbers. The following TI-83/TI-83 Plus/TI-84 Plus screen display shows that the product of $3 - 5i$ and $-7 - 5i$ is $-46 + 20i$. The i symbol is located above the decimal point key.



$$1, 2, 3, \text{ and } 6$$

Use synthetic division to determine that 2 is a zero.

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 7 & -6 \\ & & 2 & -4 & 6 \\ \hline & 1 & -2 & 3 & 0 \end{array}$$

Use the quadratic formula to solve $x^2 - 2x + 3 = 0$.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm 2i\sqrt{2}}{2} = 1 \pm i\sqrt{2}$$

The zeros of $P(x) = x^5 - 10x^4 + 65x^3 - 184x^2 + 274x - 204$ are $3 - 5i$, $3 + 5i$, 2, $1 + \sqrt{2}i$, and $1 - \sqrt{2}i$.

▸ **Try Exercise 26, page 305**

Question • Is it possible for a third-degree polynomial function with real coefficients to have two real zeros and one nonreal complex zero?

Recall that the real zeros of a polynomial function P are the x-coordinates of the x-intercepts of the graph of P. This important connection between the real zeros of a polynomial function and the x-intercepts of the graph of the polynomial function is the basis for using a graphing utility to solve equations. Careful analysis of the graph of a polynomial function and your knowledge of the properties of polynomial functions can be used to solve many polynomial equations.

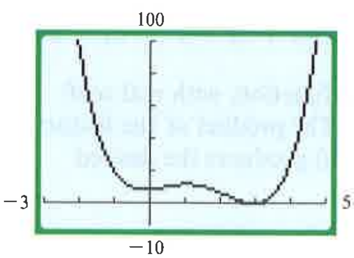
EXAMPLE 4 Solve a Polynomial Equation

Solve: $x^4 - 5x^3 + 4x^2 + 3x + 9 = 0$

Solution

Let $P(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$. The real zeros of P are the real solutions of the equation. Use a graphing utility to graph P. See Figure 3.21.

From the graph, it appears that (3, 0) is an x-intercept and the only x-intercept. Because the graph of P intersects but does not cross the x-axis at (3, 0), we know that 3 is a multiple zero of P with an even multiplicity.



$P(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$

Figure 3.21

$$\begin{array}{r|rrrrr} 3 & 1 & -5 & 4 & 3 & 9 \\ & & 3 & -6 & -6 & -9 \\ \hline & 1 & -2 & -2 & -3 & 0 \end{array}$$

• Coefficients of P

• The remainder is 0. Thus 3 is a zero.

By the Number of Zeros Theorem, there are three more zeros of P. Use synthetic division to show that 3 is also a zero of the reduced polynomial $x^3 - 2x^2 - 2x - 3$.

$$\begin{array}{r|rrrr} 3 & 1 & -2 & -2 & -3 \\ & & 3 & 3 & 3 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

• Coefficients of reduced polynomial

• The remainder is 0. Thus 3 is a zero of multiplicity 2.

We now have 3 as a double root of the original equation, and from the last line of the preceding synthetic division, the remaining solutions must

(continued)

Answer • No. Because the coefficients of the polynomial are real numbers, the nonreal complex zeros of the polynomial function must occur as conjugate pairs.

be solutions of $x^2 + x + 1 = 0$. Use the quadratic formula to solve this equation.

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

The solutions of $x^4 - 5x^3 + 4x^2 + 3x + 9 = 0$ are $3, 3, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Try Exercise 34, page 306

Finding a Polynomial Function with Given Zeros

Many of the problems in this section and in Section 3.3 dealt with the process of finding the zeros of a given polynomial function. Example 5 considers the reverse process: finding a polynomial function when the zeros are given.

EXAMPLE 5 Determine a Polynomial Function Given Its Zeros

Find each polynomial function.

a. A polynomial function of degree 3 that has 1, 2, and -3 as zeros

b. A polynomial function of degree 4 that has real coefficients and zeros $2i$ and $3 - 7i$

Solution

a. Because 1, 2, and -3 are zeros, $(x - 1)$, $(x - 2)$, and $(x + 3)$ are factors. The product of these factors produces a polynomial function with the indicated zeros.

$$P(x) = (x - 1)(x - 2)(x + 3) = (x^2 - 3x + 2)(x + 3) = x^3 - 7x + 6$$

b. By the Conjugate Pair Theorem, the polynomial function, with real coefficients, also must have $-2i$ and $3 + 7i$ as zeros. The product of the factors $x - 2i$, $x - (-2i)$, $x - (3 - 7i)$, and $x - (3 + 7i)$ produces the desired polynomial function.

$$\begin{aligned} P(x) &= (x - 2i)(x + 2i)[x - (3 - 7i)][x - (3 + 7i)] \\ &= (x^2 + 4)(x^2 - 6x + 58) \\ &= x^4 - 6x^3 + 62x^2 - 24x + 232 \end{aligned}$$

Try Exercise 60, page 306

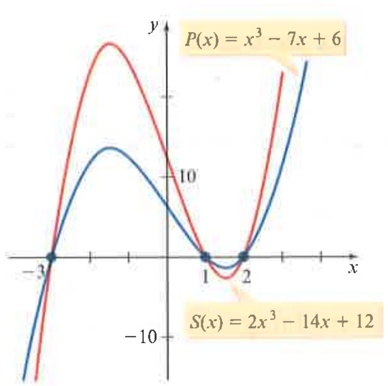


Figure 3.22

A polynomial function that has a given set of zeros is not unique. For example, $P(x) = x^3 - 7x + 6$ has zeros 1, 2, and -3, but so does any nonzero multiple of P, such as $S(x) = 2x^3 - 14x + 12$. This concept is illustrated in Figure 3.22. The graphs of the two polynomial functions are different, but they have the same zeros.

In some cases you may wish to find a polynomial function that has given zeros and satisfies a given condition. For instance, suppose you wish to find a polynomial function, of smallest degree, that has 1.5, $2 + i$, and $2 - i$ as zeros, and which also has integer coefficients. Start by finding a polynomial function, of smallest degree, with the desired zeros.

$$\begin{aligned} P(x) &= (x - 1.5)[x - (2 + i)][x - (2 - i)] \\ &= (x - 1.5)(x^2 - 4x + 5) \\ &= x^3 - 5.5x^2 + 11x - 7.5 \end{aligned}$$

P has the given zeros, but some of its coefficients are not integers. If all of the coefficients were doubled, then the new polynomial function $S = 2P$ would still have the given zeros and all of its coefficients would be integers. Thus,

$$S(x) = 2x^3 - 11x^2 + 22x - 15$$

is a polynomial function that satisfies all the given requirements.

EXERCISE SET 3.4

Concept Check

In Exercises 1 to 4, determine the number of complex zeros for each polynomial function.

1. $P(x) = x^3 - 5x^2 + 6x - 3$

2. $P(x) = 2x^5 - 2x^3 + 7x - 1$

3. $P(x) = 5x^2 + 6x - 3x^3 + 2$

4. $P(x) = 1 - \frac{1}{2}x^4$

In Exercises 5 to 20, find all the zeros of the polynomial function and write the polynomial as a product of its leading coefficient and its linear factors. (*Hint:* First determine the rational zeros.)

5. $P(x) = x^4 + x^3 - 2x^2 + 4x - 24$

6. $P(x) = x^3 - 3x^2 + 7x - 5$

7. $P(x) = 2x^4 - x^3 - 4x^2 + 10x - 4$

8. $P(x) = x^3 - 13x^2 + 65x - 125$

9. $P(x) = x^5 - 9x^4 + 34x^3 - 58x^2 + 45x - 13$

10. $P(x) = x^4 - 4x^3 + 53x^2 - 196x + 196$

11. $P(x) = 2x^4 - x^3 - 15x^2 + 23x + 15$

12. $P(x) = 3x^4 - 17x^3 - 39x^2 + 337x + 116$

13. $P(x) = 2x^4 - 14x^3 + 33x^2 - 46x + 40$

14. $P(x) = 3x^4 - 10x^3 + 15x^2 + 20x - 8$

15. $P(x) = 2x^3 - 9x^2 + 18x - 20$

16. $P(x) = 3x^4 - 19x^3 + 59x^2 - 79x + 36$

17. $P(x) = 2x^4 - x^3 - 2x^2 + 13x - 6$

18. $P(x) = 4x^4 - 4x^3 + 13x^2 - 12x + 3$

19. $P(x) = 3x^5 + 2x^4 + 10x^3 + 6x^2 - 25x - 20$

20. $P(x) = 2x^6 - 11x^5 + 5x^4 + 60x^3 - 62x^2 - 64x + 40$

In Exercises 21 to 32, use the given zero to find the remaining zeros of each polynomial function.

21. $P(x) = 2x^3 - 5x^2 + 6x - 2;\quad 1 + i$

22. $P(x) = 3x^3 - 29x^2 + 92x + 34;\quad 5 + 3i$

23. $P(x) = x^3 + 3x^2 + x + 3;\quad -i$

24. $P(x) = x^4 - 6x^3 + 71x^2 - 146x + 530;\quad 2 + 7i$

25. $P(x) = x^4 - 4x^3 + 14x^2 - 4x + 13;\quad 2 - 3i$

26. $P(x) = x^5 - 6x^4 + 22x^3 - 64x^2 + 117x - 90;\quad 3i$

27. $P(x) = x^4 - 4x^3 + 19x^2 - 30x + 50;\quad 1 + 3i$

28. $P(x) = x^5 - x^4 - 4x^3 - 4x^2 - 5x - 3;\quad i$

29. $P(x) = x^5 - 3x^4 + 7x^3 - 13x^2 + 12x - 4;\quad -2i$

30. $P(x) = x^4 - 8x^3 + 18x^2 - 8x + 17;\quad i$

31. $P(x) = x^4 - 17x^3 + 112x^2 - 333x + 377;\quad 5 + 2i$

32. $P(x) = 2x^5 - 8x^4 + 61x^3 - 99x^2 + 12x + 182;\quad 1 - 5i$

Indicates Try It Exercises

In Exercises 33 to 40, use a graph and your knowledge of the zeros of polynomial functions to determine the *exact* values of all the solutions of each equation.

33. $2x^3 - x^2 + x - 6 = 0$

34. $4x^3 + 3x^2 + 16x + 12 = 0$

35. $24x^3 - 62x^2 - 7x + 30 = 0$

36. $12x^3 - 52x^2 + 27x + 28 = 0$

37. $x^4 - 4x^3 + 5x^2 - 4x + 4 = 0$

38. $x^4 + 4x^3 + 8x^2 + 16x + 16 = 0$

39. $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$

40. $x^4 + 3x^3 - 6x^2 - 28x - 24 = 0$

In Exercises 41 to 58, find a polynomial function of smallest degree with integer coefficients that has the given zeros.

41. $4, -3, 2$

42. $-1, 1, -5$

43. $3, 2i, -2i$

44. $0, i, -i$

45. $3 + i, 3 - i, 2 + 5i, 2 - 5i$

46. $2 + 3i, 2 - 3i, -5, 2$

47. $6 + 5i, 6 - 5i, 2, 3, 5$

48. $\frac{1}{2}, 4 - i, 4 + i$

49. $\frac{3}{4}, 2 + 7i, 2 - 7i$

50. $\frac{1}{4}, -\frac{1}{5}, i, -i$

51. $2 - i\sqrt{2}, 2 + i\sqrt{2}, 3$

52. $3 + i,\ 3 - i,\ 4 - 3i,\ 4 + 3i, -i, i$

53. $5.5, -3.5, 1 - i, 1 + i$

54. $5 - 2i, 5 + 2i, \frac{3}{5}, -2$

55. $3 + 4i,\ 3 - 4i,\ 2 - i,\ 2 + i, 7$

56. $1 + 2i,\ 1 - 2i,\ 5, -3$

57. $1.5 + i,\ 1.5 - i,\ 4, -1$

58. $1 + i,\ 1 + i,\ 1 - i,\ 1 - i,\ 3, -3$

In Exercises 59 to 64, find a polynomial function *P*, with real coefficients, that has the indicated zeros and satisfies the given conditions.

59. Zeros: $2 - 5i, -4$; degree 3

60. Zeros: $3 + 2i, 7$; degree 3

61. Zeros: $4 + 3i, 5 - i$; degree 4

62. Zeros: $i, 3 - 5i$; degree 4

63. Zeros: $-2, 1, 3, 1 + 4i, 1 - 4i$; degree 5

64. Zeros: -5, 3 (multiplicity 2), $2 + i, 2 - i$; degree 5

Enrichment Exercises

In Exercises 65 to 68, find a polynomial function *P*, with real coefficients, that has the indicated zeros and satisfies the given conditions.

65. Zeros: $-1, 2, 3$; degree 3; $P(1) = 12$ [*Hint*: First find a third-degree polynomial function $T(x)$ with real coefficients that has -1, 2, and 3 as zeros. Now evaluate $T(1)$. If $T(1) = P(1) = 12$, then $T(x)$ is the desired polynomial function. If $T(1) \neq 12$, then you need to multiply $T(x)$ by $\frac{12}{T(1)}$ to produce the polynomial function that has the given zeros and whose graph passes through $(1, 12)$. That is, $P(x) = \frac{12}{T(1)}T(x)$.]

66. Zeros: $3i, 2$; degree 3; $P(3) = 27$ (See the hint in Exercise 65.)

67. Zeros: $3, -5, 2 + i$; degree 4; $P(1) = 48$ (See the hint in Exercise 65.)

68. Zeros: $\frac{1}{2}, 1 - i$; degree 3; $P(4) = 140$ (See the hint in Exercise 65.)

69. **Conjugate Pair Theorem** Verify that the function $P(x) = x^3 - x^2 - ix^2 - 9x + 9 + 9i$ has $1 + i$ as a zero and that its conjugate $1 - i$ is not a zero. Explain why this does not contradict the Conjugate Pair Theorem.

70. **Investigate the Roots of a Cubic Equation** Hieronimo Cardano, using a technique he learned from Nicolo Tartaglia, was able to solve some cubic equations.

a. Show that the cubic equation $x^3 + bx^2 + cx + d = 0$ can be transformed into the "reduced" cubic $y^3 + my = n$, where m and n are constants, depending on b, c, and d, by using the substitution $x = y - \frac{b}{3}$.

b. Cardano then showed that a solution of the reduced cubic is given by

$$\sqrt[3]{\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}-\sqrt[3]{-\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}$$

Use Cardano's procedure to solve the equation $x^3 - 6x^2 + 20x - 33 = 0$.

71. Polynomial Function Give an example of a polynomial function that has the given properties, or explain why no such polynomial function exists.

a. A polynomial function of degree 3 that has one rational zero and two irrational zeros

b. A polynomial function of degree 4 that has four irrational zeros

c. A polynomial function of degree 3, with real coefficients, that has no real zeros

d. A polynomial function of degree 4, with real coefficients, that has no real zeros

SECTION 3.5

Vertical and Horizontal Asymptotes
Sign Property of Rational Functions
General Graphing Procedure
Slant Asymptotes
Graphing Rational Functions That Have a Common Factor
Applications of Rational Functions

Graphs of Rational Functions and Their Applications

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A20.

PS1. Simplify: $\dfrac{x^2-9}{x^2-2x-15}$ [P.5]

PS2. Evaluate $\dfrac{x+4}{x^2-2x-5}$ for $x = -1$. [P.1]

PS3. Evaluate $\dfrac{2x^2+4x-5}{x+6}$ for $x = -3$. [P.1]

PS4. For what values of x does the denominator of $\dfrac{x^2-x-5}{2x^3+x^2-15x}$ equal zero? [1.4]

PS5. Determine the degree of the numerator and the degree of the denominator of $\dfrac{x^3+3x^2-5}{x^2-4}$. [P.3]

PS6. Write $\dfrac{x^3+2x^2-x-11}{x^2-2x}$ in $Q(x) + \dfrac{R(x)}{x^2-2x}$ form. [3.1]

Vertical and Horizontal Asymptotes

If $P(x)$ and $Q(x)$ are polynomials, then the function F given by

$$F(x) = \frac{P(x)}{Q(x)}$$

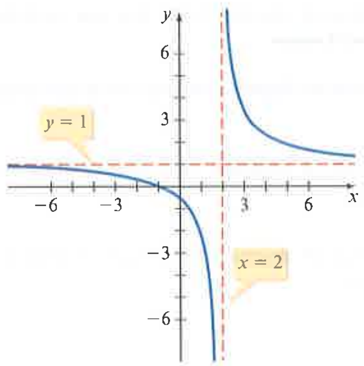
is called a **rational function.** The domain of F is the set of all real numbers except those for which $Q(x) = 0$. For example, let

$$F(x) = \frac{x^2-x-5}{2x^3+x^2-15x}$$

Setting the denominator equal to 0, we have

$$2x^3 + x^2 - 15x = 0$$
$$x(2x-5)(x+3) = 0$$

The denominator is 0 for $x = 0$, $x = \dfrac{5}{2}$, and $x = -3$. Thus the domain of F is the set of all real numbers except 0, $\dfrac{5}{2}$, and -3.



$G(x) = \dfrac{x+1}{x-2}$

Figure 3.23

The graph of $G(x) = \dfrac{x+1}{x-2}$ is given in Figure 3.23. The graph shows that G has the following properties:

- The graph has an x-intercept at $(-1, 0)$ and a y-intercept at $\left(0, -\dfrac{1}{2}\right)$.
- The graph does not exist at $x = 2$.

Note the behavior of the graph as x takes on values that are close to 2 but *less than* 2. Mathematically, we say that "x approaches 2 from the left."

x	1.9	1.95	1.99	1.995	1.999
$G(x)$	−29	−59	−299	−599	−2999

From this table and the graph, it appears that as x approaches 2 from the left, the functional values $G(x)$ decrease without bound.

- In this case, we say that "$G(x)$ approaches negative infinity."

Now observe the behavior of the graph as x takes on values that are close to 2 but *greater than* 2. Mathematically, we say that "x approaches 2 from the right."

x	2.1	2.05	2.01	2.005	2.001
$G(x)$	31	61	301	601	3001

From this table and the graph, it appears that as x approaches 2 from the right, the functional values $G(x)$ increase without bound.

- In this case, we say that "$G(x)$ approaches positive infinity."

Now consider the values of $G(x)$ as x *increases* without bound. The following table gives values of $G(x)$ for selected values of x.

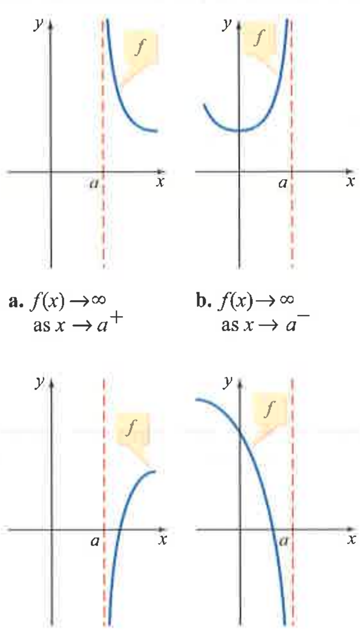
x	1000	5000	10,000	50,000	100,000
$G(x)$	1.00301	1.00060	1.00030	1.00006	1.00003

- As x increases without bound, the values of $G(x)$ become closer to 1.

Now let the values of x *decrease* without bound. The following table gives the values of $G(x)$ for selected values of x.

x	−1000	−5000	−10,000	−50,000	−100,000
$G(x)$	0.997006	0.999400	0.999700	0.999940	0.999970

- As x decreases without bound, the values of $G(x)$ become closer to 1.



a. $f(x) \to \infty$ as $x \to a^+$ **b.** $f(x) \to \infty$ as $x \to a^-$

c. $f(x) \to -\infty$ as $x \to a^+$ **d.** $f(x) \to -\infty$ as $x \to a^-$

Figure 3.24

When we are discussing functional values that increase or decrease without bound, it is convenient to use mathematical notation. The notation

$$f(x) \to \infty \quad \text{as} \quad x \to a^+$$

means that the functional values $f(x)$ increase without bound as x approaches a from the right. Recall that the symbol ∞ does not represent a real number but is used merely to describe the concept of a variable taking on larger and larger values without bound. See Figure 3.24a.

The notation

$$f(x) \to \infty \quad \text{as} \quad x \to a^-$$

means that the function values $f(x)$ increase without bound as x approaches a from the left. See Figure 3.24b.