EXERCISE SET 3.3

Concept Check

1. According to the Rational Zero Theorem, the possible rational zeros of a fourth-degree polynomial function are ±1, ±2, ±3, and ±6. Is it possible that none of these possible rational zeros are zeros of P? Explain.
Yes. All of the zeros could be irrational numbers, complex nonreal numbers, or a combination of the two.

2. According to the Rational Zero Theorem, the possible rational zeros of a fourth-degree polynomial function are ±1, ±2, ±3, and ±6. Is it possible that all eight of these possible rational zeros are zeros of P? Explain.
No. A polynomial function of degree 4 has at most four zeros.

In Exercises 3 to 10, find the zeros of the polynomial function and state the multiplicity of each zero.

3. $P(x) = (x - 4)(x + 2)^2$ 4 (multiplicity 1), -2 (multiplicity 2)

4. $P(x) = (x - 1)^3(x + 3)^2$ 1 (multiplicity 3), -3 (multiplicity 2)

5. $P(x) = x^2(3x + 5)^2$ 0 (multiplicity 2), $-\frac{5}{3}$ (multiplicity 2)

6. $P(x) = x^3(2x + 1)(3x - 12)^2$
0 (multiplicity 3), $-\frac{1}{2}$ (multiplicity 1), 4 (multiplicity 2)

7. $P(x) = (x^2 - 9)(x + 5)^2$
3 (multiplicity 1), -3 (multiplicity 1), -5 (multiplicity 2)

8. $P(x) = (x + 1)^3(x^2 - 25)^2$
-1 (multiplicity 3), 5 (multiplicity 2), -5 (multiplicity 2)

9. $P(x) = (3x - 5)^2(2x - 7)$
$\frac{5}{3}$ (multiplicity 2), $\frac{7}{2}$ (multiplicity 1)

10. $P(x) = (5x - 2)(x + 3)^4$ $\frac{2}{5}$ (multiplicity 1), -3 (multiplicity 4)

In Exercises 11 to 20, use the Rational Zero Theorem to list possible rational zeros for each polynomial function.

11. $P(x) = x^3 + 3x^2 - 6x - 8$ $\pm1, \pm2, \pm4, \pm8$

12. $P(x) = x^3 - 19x - 30$ $\pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30$

13. $P(x) = 2x^3 + x^2 - 25x + 12$
$\pm1, \pm2, \pm3, \pm4, \pm6, \pm12, \pm\frac{1}{2}, \pm\frac{3}{2}$

14. $P(x) = 3x^3 + 11x^2 - 6x - 8$
$\pm1, \pm2, \pm4, \pm8, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}$

15. $P(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$
$\pm1, \pm2, \pm4, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{1}{6}$

16. $P(x) = 2x^3 + 9x^2 - 2x - 9$
$\pm1, \pm3, \pm9, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}$

17. $P(x) = 4x^4 - 12x^3 - 3x^2 + 12x - 7$
$\pm1, \pm7, \pm\frac{1}{2}, \pm\frac{7}{2}, \pm\frac{1}{4}, \pm\frac{7}{4}$

18. $P(x) = x^5 - 3x^4 - 11x^3 + 51x^2 - 62x + 24$
$\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$

19. $P(x) = x^5 - 10x^4 + 27x^3 + 14x^2 - 128x + 96$
$\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm16, \pm24, \pm32, \pm48, \pm96$

20. $P(x) = x^4 - 9$ $\pm1, \pm3, \pm9$

In Exercises 21 to 30, find the smallest positive integer and the largest negative integer that, by the Upper- and Lower-Bound Theorem, are upper and lower bounds for the real zeros of each polynomial function.

21. $P(x) = x^3 + 3x^2 - 6x - 6$ Upper bound 2, lower bound -5

22. $P(x) = x^3 - 19x - 28$ Upper bound 5, lower bound -5

23. $P(x) = 2x^3 + x^2 - 25x + 10$ Upper bound 4, lower bound -4

24. $P(x) = 2x^3 + 4x^2 - 8x + 7$ Upper bound 2, lower bound -4

25. $P(x) = 3x^4 - 9x^3 + 2x^2 - 5x - 8$
Upper bound 4, lower bound -1

26. $P(x) = -3x^3 - 2x^2 + 4x - 31$
Upper bound 1, lower bound -3

27. $P(x) = -5x^4 + 14x^3 + 2x^2 - 12x + 325$
Upper bound 4, lower bound -3

28. $P(x) = x^5 - x^4 - 7x^3 + 7x^2 - 12x - 12$
Upper bound 4, lower bound -3

29. $P(x) = x^5 - 32$ Upper bound 2, lower bound -1

30. $P(x) = x^4 - 1$ Upper bound 1, lower bound -1

In Exercises 31 to 44, use Descartes' Rule of Signs to state the number of possible positive and negative real zeros of each polynomial function.

31. $P(x) = x^3 + 3x^2 - 6x - 8$
One positive zero, two or no negative zeros

32. $P(x) = x^3 - 19x - 30$
One positive zero, two or no negative zeros

33. $P(x) = 2x^3 + x^2 - 25x + 12$
Two or no positive zeros, one negative zero

34. $P(x) = 3x^3 + 11x^2 - 6x - 8$
One positive zero, two or no negative zeros

35. $P(x) = 2x^5 - 37x^4 + 258x^3 - 820x^2 + 1100x - 375$
Five, three, or one positive zeros; no negative zeros

36. $P(x) = 2x^4 - 3x^3 - 55x^2 + 111x - 55$
Three or one positive zeros; one negative zero

37. $P(x) = 2x^5 + 23x^4 + 90x^3 + 152x^2 + 116x + 33$
No positive zeros; five, three, or one negative zeros

38. $P(x) = x^5 + 4x^4 - 16x^3 - 56x^2 - 17x - 60$
One positive zero; four, two, or no negative zeros

39. $P(x) = x^5 - 2x^4 - 35x^3 + 158x^2 - 242x + 168$
Four, two, or no positive zeros; one negative zero

40. $P(x) = 2x^5 - 14x^4 - 4x^3 + 60x^2 - 254x + 210$
Four, two, or no positive zeros; one negative zero

Indicates Try It Exercises

41. $P(x) = 2x^6 - 19x^5 + 49x^4 - 88x^3 - 158x^2 + 656x - 280$
Five, three, or one positive zeros; one negative zero

42. $P(x) = 2x^7 - 27x^6 + 149x^5 - 464x^4 + 912x^3 - 1087x^2 + 665x - 150$
Seven, five, three, or one positive zeros; no negative zeros

43. $P(x) = 12x^7 - 112x^6 + 421x^5 - 840x^4 + 1038x^3 - 938x^2 + 629x - 210$
Seven, five, three, or one positive zeros; no negative zeros

44. $P(x) = x^7 + 2x^5 + 3x^3 + x$
No positive zeros; no negative zeros

In Exercises 45 to 68, find the zeros of each polynomial function. If a zero is a multiple zero, state its multiplicity.

45. $P(x) = x^3 + 3x^2 - 6x - 8$ $2, -1, -4$

46. $P(x) = x^3 - 19x - 30$ $5, -2, -3$

47. $P(x) = 2x^3 + x^2 - 25x + 12$ $3, -4, \frac{1}{2}$

48. $P(x) = 3x^3 + 11x^2 - 6x - 8$ $1, -4, -\frac{2}{3}$

49. $P(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$ $\frac{1}{2}, -\frac{1}{3}, -2$ (multiplicity 2)

50. $P(x) = 2x^3 + 9x^2 - 2x - 9$ $1, -1, -\frac{9}{2}$

51. $P(x) = 2x^4 - 9x^3 - 2x^2 + 27x - 12$ $\frac{1}{2}, 4, \sqrt{3}, -\sqrt{3}$

52. $P(x) = 3x^3 - x^2 - 6x + 2$ $\frac{1}{3}, \sqrt{2}, -\sqrt{2}$

53. $P(x) = x^3 - 8x^2 + 8x + 24$ $6, 1 + \sqrt{5}, 1 - \sqrt{5}$

54. $P(x) = x^3 - 7x^2 - 7x + 69$ $-3, 5 + \sqrt{2}, 5 - \sqrt{2}$

55. $P(x) = 2x^4 - 19x^3 + 51x^2 - 31x + 5$ $5, \frac{1}{2}, 2 + \sqrt{3}, 2 - \sqrt{3}$

56. $P(x) = 4x^4 - 35x^3 + 71x^2 - 4x - 6$ $3, -\frac{1}{4}, 3 + \sqrt{7}, 3 - \sqrt{7}$

57. $P(x) = 3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24$
$1, -1, -2, -\frac{2}{3}, 3 + \sqrt{3}, 3 - \sqrt{3}$

58. $P(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$
$-2, -1, \frac{1}{2}, 1$

59. $P(x) = x^3 - 3x - 2$ $2, -1$ (multiplicity 2)

60. $P(x) = 3x^4 - 4x^3 - 11x^2 + 16x - 4$ $-2, \frac{1}{3}, 1, 2$

61. $P(x) = x^4 - 5x^2 - 2x$ $0, -2, 1 + \sqrt{2}, 1 - \sqrt{2}$

62. $P(x) = x^3 - 2x + 1$ $1, \frac{-1 + \sqrt{5}}{2}, \frac{-1 - \sqrt{5}}{2}$

63. $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ -1 (multiplicity 3), 2

64. $P(x) = 6x^4 - 17x^3 - 11x^2 + 42x$ $-\frac{3}{2}, 0, 2, \frac{7}{3}$

65. $P(x) = 2x^4 - 17x^3 + 4x^2 + 35x - 24$ $-\frac{3}{2}, 1$ (multiplicity 2), 8

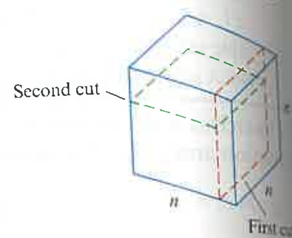
66. $P(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$ -1 (multiplicity 5)

67. $P(x) = x^3 - 16x$ $0, 4, -4$

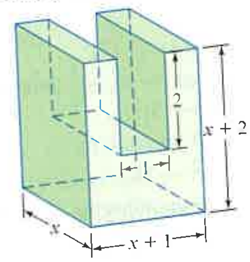
68. $P(x) = x^3 - 4x^2 - 3x$ $0, 2 + \sqrt{7}, 2 - \sqrt{7}$

69. **Find the Dimensions** A cube measures n inches on each edge. If a slice 2 inches thick is cut from one face of the cube, the resulting solid has a volume of 567 cubic inches. Find n.
$n = 9$ in.

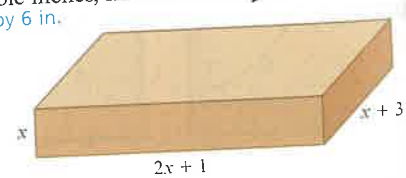
70. **Find the Dimensions** A cube measures n inches on each edge. If a slice 1 inch thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube as shown, the resulting solid has a volume of 1560 cubic inches. Find the dimensions of the original cube.
13 in. on each edge



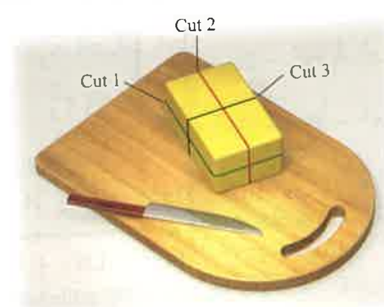
71. **Dimensions of a Solid** For what value of x will the volume of the following solid be 112 cubic inches?
$x = 4$ in.



72. **Dimensions of a Box** The length of a rectangular box is 1 inch more than twice the height of the box, and the width is 3 inches more than the height. If the volume of the box is 126 cubic inches, find the dimensions of the box.
3 by 7 by 6 in.



73. **Pieces and Cuts** One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



You might be inclined to think that every additional cut doubles the previous number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

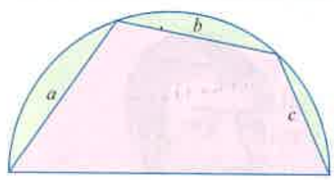
$$P(n) = \frac{n^3 + 5n + 6}{6}$$

a. Use the given function to determine the maximum number of pieces that can be produced by five straight cuts. 26 pieces

b. What is the fewest number of straight cuts that are needed to produce 64 pieces? 7 cuts

74. **Inscribed Quadrilateral** Isaac Newton discovered that if a quadrilateral with sides of lengths a, b, c, and x is inscribed in a semicircle with diameter x, then the lengths of the sides are related by the following equation.

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0$$



Given $a = 6$, $b = 5$, and $c = 4$, find x. Round to the nearest hundredth. 10.04

75. **Cannonball Stacks** Cannonballs can be stacked to form a pyramid with a square base. The total number of cannonballs T in one of these square pyramids is

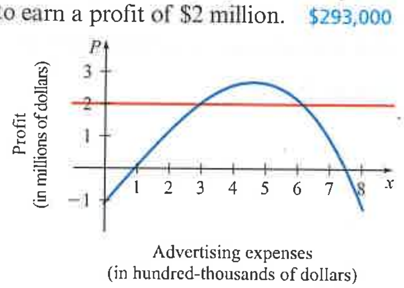
$$T = \frac{1}{6}(2n^3 + 3n^2 + n)$$

where n is the number of rows (levels). If 140 cannonballs are used to form a square pyramid, how many rows are in the pyramid? 7 rows

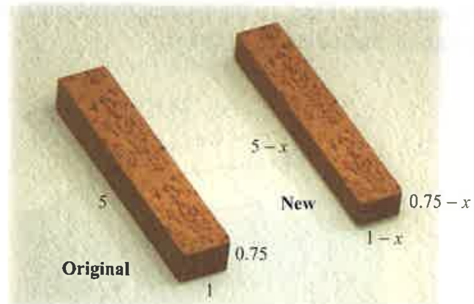
76. **Advertising Expenses** A company manufactures digital cameras. The company estimates that the profit from camera sales is

$$P(x) = -0.02x^3 + 0.01x^2 + 1.2x - 1.1$$

where P is the profit in millions of dollars and x is the amount, in hundred-thousands of dollars, spent on advertising. Determine the minimum amount, rounded to the nearest thousand dollars, the company needs to spend on advertising if it is to earn a profit of \$2 million. \$293,000

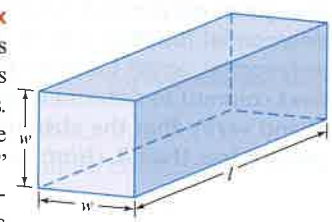


77. **Cost Cutting** A nutrition bar in the shape of a rectangular solid measures 0.75 inch by 1 inch by 5 inches. To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x inches. What value of x, rounded to the nearest thousandth of an inch, will produce a new nutrition bar with a volume that is 0.75 cubic inch less than the present bar's volume? 0.084 in.



78. **Selection of Cards** The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 ways to choose three cards from a given group. How many cards are in the group? 9 cards

79. **Dimensions of a Box** A rectangular box is square on two ends and has length plus girth of 81 inches. (The girth of a box is the shortest distance "around" the box.) Determine the possible lengths l of the box if its volume is 4900 cubic inches. Round approximate values to the nearest hundredth of an inch. In the above figure, assume that $w < l$. 25 in. or 29.05 in.



80. **Medication Level** Pseudoephedrine hydrochloride is an allergy medication. The polynomial function

$$L(t) = 0.03t^4 + 0.4t^3 - 7.3t^2 + 23.1t$$

where $0 \leq t \leq 5$, models the level of pseudoephedrine hydrochloride, in milligrams, in the bloodstream of a patient t hours after 30 milligrams of the medication have been taken.

At what times, to the nearest minute, does the level of pseudoephedrine hydrochloride in the bloodstream reach 12 milligrams? After 39 min and after 3 h 38 min

81. **Weight and Height of Giraffes** A veterinarian at a wild animal park has determined that the average weight w, in pounds, of an adult male giraffe is closely approximated by the function

$$w = 8.3h^3 - 307.5h^2 + 3914h - 15{,}230$$

where h is the giraffe's height in feet, and $15 \le h \le 18$. Use the above function to estimate the height of a giraffe that weighs 3150 pounds. Round to the nearest tenth of a foot. 16.9 ft

82. **Propane Tank Dimensions** A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 feet long and the volume of the tank is 9π cubic feet. Find, to the nearest thousandth of a foot, the length of the radius x. 1.098 ft

Enrichment Exercises

The mathematician Augustin Louis Cauchy proved a theorem that can be used to quickly establish a bound B for *all* the absolute values of the zeros (real and complex) of a given polynomial function. In Exercises 83 to 86, a polynomial function and its zeros are given. For each polynomial, apply Cauchy's Bound Theorem (shown in the next column) to determine the bound B for the polynomial and verify that the absolute value of each of the given zeros is less than B. (*Hint:* $|a + bi| = \sqrt{a^2 + b^2}$)

Cauchy's Bound Theorem

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0$ be a polynomial function with complex coefficients. The absolute value of each zero of P is less than

$$B = \left(\frac{\text{maximum of }(|a_{n-1}|, |a_{n-2}|, \ldots, |a_1|, |a_0|)}{|a_n|} + 1\right)$$

83. $P(x) = 2x^3 - 5x^2 - 28x + 15$, zeros: $-3, \frac{1}{2}, 5$
$B = 15$. The absolute value of each zero is less than B.

84. $P(x) = x^3 - 5x^2 + 2x + 8$, zeros: $-1, 2, 4$
$B = 9$. The absolute value of each zero is less than B.

85. $P(x) = x^4 - 2x^3 + 9x^2 + 2x - 10$, zeros: $1 + 3i$, $1 - 3i$, $1, -1$ $B = 11$. The absolute value of each zero is less than B.

86. $P(x) = x^4 - 4x^3 + 14x^2 - 4x + 13$, zeros: $2 + 3i$, $2 - 3i$, i, $-i$
$B = 15$. The absolute value of each zero is less than B.

Augustin Louis Cauchy
(1789–1857)

MID-CHAPTER 3 QUIZ

1. Use the Remainder Theorem to find $P(5)$ for the function $P(x) = 3x^3 + 7x^2 - 2x - 5$. 535 [3.1]

2. Use the Factor Theorem to determine whether $(x - 5)$ is a factor of $P(x) = x^3 - 6x^2 + 3x + 10$. Yes [3.1]

3. Determine the far-left and the far-right behavior of the graph of $P(x) = 4x^3 - 3x^2 - 6x - 1$. Down to the far left, up to the far right. [3.2]

4. Find the zeros of $P(x) = 3x^3 - 2x^2 - 27x + 18$. $-3, \frac{2}{3}, 3$ [3.3]

5. Use the Intermediate Value Theorem to verify that $P(x) = 2x^4 - 5x^3 + x^2 - 20x - 28$ has a zero between 3 and 4. $P(3) < 0$ and $P(4) > 0$. Therefore, by the Intermediate Value Theorem, the continuous function P has a zero between 3 and 4. [3.2]

6. Use the Rational Zero Theorem to list all possible rational zeros of $P(x) = 3x^3 + 7x^2 - 18x + 8$.
$\pm\frac{1}{3}, \pm 1, \pm\frac{2}{3}, \pm 2, \pm\frac{4}{3}, \pm 4, \pm\frac{8}{3}, \pm 8$ [3.3]

7. Use a graphing utility to estimate, to the nearest hundredth, the coordinates of the point where the function $P(x) = -x^3 + 10x^2 - 27x + 25$ has a relative minimum.
Relative minimum: $y \approx 2.94$ at $x \approx 1.88$ [3.2]

8. Find the zeros of $P(x) = 2x^4 - 19x^3 + 57x^2 - 64x + 20$. If a zero is a multiple zero, state its multiplicity.
$\frac{1}{2}$, 2 (multiplicity 2), 5 [3.3]

SECTION 3.4

Fundamental Theorem of Algebra
Number of Zeros of a Polynomial Function
Conjugate Pair Theorem
Finding a Polynomial Function with Given Zeros

Fundamental Theorem of Algebra

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A19.

PS1. What is the conjugate of $3 - 2i$? [P.6] $3 + 2i$

PS2. What is the conjugate of $2 + i\sqrt{5}$? [P.6] $2 - i\sqrt{5}$

PS3. Multiply: $(x - 1)(x - 3)(x - 4)$ [P.3] $x^3 - 8x^2 + 19x - 12$

PS4. Multiply: $[x - (2 + i)][x - (2 - i)]$ [P.3/P.6] $x^2 - 4x + 5$

PS5. Solve: $x^2 + 9 = 0$ [1.3] $-3i, 3i$

PS6. Solve: $x^2 - x + 5 = 0$ [1.3] $\frac{1}{2} \pm \frac{1}{2}i\sqrt{19}$

Fundamental Theorem of Algebra

The German mathematician Carl Friedrich Gauss (1777–1855) was the first to prove that every polynomial function has at least one complex zero. This concept is so basic to the study of algebra that it is called the **Fundamental Theorem of Algebra.** The proof of the Fundamental Theorem is beyond the scope of this text; however, it is important to understand the theorem and its consequences. As you consider each of the following theorems, keep in mind that the terms *complex coefficients* and *complex zeros* include real coefficients and real zeros because the set of real numbers is a subset of the set of complex numbers.

Fundamental Theorem of Algebra

If P is a polynomial function of degree $n \geq 1$ with complex coefficients, then P has at least one complex zero.

Math Matters

Bettmann/CORBIS

Carl Friedrich Gauss (1777–1855) has often been referred to as the Prince of Mathematics. His work covered topics in algebra, calculus, analysis, probability, number theory, non-Euclidean geometry, astronomy, and physics, to name but a few. The following quote by Eric Temple Bell gives credence to the assertion that Gauss was one of the greatest mathematicians of all time. "Archimedes, Newton, and Gauss, these three, are in a class by themselves among the great mathematicians, and it is not for ordinary mortals to attempt to range them in order of merit."*

**Men of Mathematics*, by E. T. Bell, New York, Simon and Schuster, 1937.

Number of Zeros of a Polynomial Function

Let P be a polynomial function of degree $n \geq 1$ with complex coefficients. The Fundamental Theorem implies that P has a complex zero—say, c_1. The Factor Theorem implies that

$$P(x) = (x - c_1)Q(x)$$

where $Q(x)$ is a polynomial of degree 1 less than the degree of P. Recall that the polynomial $Q(x)$ is called a *reduced polynomial.* Assuming that the degree of $Q(x)$ is 1 or more, the Fundamental Theorem implies that it also must have a zero. A continuation of this reasoning process leads to the following theorem.

Linear Factor Theorem

If P is a polynomial function of degree $n \geq 1$ with leading coefficient $a_n \neq 0$,

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x^1 + a_0$$

then P has exactly n linear factors

$$P(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.