SECTION 3.3

Zeros of Polynomial Functions

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A19.

PS1. Find the zeros of $P(x) = 6x^2 - 25x + 14$. [1.3, 2.4]

PS2. Use synthetic division to divide $2x^3 + 3x^2 + 4x - 7$ by $x + 2$. [3.1]

PS3. Use synthetic division to divide $3x^4 - 21x^2 - 3x - 5$ by $x - 3$. [3.1]

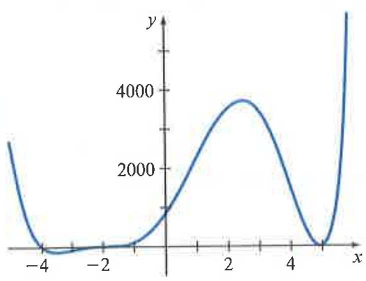
PS4. List all natural numbers that are factors of 12. [P.1]

PS5. List all integers that are factors of 27. [P.1]

PS6. Given $P(x) = 4x^3 - 3x^2 - 2x + 5$, find $P(-x)$. [2.5]

Multiple Zeros of a Polynomial Function

Recall that if P is a polynomial function then the values of x for which $P(x)$ is equal to 0 are called the *zeros* of P or the **roots** of the equation $P(x) = 0$. A zero of a polynomial function may be a **multiple zero.** For example, $P(x) = x^2 + 6x + 9$ can be expressed in factored form as $(x + 3)(x + 3)$. Setting each factor equal to zero yields $x = -3$ in both cases. Thus $P(x) = x^2 + 6x + 9$ has a zero of -3 that occurs twice. The following definition will be most useful when we are discussing multiple zeros.



$P(x) = (x - 5)^2(x + 2)^3(x + 4)$

Figure 3.18

Definition of Multiple Zeros of a Polynomial Function

If a polynomial function P has $(x - r)$ as a factor exactly k times, then r is a **zero of multiplicity k** of the polynomial function P.

EXAMPLE

The graph of the polynomial function

$$P(x) = (x - 5)^2(x + 2)^3(x + 4)$$

is shown in Figure 3.18. This polynomial function has

- 5 as a zero of multiplicity 2.
- -2 as a zero of multiplicity 3.
- -4 as a zero of multiplicity 1.

A zero of multiplicity 1 is generally referred to as a **simple zero.**

When searching for the zeros of a polynomial function, it is important that we know how many zeros to expect. This question is answered completely in Section 3.4. For the work in this section, the following result is helpful.

Number of Zeros of a Polynomial Function

A polynomial function P of degree n has at most n zeros, where each zero of multiplicity k is counted k times.

Rational Zero Theorem

The rational zeros of polynomial functions with integer coefficients can be found with the aid of the following theorem.

Study tip

The Rational Zero Theorem is one of the most important theorems of this chapter. It enables us to narrow the search for rational zeros to a finite list.

Rational Zero Theorem

If $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has *integer* coefficients ($a_n \neq 0$) and $\frac{p}{q}$ is a rational zero (in simplest form) of P, then

- p is a factor of the constant term a_0.
- q is a factor of the leading coefficient a_n.

The Rational Zero Theorem often is used to make a list of all possible rational zeros of a polynomial function. The list consists of all rational numbers of the form $\frac{p}{q}$, where p is an integer factor of the constant term a_0 and q is an integer factor of the leading coefficient a_n.

Caution

The Rational Zero Theorem gives the *possible* rational zeros of a polynomial function. That is, if P has a rational zero, then it must be one indicated by the theorem. However, P may not have any rational zeros. In the case of the polynomial function in Example 1, the only rational zeros are $-\frac{1}{4}$ and 2. The remaining rational numbers in the list are not zeros of P.

EXAMPLE 1 Apply the Rational Zero Theorem

Use the Rational Zero Theorem to list all possible rational zeros of

$$P(x) = 4x^4 + x^3 - 40x^2 + 38x + 12$$

Solution

List all integers p that are factors of 12 and all integers q that are factors of 4.

$$p: \quad \pm1, \pm2, \pm3, \pm4, \pm6, \pm12$$

$$q: \quad \pm1, \pm2, \pm4$$

Form all possible rational numbers using ±1, ±2, ±3, ±4, ±6, and ±12 for the numerator and ±1, ±2, and ±4 for the denominator. By the Rational Zero Theorem, the possible rational zeros are

$$\pm1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm2, \pm3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm4, \pm6, \pm12$$

It is not necessary to list a factor that is already listed in reduced form. For example, $\pm\frac{6}{4}$ is not listed because it is equal to $\pm\frac{3}{2}$.

Try Exercise 14, page 295

Question • If $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has integer coefficients and a leading coefficient of $a_n = 1$, must all the rational zeros of P be integers?

Upper and Lower Bounds for Real Zeros

A real number b is called an **upper bound** of the zeros of the polynomial function P if no zero is greater than b. A real number b is called a **lower bound** of the zeros of P if no zero is less than b. The following theorem is often used to find positive upper bounds and negative lower bounds for the real zeros of a polynomial function.

Answer • Yes. By the Rational Zero Theorem, the rational zeros of P are of the form $\frac{p}{q}$ where p is an integer factor of a_0 and q is an integer factor of a_n. Thus $q = \pm1$ and $\frac{p}{q} = \frac{p}{\pm1} = \pm p$.

Upper- and Lower-Bound Theorem

Let P be a polynomial function with real coefficients. Use synthetic division to divide P by $x - b$, where b is a nonzero real number.

Upper bound **a.** If $b > 0$ and the leading coefficient of P is positive, then b is an upper bound for the real zeros of P provided none of the numbers in the bottom row of the synthetic division are negative.

b. If $b > 0$ and the leading coefficient of P is negative, then b is an upper bound for the real zeros of P provided none of the numbers in the bottom row of the synthetic division are positive.

Lower bound If $b < 0$ and the numbers in the bottom row of the synthetic division alternate in sign (the number 0 can be considered positive or negative as needed to produce an alternating sign pattern), then b is a lower bound for the real zeros of P.

Study tip

When you check for bounds, you do not need to limit your choices to the possible zeros given by the Rational Zero Theorem. For instance, in Example 2, the integer −4 is a lower bound; however, −4 is not one of the possible zeros of P as given by the Rational Zero Theorem.

Upper and lower bounds are not unique. For example, if b is an upper bound for the real zeros of P, then any number greater than b is also an upper bound. Likewise, if a is a lower bound for the real zeros of P, then any number less than a is also a lower bound.

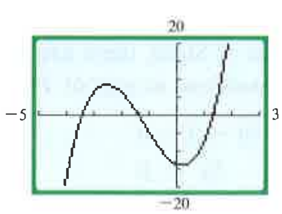
Integrating Technology

Note in Figure 3.19 that the zeros of

$$P(x) = 2x^3 + 7x^2 - 4x - 14$$

are between −4 (a lower bound) and 2 (an upper bound).

You can use the Upper- and Lower-Bound Theorem to help determine Xmin and Xmax for the viewing window of a graphing utility. This will ensure that all the x-intercepts of the polynomial function will be shown.



$P(x) = 2x^3 + 7x^2 - 4x - 14$

Figure 3.19

EXAMPLE 2 Find Upper and Lower Bounds

According to the Upper- and Lower-Bound Theorem, what is the smallest positive integer that is an upper bound and the largest negative integer that is a lower bound of the real zeros of $P(x) = 2x^3 + 7x^2 - 4x - 14$?

Solution

To find the smallest positive-integer upper bound, use synthetic division with 1, 2, . . . , as test values.

1	2	7	−4	−14
		2	9	5
	2	9	5	−9

2	2	7	−4	−14	
		4	22	36	
	2	11	18	22	• No negative numbers

According to the Upper- and Lower-Bound Theorem, 2 is the smallest positive-integer upper bound.

Now find the largest negative-integer lower bound.

−1	2	7	−4	−14
		−2	−5	9
	2	5	−9	−5
−3	2	7	−4	−14
		−6	−3	21
	2	1	−7	7

−2	2	7	−4	−14	
		−4	−6	20	
	2	3	−10	6	
−4	2	7	−4	−14	
		−8	4	0	
	2	−1	0	−14	• Alternating signs

According to the Upper- and Lower-Bound Theorem, −4 is the largest negative-integer lower bound.

▶ **Try Exercise 22, page 295**

Descartes' Rule of Signs

Descartes' Rule of Signs is another theorem that is often used to obtain information about the zeros of a polynomial function. In Descartes' Rule of Signs, the number of **variations in sign** of the coefficients of $P(x)$ or $P(-x)$ refers to sign changes of the coefficients from positive to negative or from negative to positive that we find when we examine successive terms of the function. The terms are assumed to appear in order of descending powers of x. For example, the polynomial function

$$P(x) = +3x^4 - 5x^3 - 7x^2 + x - 7$$

1 2 3

has three variations in sign. The polynomial function

$$\begin{aligned} P(-x) &= +3(-x)^4 - 5(-x)^3 - 7(-x)^2 + (-x) - 7 \\ &= +\ 3x^4 + 5x^3 - 7x^2 - x - 7 \end{aligned}$$

1

has one variation in sign.

Terms that have a coefficient of 0 are not counted as variations in sign and may be ignored. For example,

$$P(x) = -x^5 + 4x^2 + 1$$

1

has one variation in sign.

Math Matters

Descartes' Rule of Signs first appeared in his *La Géométrie* (1673). Although a proof of Descartes' Rule of Signs is beyond the scope of this course, we can see that a polynomial function with no variations in sign cannot have a positive zero. For instance, consider $P(x) = x^3 + x^2 + x + 1$. Each term of P is positive for any positive value of x. Thus P is never zero for $x > 0$.

Descartes' Rule of Signs

Let P be a polynomial function with real coefficients and with the terms arranged in order of decreasing powers of x.

- The number of positive real zeros of P is equal to the number of variations in sign of $P(x)$ or to that number decreased by an even integer.
- The number of negative real zeros of P is equal to the number of variations in sign of $P(-x)$ or to that number decreased by an even integer.

EXAMPLE 3 Apply Descartes' Rule of Signs

Use Descartes' Rule of Signs to determine both the number of possible positive and the number of possible negative real zeros of each polynomial function.

a. $P(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$ **b.** $P(x) = 2x^5 + 3x^3 + 5x^2 + 8x + 7$

Solution

a. $P(x) = +x^4 - 5x^3 + 5x^2 + 5x - 6$

1 2 3

There are three variations in sign. By Descartes' Rule of Signs, there are either three or one positive real zeros. Now examine the variations in sign of $P(-x)$.

$$\begin{aligned} P(-x) &= (-x)^4 - 5(-x)^3 + 5(-x)^2 + 5(-x) - 6 \\ &= x^4 + 5x^3 + 5x^2 - 5x - 6 \end{aligned}$$

1

There is one variation in sign of $P(-x)$. By Descartes' Rule of Signs, there is one negative real zero.

b. $P(x) = 2x^5 + 3x^3 + 5x^2 + 8x + 7$ has no variations in sign, so there are no positive real zeros.

$$\begin{aligned} P(-x) &= 2(-x)^5 + 3(-x)^3 + 5(-x)^2 + 8(-x) + 7 \\ &= -2x^5 \underbrace{- \ 3x^3}_{1} \underbrace{+ \ 5x^2}_{2} \underbrace{- \ 8x}_{3} + 7 \end{aligned}$$

$P(-x)$ has three variations in sign, so there are either three or one negative real zeros.

Try Exercise 32, page 295

Question • If $P(x) = ax^2 + bx + c$ has two variations in sign, must P have two positive real zeros?

In applying Descartes' Rule of Signs, we count each zero of multiplicity k as k zeros. For instance,

$$P(x) = x^2 - 10x + 25$$

has two variations in sign. Thus, by Descartes' Rule of Signs, P must have either two or no positive real zeros. Factoring P produces $(x - 5)^2$, from which we can observe that 5 is a positive zero of multiplicity 2.

Zeros of a Polynomial Function

Guidelines for Finding the Zeros of a Polynomial Function with Integer Coefficients

1. ***Gather general information*** Determine the degree n of the polynomial function. The number of distinct zeros of the polynomial function is at most n. Apply Descartes' Rule of Signs to find the possible number of positive zeros and the possible number of negative zeros.
2. ***Check suspects*** Apply the Rational Zero Theorem to list rational numbers that are possible zeros. Use synthetic division to test numbers in your list. If you find an upper or a lower bound, then eliminate from your list any number that is greater than the upper bound or less than the lower bound.
3. ***Work with the reduced polynomials*** Each time a zero is found, you obtain a reduced polynomial.
   - If a reduced polynomial is of degree 2, find its zeros either by factoring or by applying the quadratic formula.
   - If the degree of a reduced polynomial is 3 or greater, repeat the preceding steps for this polynomial.

Example 4 illustrates the procedure discussed in the preceding guidelines.

Answer • No. According to Descartes' Rule of Signs, P will have either two positive real zeros or no positive real zeros.

EXAMPLE 4 Find the Zeros of a Polynomial Function

Find the zeros of $P(x) = 3x^4 + 23x^3 + 56x^2 + 52x + 16$.

Solution

1. ***Gather general information*** The degree of P is 4. Thus the number of zeros of P is at most 4. By Descartes' Rule of Signs, there are no positive zeros and there are four, two, or no negative zeros.

2. ***Check suspects*** By the Rational Zero Theorem, the possible negative rational zeros of P are

$$\frac{p}{q}: \ -1, -2, -4, -8, -16, -\frac{1}{3}, -\frac{2}{3}, -\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}$$

Use synthetic division to test the possible rational zeros. The following work shows that -4 is a zero of P.

−4	3	23	56	52	16
		−12	−44	−48	−16
	3	11	12	4	0

Coefficients of the first reduced polynomial

3. ***Work with the reduced polynomials*** Because -4 is a zero, $(x + 4)$ and the first reduced polynomial $(3x^3 + 11x^2 + 12x + 4)$ are both factors of P. Thus

$$P(x) = (x + 4)(3x^3 + 11x^2 + 12x + 4)$$

All remaining zeros of P must be zeros of $3x^3 + 11x^2 + 12x + 4$. The Rational Zero Theorem indicates that the only possible negative rational zeros of $3x^3 + 11x^2 + 12x + 4$ are

$$\frac{p}{q}: \ -1, -2, -4, -\frac{1}{3}, -\frac{2}{3}, -\frac{4}{3}$$

Synthetic division is again used to test possible zeros.

−2	3	11	12	4
		−6	−10	−4
	3	5	2	0

Coefficients of the second reduced polynomial

Because -2 is a zero, $(x + 2)$ is also a factor of P. Thus

$$P(x) = (x + 4)(x + 2)(3x^2 + 5x + 2)$$

The remaining zeros of P must be zeros of $3x^2 + 5x + 2$.

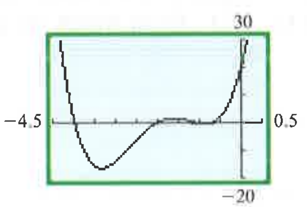
$$3x^2 + 5x + 2 = 0$$
$$(3x + 2)(x + 1) = 0$$
$$x = -\frac{2}{3} \quad \text{and} \quad x = -1$$

The zeros of $P(x) = 3x^4 + 23x^3 + 56x^2 + 52x + 16$ are $-4, -2, -\frac{2}{3}$, and -1.

▸ Try Exercise 46, page 296

Integrating Technology

If you have a graphing utility, you can produce a graph similar to the one below. By looking at the x-intercepts of the graph, you can reject as possible zeros some of the values suggested by the Rational Zero Theorem. This will reduce the amount of work that is necessary to find the zeros of the polynomial function.

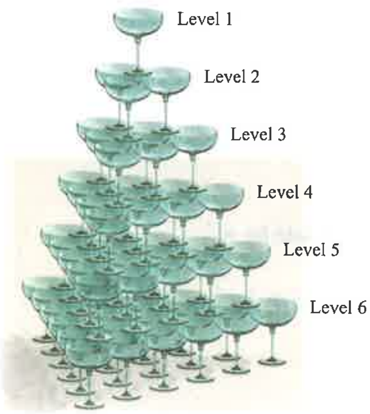


$P(x) = 3x^4 + 23x^3 + 56x^2 + 52x + 16$

Applications of Polynomial Functions

In the following example we use an upper bound to eliminate several of the possible zeros that are given by the Rational Zero Theorem.

EXAMPLE 5 Solve an Application



Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T = \frac{1}{6}(k^3 + 3k^2 + 2k)$$

where k is the number of levels in the pyramid. If 220 glasses are used to form a triangular pyramid, how many levels are in the pyramid?

Solution

We need to solve $220 = \frac{1}{6}(k^3 + 3k^2 + 2k)$ for k. Multiplying each side of the equation by 6 produces $1320 = k^3 + 3k^2 + 2k$, which can be written as $k^3 + 3k^2 + 2k - 1320 = 0$. The number 1320 has many natural number divisors, but we can eliminate many of these by showing that 12 is an upper bound.

12	1	3	2	−1320
		12	180	2184
	1	15	182	864

No number in the bottom row is negative. Thus 12 is an upper bound.

The only natural number divisors of 1320 that are less than 12 are 1, 2, 3, 4, 5, 6, 8, 10, and 11. The following synthetic division shows that 10 is a zero of $k^3 + 3k^2 + 2k - 1320$.

10	1	3	2	−1320
		10	130	1320
	1	13	132	0

The pyramid has 10 levels. There is no need to seek additional solutions, because the number of levels is uniquely determined by the number of glasses.

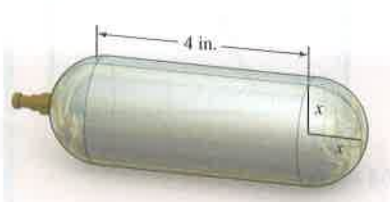
Note

The reduced polynomial $k^2 + 13k + 132$ has zeros of $k = \frac{-13 \pm i\sqrt{359}}{2}$. These zeros are not solutions of this application because the number of levels must be a natural number.

▶ Try Exercise 78, page 297

The procedures developed in this section will not find all solutions of every polynomial equation. However, a graphing utility can be used to estimate the real solutions of any polynomial equation. In Example 6 we utilize a graphing utility to solve an application.

EXAMPLE 6 Use a Graphing Utility to Solve an Application



A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 inches long, and the volume of the cartridge is 2π cubic inches (approximately 6.3 cubic inches). In the figure at the left, the common interior radius of the cylinder and the hemispheres is denoted by x. Use a graphing utility to estimate, to the nearest hundredth of an inch, the length of the radius x.

(continued)

Solution

The volume of the cartridge is equal to the volume of the two hemispheres plus the volume of the cylinder. Recall that the volume of a sphere of radius x is given by $\frac{4}{3}\pi x^3$. Therefore, the volume of a hemisphere is $\frac{1}{2}\left(\frac{4}{3}\pi x^3\right)$. The volume of a right circular cylinder is $\pi x^2 h$, where x is the radius of the base and h is the height of the cylinder. Thus the volume V of the cartridge is given by

$$V = \frac{1}{2}\left(\frac{4}{3}\pi x^3\right) + \frac{1}{2}\left(\frac{4}{3}\pi x^3\right) + \pi x^2 h$$

$$= \frac{4}{3}\pi x^3 + \pi x^2 h$$

Replacing V with 2π and h with 4 yields

$$2\pi = \frac{4}{3}\pi x^3 + 4\pi x^2$$

$$2 = \frac{4}{3}x^3 + 4x^2 \qquad \text{• Divide each side by } \pi.$$

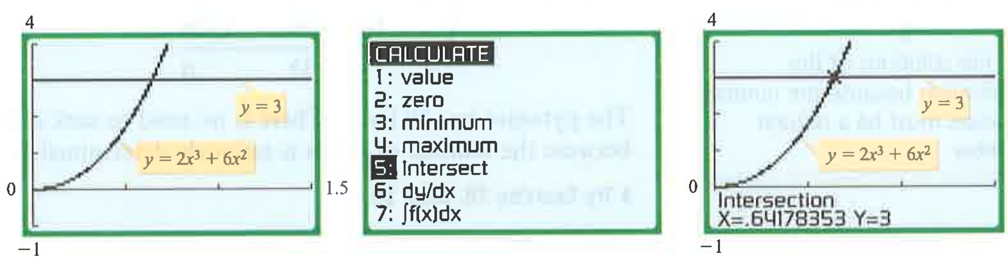
$$3 = 2x^3 + 6x^2 \qquad \text{• Multiply each side by } \frac{3}{2}.$$

Here are two methods that can be used to solve

$$3 = 2x^3 + 6x^2 \qquad (1)$$

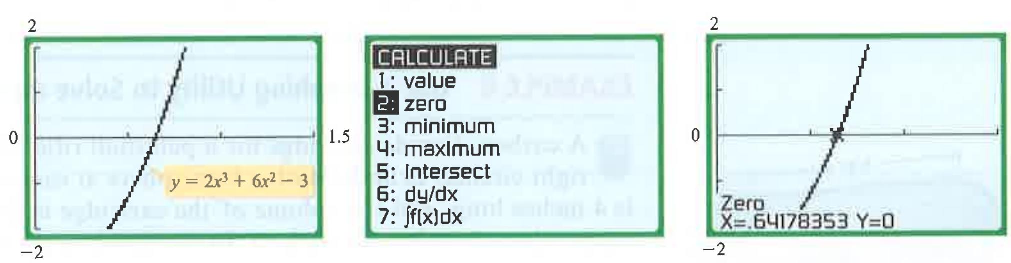
for x with the aid of a graphing utility.

1. ***Intersection method*** Use a graphing utility to graph $y = 2x^3 + 6x^2$ and $y = 3$ on the same screen, with $x > 0$. The x-coordinate of the point of intersection of the two graphs is the desired solution. The graphs intersect at $x \approx 0.64$ inch. See the following figures.



The length of the radius is approximately 0.64 inch.

2. ***Intercept method*** Rewrite Equation (1) as $2x^3 + 6x^2 - 3 = 0$. Graph $y = 2x^3 + 6x^2 - 3$ with $x > 0$. Use a graphing utility to find the x-intercept of the graph. This method also shows that $x \approx 0.64$ inch.



The length of the radius is approximately 0.64 inch.

▶ **Try Exercise 82, page 298**

EXERCISE SET 3.3

Concept Check

1. According to the Rational Zero Theorem, the possible rational zeros of a fourth-degree polynomial function are ± 1, ± 2, ± 3, and ± 6. Is it possible that none of these possible rational zeros are zeros of P? Explain.

2. According to the Rational Zero Theorem, the possible rational zeros of a fourth-degree polynomial function are ± 1, ± 2, ± 3, and ± 6. Is it possible that all eight of these possible rational zeros are zeros of P? Explain.

In Exercises 3 to 10, find the zeros of the polynomial function and state the multiplicity of each zero.

3. $P(x) = (x - 4)(x + 2)^2$
4. $P(x) = (x - 1)^3(x + 3)^2$
5. $P(x) = x^2(3x + 5)^2$
6. $P(x) = x^3(2x + 1)(3x - 12)^2$
7. $P(x) = (x^2 - 9)(x + 5)^2$
8. $P(x) = (x + 1)^3(x^2 - 25)^2$
9. $P(x) = (3x - 5)^2(2x - 7)$
10. $P(x) = (5x - 2)(x + 3)^4$

In Exercises 11 to 20, use the Rational Zero Theorem to list possible rational zeros for each polynomial function.

11. $P(x) = x^3 + 3x^2 - 6x - 8$
12. $P(x) = x^3 - 19x - 30$
13. $P(x) = 2x^3 + x^2 - 25x + 12$
14. $P(x) = 3x^3 + 11x^2 - 6x - 8$
15. $P(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$
16. $P(x) = 2x^3 + 9x^2 - 2x - 9$
17. $P(x) = 4x^4 - 12x^3 - 3x^2 + 12x - 7$
18. $P(x) = x^5 - 3x^4 - 11x^3 + 51x^2 - 62x + 24$
19. $P(x) = x^5 - 10x^4 + 27x^3 + 14x^2 - 128x + 96$
20. $P(x) = x^4 - 9$

In Exercises 21 to 30, find the smallest positive integer and the largest negative integer that, by the Upper- and Lower-Bound Theorem, are upper and lower bounds for the real zeros of each polynomial function.

21. $P(x) = x^3 + 3x^2 - 6x - 6$
22. $P(x) = x^3 - 19x - 28$
23. $P(x) = 2x^3 + x^2 - 25x + 10$
24. $P(x) = 2x^3 + 4x^2 - 8x + 7$
25. $P(x) = 3x^4 - 9x^3 + 2x^2 - 5x - 8$
26. $P(x) = -3x^3 - 2x^2 + 4x - 31$
27. $P(x) = -5x^4 + 14x^3 + 2x^2 - 12x + 325$
28. $P(x) = x^5 - x^4 - 7x^3 + 7x^2 - 12x - 12$
29. $P(x) = x^5 - 32$
30. $P(x) = x^4 - 1$

In Exercises 31 to 44, use Descartes' Rule of Signs to state the number of possible positive and negative real zeros of each polynomial function.

31. $P(x) = x^3 + 3x^2 - 6x - 8$
32. $P(x) = x^3 - 19x - 30$
33. $P(x) = 2x^3 + x^2 - 25x + 12$
34. $P(x) = 3x^3 + 11x^2 - 6x - 8$
35. $P(x) = 2x^5 - 37x^4 + 258x^3 - 820x^2 + 1100x - 375$
36. $P(x) = 2x^4 - 3x^3 - 55x^2 + 111x - 55$
37. $P(x) = 2x^5 + 23x^4 + 90x^3 + 152x^2 + 116x + 33$
38. $P(x) = x^5 + 4x^4 - 16x^3 - 56x^2 - 17x - 60$
39. $P(x) = x^5 - 2x^4 - 35x^3 + 158x^2 - 242x + 168$
40. $P(x) = 2x^5 - 14x^4 - 4x^3 + 60x^2 - 254x + 210$

Indicates Try It Exercises

41. $P(x) = 2x^6 - 19x^5 + 49x^4 - 88x^3 - 158x^2 + 656x - 280$

42. $P(x) = 2x^7 - 27x^6 + 149x^5 - 464x^4 + 912x^3 - 1087x^2 + 665x - 150$

43. $P(x) = 12x^7 - 112x^6 + 421x^5 - 840x^4 + 1038x^3 - 938x^2 + 629x - 210$

44. $P(x) = x^7 + 2x^5 + 3x^3 + x$

In Exercises 45 to 68, find the zeros of each polynomial function. If a zero is a multiple zero, state its multiplicity.

45. $P(x) = x^3 + 3x^2 - 6x - 8$

46. $P(x) = x^3 - 19x - 30$

47. $P(x) = 2x^3 + x^2 - 25x + 12$

48. $P(x) = 3x^3 + 11x^2 - 6x - 8$

49. $P(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$

50. $P(x) = 2x^3 + 9x^2 - 2x - 9$

51. $P(x) = 2x^4 - 9x^3 - 2x^2 + 27x - 12$

52. $P(x) = 3x^3 - x^2 - 6x + 2$

53. $P(x) = x^3 - 8x^2 + 8x + 24$

54. $P(x) = x^3 - 7x^2 - 7x + 69$

55. $P(x) = 2x^4 - 19x^3 + 51x^2 - 31x + 5$

56. $P(x) = 4x^4 - 35x^3 + 71x^2 - 4x - 6$

57. $P(x) = 3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24$

58. $P(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$

59. $P(x) = x^3 - 3x - 2$

60. $P(x) = 3x^4 - 4x^3 - 11x^2 + 16x - 4$

61. $P(x) = x^4 - 5x^2 - 2x$

62. $P(x) = x^3 - 2x + 1$

63. $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$

64. $P(x) = 6x^4 - 17x^3 - 11x^2 + 42x$

65. $P(x) = 2x^4 - 17x^3 + 4x^2 + 35x - 24$

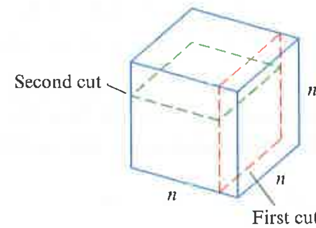
66. $P(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

67. $P(x) = x^3 - 16x$

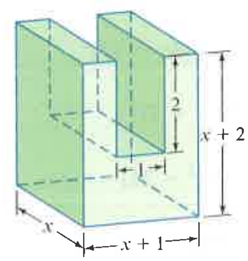
68. $P(x) = x^3 - 4x^2 - 3x$

69. Find the Dimensions A cube measures n inches on each edge. If a slice 2 inches thick is cut from one face of the cube, the resulting solid has a volume of 567 cubic inches. Find n.

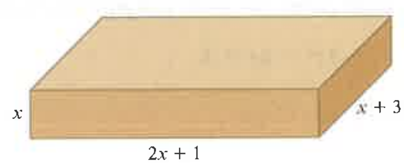
70. Find the Dimensions A cube measures n inches on each edge. If a slice 1 inch thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube as shown, the resulting solid has a volume of 1560 cubic inches. Find the dimensions of the original cube.



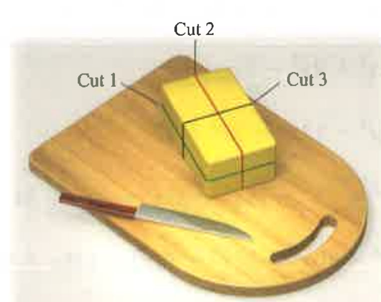
71. Dimensions of a Solid For what value of x will the volume of the following solid be 112 cubic inches?



72. Dimensions of a Box The length of a rectangular box is 1 inch more than twice the height of the box, and the width is 3 inches more than the height. If the volume of the box is 126 cubic inches, find the dimensions of the box.



73. Pieces and Cuts One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



You might be inclined to think that every additional cut doubles the previous number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

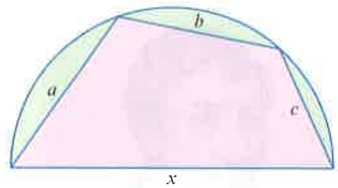
$$P(n) = \frac{n^3 + 5n + 6}{6}$$

a. Use the given function to determine the maximum number of pieces that can be produced by five straight cuts.

b. What is the fewest number of straight cuts that are needed to produce 64 pieces?

74. **Inscribed Quadrilateral** Isaac Newton discovered that if a quadrilateral with sides of lengths a, b, c, and x is inscribed in a semicircle with diameter x, then the lengths of the sides are related by the following equation.

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0$$



Given $a = 6$, $b = 5$, and $c = 4$, find x. Round to the nearest hundredth.

75. **Cannonball Stacks** Cannonballs can be stacked to form a pyramid with a square base. The total number of cannonballs T in one of these square pyramids is

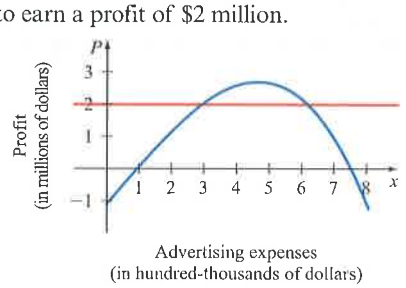
$$T = \frac{1}{6}(2n^3 + 3n^2 + n)$$

where n is the number of rows (levels). If 140 cannonballs are used to form a square pyramid, how many rows are in the pyramid?

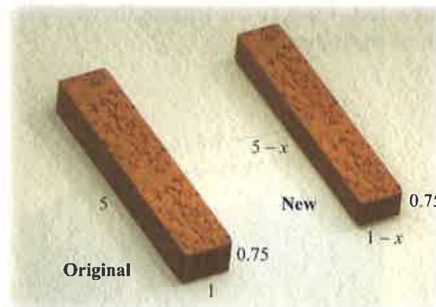
76. **Advertising Expenses** A company manufactures digital cameras. The company estimates that the profit from camera sales is

$$P(x) = -0.02x^3 + 0.01x^2 + 1.2x - 1.1$$

where P is the profit in millions of dollars and x is the amount, in hundred-thousands of dollars, spent on advertising. Determine the minimum amount, rounded to the nearest thousand dollars, the company needs to spend on advertising if it is to earn a profit of $2 million.

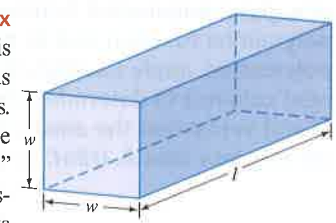


77. **Cost Cutting** A nutrition bar in the shape of a rectangular solid measures 0.75 inch by 1 inch by 5 inches. To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x inches. What value of x, rounded to the nearest thousandth of an inch, will produce a new nutrition bar with a volume that is 0.75 cubic inch less than the present bar's volume?



78. **Selection of Cards** The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 ways to choose three cards from a given group. How many cards are in the group?

79. **Dimensions of a Box** A rectangular box is square on two ends and has length plus girth of 81 inches. (The girth of a box is the shortest distance "around" the box.) Determine the possible lengths l of the box if its volume is 4900 cubic inches. Round approximate values to the nearest hundredth of an inch. In the above figure, assume that $w < l$.



80. **Medication Level** Pseudoephedrine hydrochloride is an allergy medication. The polynomial function

$$L(t) = 0.03t^4 + 0.4t^3 - 7.3t^2 + 23.1t$$

where $0 \leq t \leq 5$, models the level of pseudoephedrine hydrochloride, in milligrams, in the bloodstream of a patient t hours after 30 milligrams of the medication have been taken.

At what times, to the nearest minute, does the level of pseudoephedrine hydrochloride in the bloodstream reach 12 milligrams?

81. **Weight and Height of Giraffes** A veterinarian at a wild animal park has determined that the average weight w, in pounds, of an adult male giraffe is closely approximated by the function

$$w = 8.3h^3 - 307.5h^2 + 3914h - 15{,}230$$

where h is the giraffe's height in feet, and $15 \leq h \leq 18$. Use the above function to estimate the height of a giraffe that weighs 3150 pounds. Round to the nearest tenth of a foot.

82. **Propane Tank Dimensions** A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 feet long and the volume of the tank is 9π cubic feet. Find, to the nearest thousandth of a foot, the length of the radius x.

Enrichment Exercises

The mathematician Augustin Louis Cauchy proved a theorem that can be used to quickly establish a bound B for *all* the absolute values of the zeros (real and complex) of a given polynomial function. In Exercises 83 to 86, a polynomial function and its zeros are given. For each polynomial, apply Cauchy's Bound Theorem (shown in the next column) to determine the bound B for the polynomial and verify that the absolute value of each of the given zeros is less than B. (*Hint:* $|a + bi| = \sqrt{a^2 + b^2}$)

Cauchy's Bound Theorem

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0$ be a polynomial function with complex coefficients. The absolute value of each zero of P is less than

$$B = \left(\frac{\text{maximum of } (|a_{n-1}|, |a_{n-2}|, \ldots, |a_1|, |a_0|)}{|a_n|} + 1\right)$$

83. $P(x) = 2x^3 - 5x^2 - 28x + 15$, zeros: $-3, \frac{1}{2}, 5$

84. $P(x) = x^3 - 5x^2 + 2x + 8$, zeros: $-1, 2, 4$

85. $P(x) = x^4 - 2x^3 + 9x^2 + 2x - 10$, zeros: $1 + 3i$, $1 - 3i$, $1, -1$

86. $P(x) = x^4 - 4x^3 + 14x^2 - 4x + 13$, zeros: $2 + 3i$, $2 - 3i$, i, $-i$

Bettmann/CORBIS

Augustin Louis Cauchy
(1789–1857)

MID-CHAPTER 3 QUIZ

1. Use the Remainder Theorem to find $P(5)$ for the function $P(x) = 3x^3 + 7x^2 - 2x - 5$.

2. Use the Factor Theorem to determine whether $(x - 5)$ is a factor of $P(x) = x^3 - 6x^2 + 3x + 10$.

3. Determine the far-left and the far-right behavior of the graph of $P(x) = 4x^3 - 3x^2 - 6x - 1$.

4. Find the zeros of $P(x) = 3x^3 - 2x^2 - 27x + 18$.

5. Use the Intermediate Value Theorem to verify that $P(x) = 2x^4 - 5x^3 + x^2 - 20x - 28$ has a zero between 3 and 4.

6. Use the Rational Zero Theorem to list all possible rational zeros of $P(x) = 3x^3 + 7x^2 - 18x + 8$.

7. Use a graphing utility to estimate, to the nearest hundredth, the coordinates of the point where the function $P(x) = -x^3 + 10x^2 - 27x + 25$ has a relative minimum.

8. Find the zeros of $P(x) = 2x^4 - 19x^3 + 57x^2 - 64x + 20$. If a zero is a multiple zero, state its multiplicity.