3.3 Completing the Square

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Learning Target Solve quadratic equations and rewrite quadratic functions by completing the square.

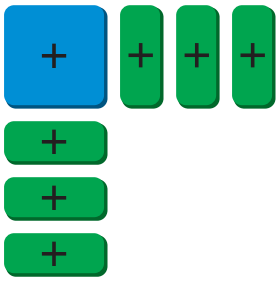
Success Criteria

- I can solve quadratic equations using square roots.
- I can solve quadratic equations by completing the square.
- I can apply completing the square to write quadratic functions in vertex form.

EXPLORE IT! Using Algebra Tiles to Complete the Square

Work with a partner.

a. Write the expression modeled by the algebra tiles. How can you *complete the square*?



b. Use the model to find the value of c so that the expression

$$x^2 + 6x + c$$

is a perfect square trinomial. Then write the expression as the square of a binomial.

Math Practice

Find General Methods
How can completing the square help you solve quadratic equations?

c. Use the method outlined in parts (a) and (b) to complete the table.

Expression	Value of c needed to complete the square	Expression written as a binomial squared
$x^2 + 2x + c$		
$x^2 + 4x + c$		
$x^2 + 8x + c$		
$x^2 + 10x + c$		

d. Look for patterns in the table in part (c). Consider the general statement $x^2 + bx + c = (x + d)^2$. In each case,

- how are b and d related?
- how are c and d related?
- how are b and c related?

e. How can you complete the square for a quadratic expression of the form $x^2 + bx$?

f. Solve $x^2 + 8x + 16 = 9$ by taking the square root of each side. Can you use this method to solve the equivalent equation $x^2 + 8x = -7$? What does this imply about solving quadratic equations?

Solving Quadratic Equations Using Square Roots

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Vocabulary VOCAB

completing the square, *p. 110*

Previously, you have solved equations of the form $u^2 = d$ by taking the square root of each side. This method also works when one side of an equation is a perfect square trinomial and the other side is a constant.

EXAMPLE 1 Solving a Quadratic Equation Using Square Roots

WATCH

Solve $x^2 - 16x + 64 = 100$ using square roots.

SOLUTION

ANOTHER WAY

You can also solve the equation by writing it in standard form as $x^2 - 16x - 36 = 0$ and factoring.

$x^2 - 16x + 64 = 100$ Write the equation.

$(x - 8)^2 = 100$ Write the left side as a binomial squared.

$x - 8 = \pm 10$ Take square root of each side.

$x = 8 \pm 10$ Add 8 to each side.

So, the solutions are $x = 8 + 10 = 18$ and $x = 8 - 10 = -2$.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the equation using square roots. Check your solution(s).

1. $x^2 + 4x + 4 = 36$ **2.** $x^2 - 6x + 9 = 1$ **3.** $x^2 - 22x + 121 = 81$

Solving Quadratic Equations by Completing the Square

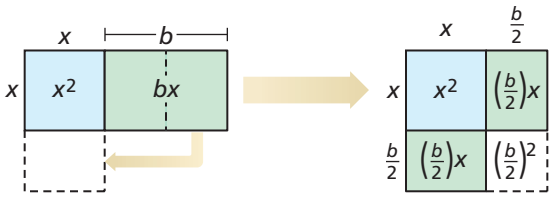
In Example 1, the expression $x^2 - 16x + 64$ is a perfect square trinomial because it can be written as $(x - 8)^2$. In some situations, you will need to add a constant to an expression of the form $x^2 + bx$ to make it a perfect square trinomial. This process is called **completing the square**.

KEY IDEA

Completing the Square

Words To complete the square for the expression $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.

Diagrams In each diagram, the combined area of the shaded regions is $x^2 + bx$. Adding $\left(\frac{b}{2}\right)^2$ completes the square in the second diagram.



Algebra $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)\left(x + \frac{b}{2}\right) = \left(x + \frac{b}{2}\right)^2$

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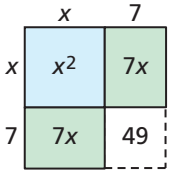
EXAMPLE 2 Completing the Square

WATCH

Complete the square for $x^2 + 14x$. Then factor the trinomial.

SOLUTION

In this binomial, $b = 14$.



Step 1 Find $\left(\frac{b}{2}\right)^2$. $\left(\frac{b}{2}\right)^2 = \left(\frac{14}{2}\right)^2 = 7^2 = 49$

Step 2 Add the result to $x^2 + bx$. $x^2 + 14x + 49$

▶ $x^2 + 14x + 49 = (x + 7)(x + 7) = (x + 7)^2$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Complete the square for the expression. Then factor the trinomial.

4. $x^2 + 8x$ **5.** $x^2 - 2x$ **6.** $x^2 - 9x$

7. MP **REASONING** You complete the square for an expression of the form $x^2 + bx$ by adding 100. What is the value of b?

Completing the square can be used to solve any quadratic equation. First, write the equation in the form $x^2 + bx = d$. Then, when you complete the square, you must add the same number to *both* sides of the equation.

EXAMPLE 3 Solving $ax^2 + bx + c = 0$ when $a = 1$

WATCH

Solve $x^2 - 10x + 7 = 0$ by completing the square.

SOLUTION

$x^2 - 10x + 7 = 0$	Write the equation.
$x^2 - 10x = -7$	Write left side in the form $x^2 + bx$.
$x^2 - 10x + 25 = -7 + 25$	Add $\left(\frac{b}{2}\right)^2 = \left(\frac{-10}{2}\right)^2 = 25$ to each side.
$(x - 5)^2 = 18$	Write left side as a binomial squared.
$x - 5 = \pm\sqrt{18}$	Take square root of each side.
$x = 5 \pm \sqrt{18}$	Add 5 to each side.
$x = 5 \pm 3\sqrt{2}$	Simplify radical.

▶ The solutions are $x = 5 + 3\sqrt{2}$ and $x = 5 - 3\sqrt{2}$.

Math Practice

Find Entry Points

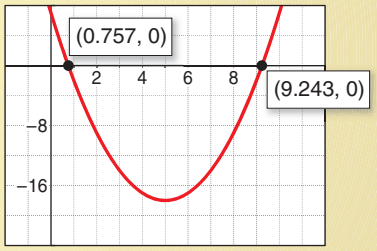
How can you determine when it is convenient to solve by completing the square?

Check

You can check your solutions by graphing $y = x^2 - 10x + 7$.

$9.243 \approx 5 + 3\sqrt{2}$

$0.757 \approx 5 - 3\sqrt{2}$



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EXAMPLE 4 Solving $ax^2 + bx + c = 0$ when $a \neq 1$

WATCH

Solve $3x^2 + 12x + 15 = 0$ by completing the square.

SOLUTION

The coefficient a is not 1, so you must first divide each side of the equation by a.

$3x^2 + 12x + 15 = 0$	Write the equation.
$x^2 + 4x + 5 = 0$	Divide each side by 3.
$x^2 + 4x = -5$	Write left side in the form $x^2 + bx$.
$x^2 + 4x + 4 = -5 + 4$	Add $\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 4$ to each side.
$(x + 2)^2 = -1$	Write left side as a binomial squared.
$x + 2 = \pm\sqrt{-1}$	Take square root of each side.
$x = -2 \pm \sqrt{-1}$	Subtract 2 from each side.
$x = -2 \pm i$	Write in terms of i.

▶ The solutions are $x = -2 + i$ and $x = -2 - i$.

Writing Quadratic Functions in Vertex Form

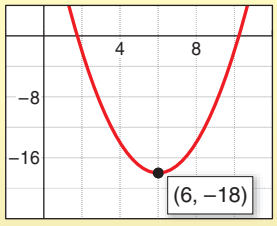
Recall that the vertex form of a quadratic function is $y = a(x - h)^2 + k$, where (h, k) is the vertex of the graph of the function. You can write a quadratic function in vertex form by completing the square.

EXAMPLE 5 Writing a Quadratic Function in Vertex Form

WATCH

Write $y = x^2 - 12x + 18$ in vertex form. Then identify the vertex.

Check



SOLUTION

$y = x^2 - 12x + 18$	Write the function.
$y + ? = (x^2 - 12x + ?) + 18$	Prepare to complete the square.
$y + 36 = (x^2 - 12x + 36) + 18$	Add $\left(\frac{b}{2}\right)^2 = \left(\frac{-12}{2}\right)^2 = 36$ to each side.
$y + 36 = (x - 6)^2 + 18$	Write $x^2 - 12x + 36$ as a binomial squared.
$y = (x - 6)^2 - 18$	Solve for y.

▶ The vertex form of the function is $y = (x - 6)^2 - 18$. The vertex is $(6, -18)$.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

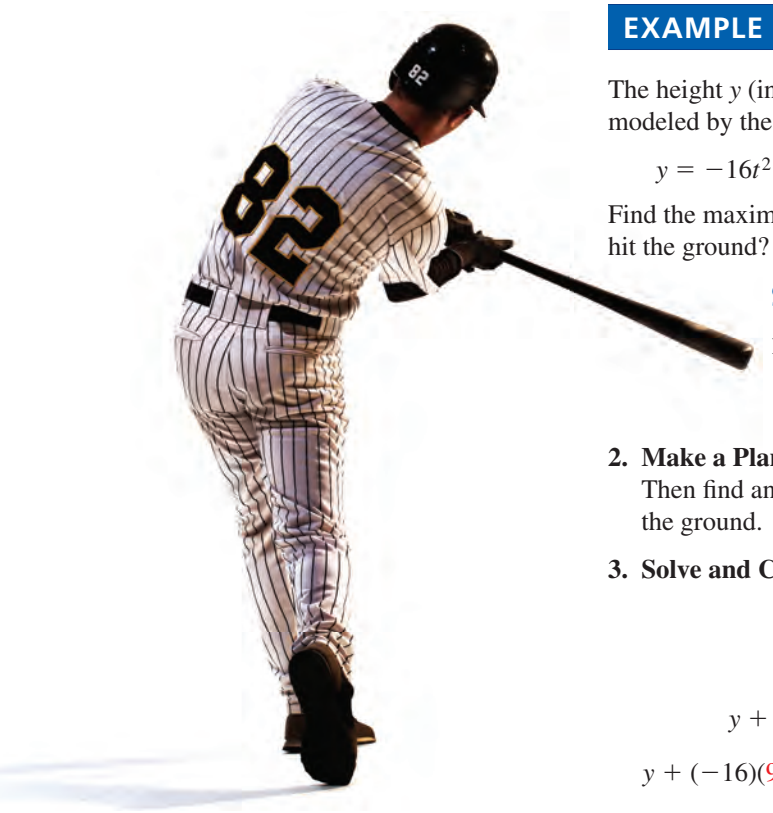
Solve the equation by completing the square.

8. $x^2 - 4x + 8 = 0$ **9.** $x^2 + 8x - 5 = 0$ **10.** $-3x^2 - 18x - 6 = 0$

11. $4x^2 + 32x = -68$ **12.** $6x(x + 2) = -42$ **13.** $2x(x - 2) = 200$

Write the quadratic function in vertex form. Then identify the vertex.

14. $y = x^2 - 8x + 18$ **15.** $y = x^2 + 6x + 4$ **16.** $f(x) = x^2 - 2x - 6$



EXAMPLE 6 Modeling Real Life

WATCH

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The height y (in feet) of a baseball t seconds after it is hit can be modeled by the function

$$y = -16t^2 + 96t + 3.$$

Find the maximum height of the baseball. How long does the ball take to hit the ground?

SOLUTION

1. **Understand the Problem** You are given a quadratic function that represents the height of a ball. You are asked to determine the maximum height of the ball and how long it is in the air.

2. **Make a Plan** Write the function in vertex form to identify the maximum height. Then find and interpret the zeros to determine how long the ball takes to hit the ground.

3. **Solve and Check** Write the function in vertex form by completing the square.

$$y = -16t^2 + 96t + 3 \qquad \text{Write the function.}$$

$$y = -16(t^2 - 6t) + 3 \qquad \text{Factor } -16 \text{ from first two terms.}$$

$$y + ? = -16(t^2 - 6t + ?) + 3 \qquad \text{Prepare to complete the square.}$$

$$y + (-16)(9) = -16(t^2 - 6t + 9) + 3 \qquad \text{Add } (-16)(9) \text{ to each side.}$$

$$y - 144 = -16(t - 3)^2 + 3 \qquad \text{Write } t^2 - 6t + 9 \text{ as a binomial squared.}$$

$$y = -16(t - 3)^2 + 147 \qquad \text{Solve for } y.$$

The vertex is (3, 147). Find the zeros of the function.

$$0 = -16(t - 3)^2 + 147 \qquad \text{Substitute 0 for } y.$$

$$-147 = -16(t - 3)^2 \qquad \text{Subtract 147 from each side.}$$

$$9.1875 = (t - 3)^2 \qquad \text{Divide each side by } -16.$$

$$\pm\sqrt{9.1875} = t - 3 \qquad \text{Take square root of each side.}$$

$$3 \pm \sqrt{9.1875} = t \qquad \text{Add 3 to each side.}$$

Reject the negative solution, $3 - \sqrt{9.1875} \approx -0.03$, because time must be positive.

▶ So, the maximum height of the ball is 147 feet, and it takes $3 + \sqrt{9.1875} \approx 6$ seconds for the ball to hit the ground.

ANOTHER WAY

You can use the coefficients of the original function $y = f(x)$ to find the maximum height.

$$f\left(-\frac{b}{2a}\right) = f\left(-\frac{96}{2(-16)}\right)$$

$$= f(3)$$

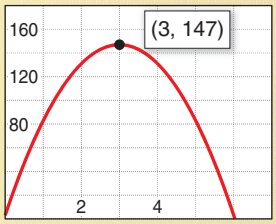
$$= 147$$

Math Practice

Interpret Results

The zeros can be written as $3 \pm \sqrt{9.1875}$ or as $3 \pm \frac{7\sqrt{3}}{4}$. Which form is more helpful when you interpret the solutions in this context?

Check The vertex indicates that the maximum height of 147 feet occurs when $t = 3$. This makes sense because the graph of the function is parabolic with zeros near $t = 0$ and $t = 6$. You can use a graph to check the maximum height.



SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

17. **WHAT IF?** The height of the baseball can be modeled by $y = -16t^2 + 80t + 2$. Find the maximum height of the baseball. How long does the ball take to hit the ground?

3.3 Practice WITH CalcChat® AND CalcView®

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In Exercises 1–8, solve the equation using square roots. Check your solution(s). ▶ *Example 1*

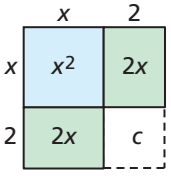
1. $x^2 - 8x + 16 = 25$
2. $r^2 - 10r + 25 = 1$
3. $x^2 - 18x + 81 = 5$
4. $m^2 + 8m + 16 = 45$
5. $y^2 - 24y + 144 = -100$
6. $x^2 - 26x + 169 = -13$
7. $4w^2 + 4w + 1 = 75$
8. $4x^2 - 8x + 4 = 1$

In Exercises 9–18, complete the square for the expression. Then factor the trinomial. ▶ *Example 2*

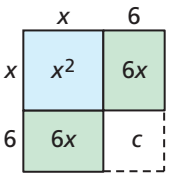
9. $x^2 + 10x$
10. $x^2 + 20x$
11. $y^2 - 12y$
12. $t^2 - 22t$
13. $x^2 - 6x$
14. $x^2 + 24x$
15. $z^2 - 5z$
16. $x^2 + 9x$
17. $w^2 + 13w$
18. $s^2 - 17s$

In Exercises 19 and 20, find the value of c. Then write an expression represented by the diagram.

19.



20.



In Exercises 21–32, solve the equation by completing the square. ▶ *Examples 3 and 4*

21. $x^2 + 6x + 3 = 0$
22. $s^2 + 2s - 6 = 0$
23. $x^2 + 4x - 2 = 0$
24. $t^2 - 8t - 5 = 0$
25. $z(z + 9) = 1$
26. $x(x + 8) = -20$
27. $7t^2 + 28t + 56 = 0$
28. $6r^2 + 6r + 12 = 0$
29. $5x(x + 6) = -50$
30. $4w(w - 3) = 24$
31. $4x^2 - 30x = 12 + 10x$
32. $3s^2 + 8s = 2s - 9$

MP STRUCTURE In Exercises 33–42, decide whether to use factoring, square roots, or completing the square to solve the equation. Explain your reasoning. Then solve the equation.

33. $x^2 - 4x - 21 = 0$
34. $x^2 - 18x + 64 = 0$
35. $(x + 4)^2 = 16$
36. $(x - 7)^2 = -9$
37. $x^2 + 12x + 50 = 0$
38. $x^2 + 13x + 22 = 0$
39. $2x^2 + 4x - 3 = 0$
40. $3x^2 + 12x + 13 = 0$
41. $x^2 + 121 = 0$
42. $4x^2 - 20 = 0$

In Exercises 43–52, write the quadratic function in vertex form. Then identify the vertex. ▶ *Example 5*

43. $y = x^2 - 8x + 19$
44. $y = x^2 - 4x - 1$
45. $g(x) = x^2 + 12x + 37$
46. $h(x) = x^2 + 20x + 90$
47. $f(x) = x^2 - 3x + 4$
48. $g(x) = x^2 + 7x + 2$
49. $y = -x^2 - 2x - 9$
50. $y = -x^2 + 6x - 16$
51. $f(x) = 2x^2 - 8x - 13$
52. $g(x) = 5x^2 + 25x + 32$

53. **ERROR ANALYSIS** Describe and correct the error in completing the square for $x^2 + 30x$.

✗

$x^2 + 30x$

$x^2 + 30x + \frac{30}{2}$

$x^2 + 30x + 15$

54. **ERROR ANALYSIS** Describe and correct the error in writing $y = 4x^2 + 24x - 11$ in vertex form.

✗

$y = 4x^2 + 24x - 11$

$y = 4(x^2 + 6x) - 11$

$y + 9 = 4(x^2 + 6x + 9) - 11$

$y + 9 = 4(x + 3)^2 - 11$

$y = 4(x + 3)^2 - 20$

55. **MODELING REAL LIFE** The height h (in feet) of a badminton birdie t seconds after it is hit can be modeled by $h = -16t^2 + 32t + 4$. ▶ *Example 6*

a. Find the maximum height of the birdie.

b. How long does the birdie take to hit the ground?

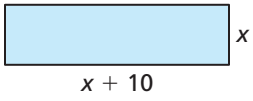
56. MODELING REAL LIFE A drum major throws a baton into the air and catches it. The height h (in feet) of the baton t seconds after it is thrown can be modeled by $h = -16t^2 + 48t + 6$.

a. Find the maximum height of the baton.

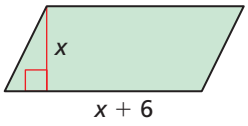
b. The drum major catches the baton when it is 4 feet above the ground. How long is the baton in the air?

CONNECTING CONCEPTS **In Exercises 57–60, find the value of x.**

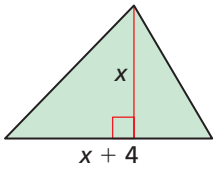
57. Area of rectangle $= 50$



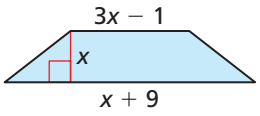
58. Area of parallelogram $= 48$



59. Area of triangle $= 40$



60. Area of trapezoid $= 20$



61. COMPARING METHODS An online skateboard shop charges \$70 per skateboard and sells 50 skateboards per week. For each \$1 decrease in price, one additional skateboard per week is sold. The shop's revenue can be modeled by $y = (70 - x)(50 + x)$.

a. Find the maximum weekly revenue by using the intercept form of the function and by writing the function in vertex form.

b. Which method do you prefer? Explain.

62. WRITING At Buckingham Fountain in Chicago, the height h (in feet) of the water above the main nozzle can be modeled by $h = -16t^2 + 89.6t$, where t is the time (in seconds) since the water has left the nozzle. Describe three different ways you can find the maximum height the water reaches. Then choose a method and find the maximum height of the water.

63. MAKING AN ARGUMENT Your friend says the equation $x^2 + 10x = -20$ can be solved by either completing the square or factoring. Is your friend correct? Explain.

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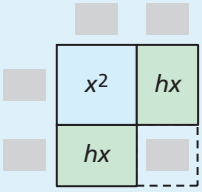
64. HOW DO YOU SEE IT? The diagram represents completing the square for an expression. What is the expression? Complete the diagram and write the resulting perfect square trinomial.



65. COLLEGE PREP Which of the following are solutions of the equation $x^2 - 2ax + a^2 = b^2$? Select all that apply.

Ⓐ ab Ⓑ $-a - b$

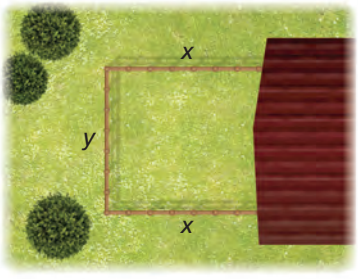
Ⓒ b Ⓓ a

Ⓔ $a - b$ Ⓕ $a + b$

66. THOUGHT PROVOKING The equation of a parabola is $y^2 + 10y + 20x - 15 = 0$. Write the equation in standard form. Identify the vertex, focus, and directrix.

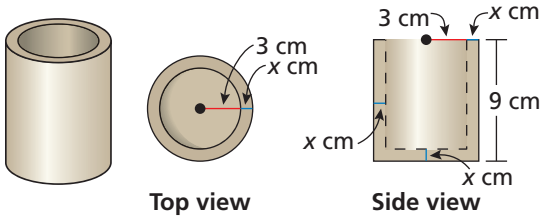
67. CRITICAL THINKING Write a function g in standard form whose graph has the same x-intercepts as the graph of $f(x) = 2x^2 + 8x + 2$. Find the zeros of each function by completing the square. Graph each function.

68. MP PROBLEM SOLVING A farmer is building a rectangular pen along the side of a barn for animals. The barn will serve as one side of the pen. The farmer has 120 feet of fence to enclose an area of 1512 square feet and wants each side of the pen to be at least 20 feet long. What are the dimensions of the pen?



69. CRITICAL THINKING Solve $x^2 + bx + c = 0$ by completing the square. Your answer will be an expression for x in terms of b and c. What does your result represent?

70. **DIG DEEPER** In your pottery class, you are given a lump of clay with a volume of 200 cubic centimeters and are asked to make a cylindrical pencil holder. The pencil holder should be 9 centimeters high and have an inner radius of 3 centimeters. What thickness x should your pencil holder have if you want to use all of the clay?



71. **PERFORMANCE TASK** A company wants to design a new smartphone. The ratio of the screen's height to its width should be 18 : 9. The total area of the screen and the border should be about 120 square centimeters.

a. Decide what the thickness of the border will be on each side. Then find the dimensions of the screen. Justify your answer using a quadratic equation.

b. Make a sketch of your design. Be sure to label the appropriate dimensions.

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REVIEW & REFRESH

WATCH

In Exercises 72–74, graph the function. Label the vertex, axis of symmetry, and x-intercepts.

72. $g(x) = 6(x - 4)^2$

73. $f(x) = x^2 + 2x + 5$

74. $f(x) = 2(x + 10)(x - 12)$

In Exercises 75–78, solve the inequality. Graph the solution.

75. $2x - 3 < 5$

76. $4 - 8y \geq 12$

77. $\frac{n}{3} + 6 > 1$

78. $-\frac{2s}{5} \leq 8$

In Exercises 79–82, perform the operation. Write the answer in standard form.

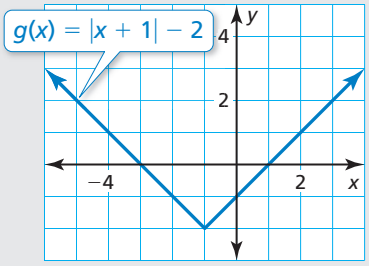
79. $(2 + 5i) + (-4 + 3i)$

80. $(3 + 9i) - (1 - 7i)$

81. $(2 + 4i)(-3 - 5i)$

82. $(9 - 2i)^2$

83. Identify the function family to which g belongs. Compare the graph of the function to the graph of its parent function.



84. Write $y = x^2 + 2x - 48$ in vertex form. Then identify the vertex.

85. **MP STRUCTURE** For what value of m are the graphs of $-2y = 3x - 8$ and $y = mx - 6$ parallel? perpendicular?

In Exercises 86 and 87, write a function g whose graph represents the indicated transformation of the graph of f.

86. $f(x) = 3|x + 5|$; reflection in the x-axis

87. $f(x) = \frac{1}{3}x - \frac{2}{3}$; translation 4 units left

88. Write a function that models the data.

x	1	2	3	4	5
y	9	22	45	78	121

89. Solve $-4t^2 + 16t - 36 = 0$ by completing the square.

90. **MODELING REAL LIFE** A museum has a café with a rectangular patio. The museum wants to add 464 square feet to the area of the patio by expanding the existing patio as shown. By what distance x should the length of the patio be expanded?