63. **House of Cards** The number of cards C needed to build a house of cards with r rows (levels) is given by the polynomial function $C(r) = 1.5r^2 + 0.5r$.

Topham/The Image Works

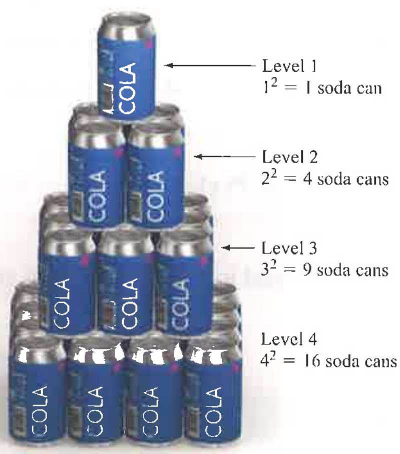
Use the Remainder Theorem to determine the number of cards needed to build a house of cards with

a. $r = 7$ rows

b. $r = 12$ rows

64. **Display of Soda Cans** The number S of soda cans needed to build a square pyramid display with n levels is given by the polynomial function

$$S(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$



A square pyramid display with n^2 soda cans in level n

Use the Remainder Theorem to determine the number of soda cans needed to build a square pyramid display with

a. $n = 6$ levels

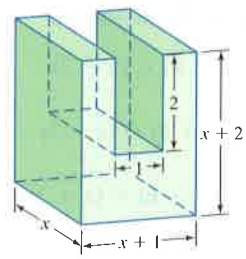
b. $n = 12$ levels

65. **Election of Class Officers** The number of ways a class of n students can elect a president, a vice president, a secretary, and a treasurer is given by $P(n) = n^4 - 6n^3 + 11n^2 - 6n$, where $n \geq 4$. Use the Remainder Theorem to determine the number of ways the class can elect officers if the class consists of

a. $n = 8$ students

b. $n = 18$ students

66. **Volume of a Solid** The volume, in cubic inches, of the following solid is given by $V(x) = x^3 + 3x^2$.

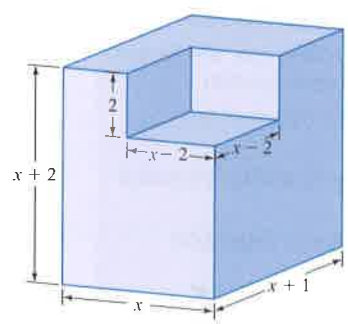


Use the Remainder Theorem to determine the volume of the solid if

a. $x = 7$ inches

b. $x = 11$ inches

67. **Volume of a Solid** The volume, in cubic inches, of the following solid is given by $V(x) = x^3 + x^2 + 10x - 8$.

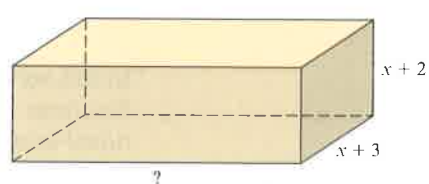


Use the Remainder Theorem to determine the volume of the solid if

a. $x = 6$ inches

b. $x = 9$ inches

68. **Volume of a Box** A rectangular box has a volume of $V(x) = x^3 + 10x^2 + 31x + 30$ cubic inches. The height of the box is $x + 2$ inches. The width of the box is $x + 3$ inches. Find the length of the box in terms of x.



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Enrichment Exercises

69. Use synthetic division to divide each of the following polynomials by $x - 1$.

$$x^3 - 1, x^5 - 1, x^7 - 1$$

Use the pattern suggested by these quotients to write the quotient of $(x^9 - 1) \div (x - 1)$.

In Exercises 70 to 73, determine the value of k so that the divisor is a factor of the dividend.

70. $(x^3 - x^2 - 14x + k) \div (x - 2)$

71. $(2x^3 + x^2 - 25x + k) \div (x - 3)$

72. $(3x^3 + 14x^2 + kx - 6) \div (x + 2)$

73. $(x^4 + 3x^3 - 8x^2 + kx + 16) \div (x + 4)$

74. Use the Factor Theorem to show that for any positive integer n

$$P(x) = x^n - 1$$

has $x - 1$ as a factor.

75. Find the remainder of

$$5x^{48} + 6x^{10} - 5x + 7$$

divided by $x - 1$.

76. Find the remainder of

$$18x^{80} - 6x^{50} + 4x^{20} - 2$$

divided by $x + 1$.

77. Determine whether i is a zero of

$$P(x) = x^3 - 3x^2 + x - 3$$

78. Determine whether $-2i$ is a zero of

$$P(x) = x^4 - 2x^3 + x^2 - 8x - 12$$

SECTION 3.2

- Far-Left and Far-Right Behavior
- Maximum and Minimum Values
- Real Zeros of a Polynomial Function
- Intermediate Value Theorem
- Real Zeros, x-Intercepts, and Factors of a Polynomial Function
- Even and Odd Powers of $(x - c)$ Theorem
- Procedure for Graphing Polynomial Functions
- Cubic and Quartic Regression Functions

Polynomial Functions of Higher Degree

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A18.

PS1. Find the minimum value of $P(x) = x^2 - 4x + 6$. [2.4]

PS2. Find the maximum value of $P(x) = -2x^2 - x + 1$. [2.4]

PS3. Find the interval on which $P(x) = x^2 + 2x + 7$ is increasing. [2.4]

PS4. Find the interval on which $P(x) = -2x^2 + 4x + 5$ is decreasing. [2.4]

PS5. Factor: $x^4 - 5x^2 + 4$ [P.4]

PS6. Find the x-intercepts of the graph of $P(x) = 6x^2 - x - 2$. [2.4]

Table 3.1 summarizes information developed in Chapter 2 about graphs of polynomial functions of degree 0, 1, or 2.

Table 3.1

Polynomial Function $P(x)$	Graph
$P(x) = a$ (degree 0)	Horizontal line through $(0, a)$
$P(x) = ax + b$ (degree 1), $a \neq 0$	Line with y-intercept $(0, b)$ and slope a
$P(x) = ax^2 + bx + c$ (degree 2), $a \neq 0$	Parabola with vertex $\left(-\frac{b}{2a}, P\left(-\frac{b}{2a}\right)\right)$

In this section, we will focus on polynomial functions of degree 3 or higher. These functions can be graphed by the technique of plotting points; however, some additional knowledge about polynomial functions will make graphing easier.

CALCULUS CONNECTION

All polynomial functions have graphs that are **smooth continuous curves.** The terms *smooth* and *continuous* are defined rigorously in calculus, but for the

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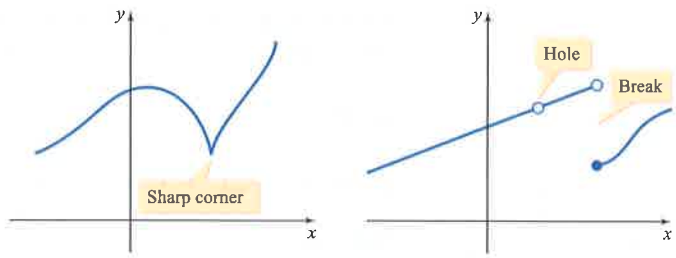
> **Note**
>
> The **general form of a polynomial** is given by
>
> $$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$
>
> In this text, the coefficients $a_n, a_{n-1}, \ldots, a_0$ are all real numbers unless specifically stated otherwise.

present, a smooth curve is a curve that does not have sharp corners, like the graph shown in Figure 3.5a. A continuous curve does not have a break or hole, like the graph shown in Figure 3.5b.



a. Continuous, but not smooth **b.** Not continuous

Figure 3.5

Far-Left and Far-Right Behavior

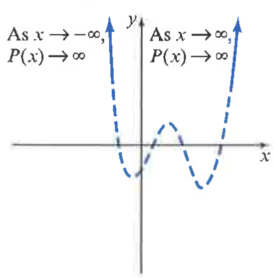
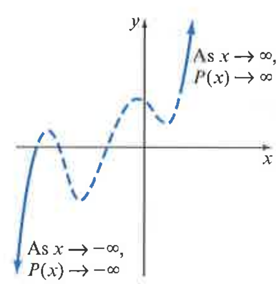
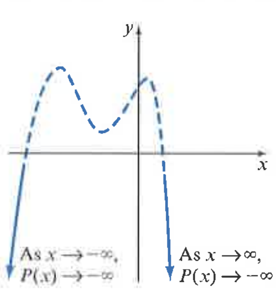
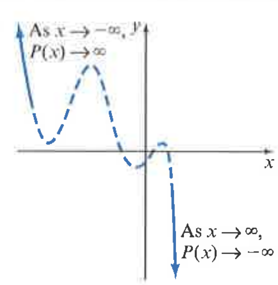
> **Note**
>
> The leading term of a polynomial function in x is the nonzero term that contains the largest power of x. The leading coefficient of a polynomial function is the coefficient of the leading term.

The graph of a polynomial function may have several up and down fluctuations; however, the graph of every polynomial function eventually will increase or decrease without bound as $|x|$ becomes larger. The **leading term** $a_n x^n$ is said to be the **dominant term** of $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ because, as $|x|$ becomes larger, the absolute value of $a_n x^n$ will be much larger than the absolute value of any of the other terms. Because of this condition, you can determine the **far-left** and **far-right behavior** of the polynomial by examining the **leading coefficient** a_n and the degree n of the polynomial.

Table 3.2 shows the far-left and far-right behavior of a polynomial function P with leading term $a_n x^n$.

Table 3.2 **The Leading Term Test**

The far-left and far-right behavior of the graph of the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ can be determined by examining its leading term $a_n x^n$.

	n **Is Even**	n **Is Odd**
$a_n > 0$	If $a_n > 0$ and n is even, then the graph of P goes up to the far left and up to the far right. 	If $a_n > 0$ and n is odd, then the graph of P goes down to the far left and up to the far right. 
$a_n < 0$	If $a_n < 0$ and n is even, then the graph of P goes down to the far left and down to the far right. 	If $a_n < 0$ and n is odd, then the graph of P goes up to the far left and down to the far right. 

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EXAMPLE 1 Determine the Far-Left and Far-Right Behavior of a Polynomial Function

Examine the leading term to determine the far-left and far-right behavior of the graph of each polynomial function.

a. $P(x) = x^3 - x$ **b.** $S(x) = \frac{1}{2}x^4 - \frac{5}{2}x^2 + 2$

c. $T(x) = -2x^3 + x^2 + 7x - 6$ **d.** $U(x) = 9 + 8x^2 - x^4$

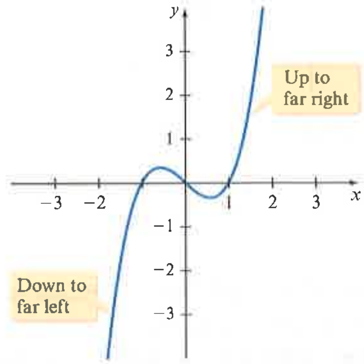
Solution

a. Because $a_n = 1$ is *positive* and $n = 3$ is *odd*, the graph of P goes down to the far left and up to the far right. See Figure 3.6.

b. Because $a_n = \frac{1}{2}$ is *positive* and $n = 4$ is *even*, the graph of S goes up to the far left and up to the far right. See Figure 3.7.

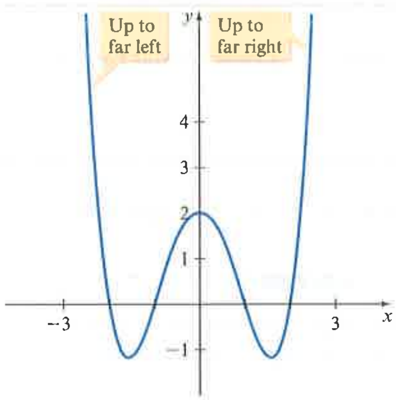
c. Because $a_n = -2$ is *negative* and $n = 3$ is *odd*, the graph of T goes up to the far left and down to the far right. See Figure 3.8.

d. The leading term of U is $-x^4$ and the leading coefficient is -1. Because $a_n = -1$ is *negative* and $n = 4$ is *even*, the graph of U goes down to the far left and down to the far right. See Figure 3.9.



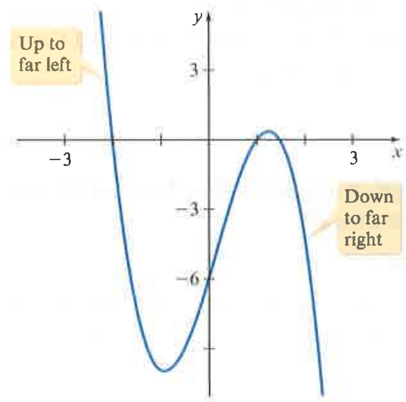
$P(x) = x^3 - x$

Figure 3.6



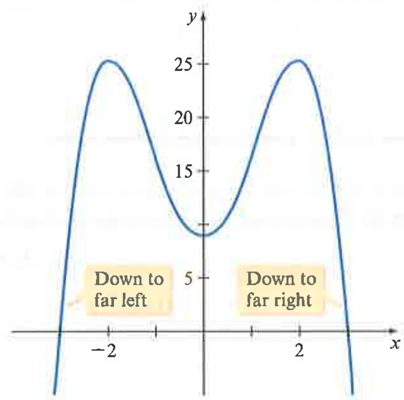
$S(x) = \frac{1}{2}x^4 - \frac{5}{2}x^2 + 2$

Figure 3.7



$T(x) = -2x^3 + x^2 + 7x - 6$

Figure 3.8



$U(x) = -x^4 + 8x^2 + 9$

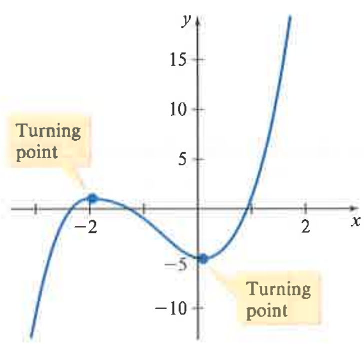
Figure 3.9

Try Exercise 6, page 283

Maximum and Minimum Values

Figure 3.10 illustrates the graph of a polynomial function of degree 3 with two **turning points,** points at which the function changes from an increasing function to a decreasing function, or vice versa. In general, the graph of a polynomial function of degree n has at most $n - 1$ turning points.

Turning points can be related to the concepts of maximum and minimum values of a function. These concepts were introduced in the discussion of graphs of second-degree equations in two variables earlier in the text. Recall that the minimum value of a function f is the smallest range value of f. It is often called



$P(x) = 2x^3 + 5x^2 - x - 5$

Figure 3.10

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the **absolute minimum.** The maximum value of a function f is the largest range value of f. The maximum value of a function is also called the **absolute maximum.** For the function whose graph is shown in Figure 3.11, the y value of point E is the absolute minimum. There are no y values less than y_5.

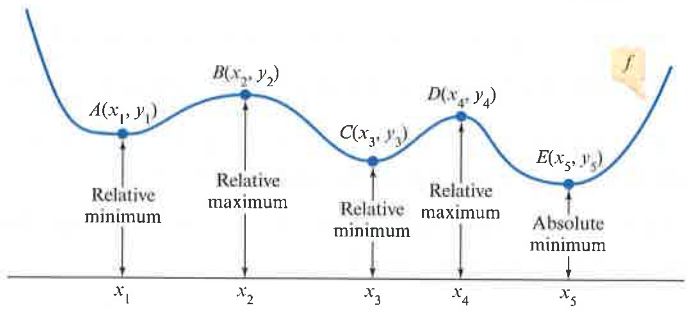

Figure 3.11

Now consider y_1, the y value of turning point A in Figure 3.11. It is not the smallest y value of every point on the graph of f; however, it is the smallest y value if we *localize* our field of view to a small open interval containing x_1. It is for this reason that we refer to y_1 as a **local minimum,** or **relative minimum,** of f. The y value of point C is also a relative minimum of f.

The function does not have an absolute maximum because it goes up both to the far left and to the far right. The y value of point B is a relative maximum, as is the y value of point D. The formal definitions of **relative maximum** and **relative minimum** are presented below.

Definition of Relative Minimum and Relative Maximum

If there is an open interval I containing c on which

- $f(c) \leq f(x)$ for all x in I, then $f(c)$ is a **relative minimum** of f.
- $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a **relative maximum** of f.

Question • Is the absolute minimum y_5 shown in Figure 3.11 also a relative minimum of f?

Integrating Technology

Use a Graphing Calculator and WolframAlpha to Determine a Relative Maximum of a Function

To use a TI-83/TI-83 Plus/TI-84 Plus calculator to estimate the relative maximum of

$$P(x) = 0.3x^3 - 2.8x^2 + 6.4x + 2$$

(continued)

Answer • Yes, the absolute minimum y_5 also satisfies the requirements of a relative minimum.

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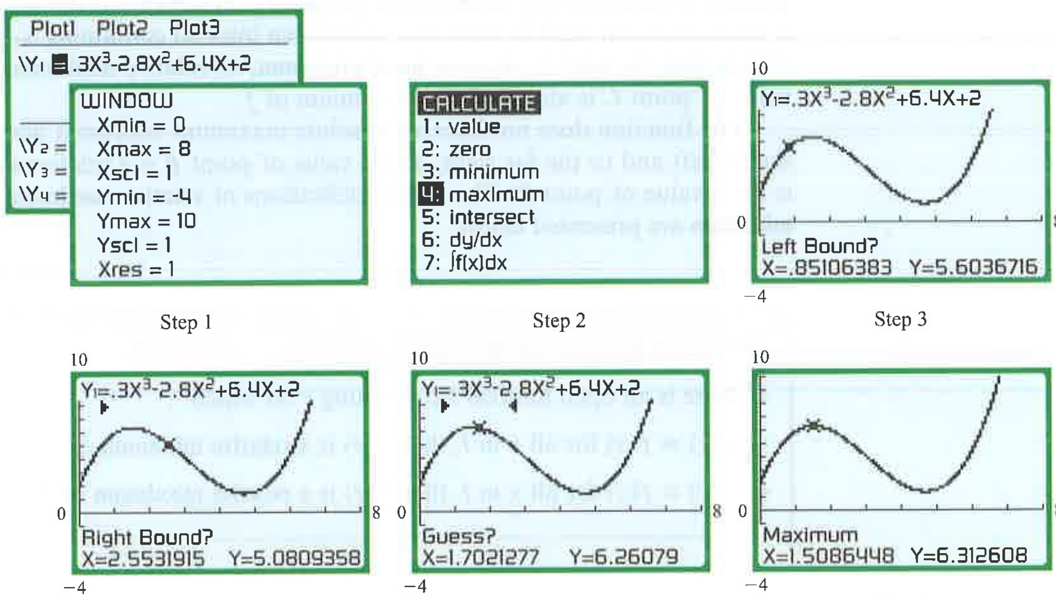
follow these steps:

1. Enter the function in the Y= menu. Choose your window settings.
2. Select 4: maximum from the CALC menu, which is located above the TRACE key. The graph of Y_1 is displayed.
3. Press ◀ or ▶ repeatedly to select an x value that is to the left of the relative maximum point. Press ENTER. A left bound is displayed in the bottom left corner.
4. Press ▶ repeatedly to select an x value that is to the right of the relative maximum point. Press ENTER. A right bound is displayed in the bottom left corner.
5. The word **Guess?** is now displayed in the bottom left corner. Press ◀ repeatedly to move to a point near the maximum point. Press ENTER.
6. The cursor appears on the relative maximum point, and the coordinates of the relative maximum point are displayed. In this example, the y value 6.312608 is the approximate relative maximum of the function P. (*Note:* If your window settings, bounds, or guess are different from those shown here, then your final results may differ slightly from the final results shown in step 6.)



Step 1 Step 2 Step 3

Step 4 Step 5 Step 6

Scan the following QR code to access WolframAlpha on a mobile device.

www.wolframalpha.com

WolframAlpha can be used to graph polynomial functions and to determine the relative maxima and the relative minima of a polynomial function. For instance, to graph

$$P(x) = 0.3x^3 - 2.8x^2 + 6.4x + 2$$

enter the text shown in the following WolframAlpha input field.

```
plot 0.3x^3 - 2.8x^2 + 6.4x + 2
```

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The caret symbols are used to denote exponentiation. Click on the equal sign icon to display the graph. Use the text in the following input field to find that the relative maximum of P is approximately 6.31261 and it occurs at $x \approx 1.50864$.

```
local maximum 0.3x^3 - 2.8x^2 + 6.4x + 2
```

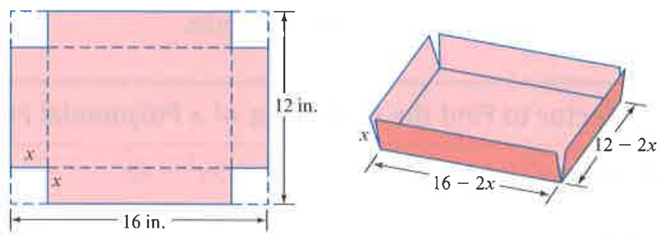
WolframAlpha can also be used to evaluate a function for a given domain value. For example, the text in the following input field yields 4.1 as the value of $P(3)$.

```
0.3x^3 - 2.8x^2 + 6.4x + 2, x = 3
```

The following example illustrates the role a maximum plays in an application.

EXAMPLE 2 Solve an Application

A rectangular piece of cardboard measures 12 inches by 16 inches. An open box is formed by cutting squares that measure x inches by x inches from each of the corners of the cardboard and folding up the sides, as shown below.



a. Express the volume V of the box as a function of x.

b. Determine (to the nearest tenth of an inch) the x value that maximizes the volume.

Solution

a. The height, width, and length of the open box are x, $12 - 2x$, and $16 - 2x$. The volume is given by

$$V(x) = x(12 - 2x)(16 - 2x)$$

$$V(x) = 4x^3 - 56x^2 + 192x$$

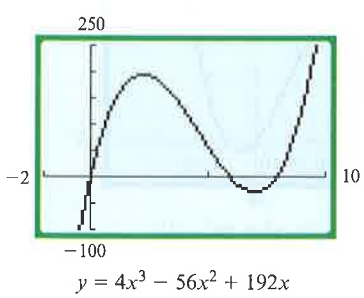


Figure 3.12

b. Use a graphing utility to graph $y = V(x)$. The graph is shown in Figure 3.12.

Note that we are interested only in the part of the graph for which $0 < x < 6$. This is because the length of each side of the box must be positive. In other words,

$$x > 0, \quad 12 - 2x > 0, \quad \text{and} \quad 16 - 2x > 0$$
$$x < 6 \qquad\qquad x < 8$$

(continued)

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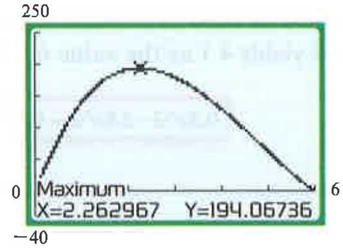
Integrating Technology

A TI graphing calculator program is available that simulates the construction of a box by cutting out squares from each corner of a rectangular piece of cardboard. This program, CUTOUT, can be found at our online study center at www.cengagebrain.com.

The domain of V is the intersection of the solution sets of the three inequalities. Thus the domain is $\{x \mid 0 < x < 6\}$.

Now use a graphing utility to find that V attains its relative maximum of about 194.06736 cubic inches when $x \approx 2.3$ inches. See Figure 3.13.



$y = 4x^3 - 56x^2 + 192x,\ 0 < x < 6$

Figure 3.13

▶ Try Exercise 70, page 285

Real Zeros of a Polynomial Function

Sometimes the real zeros of a polynomial function can be determined by using the factoring procedures developed in previous chapters. We illustrate this concept in the next example.

EXAMPLE 3 Factor to Find the Real Zeros of a Polynomial Function

Factor to find the three real zeros of $P(x) = x^3 + 3x^2 - 4x$.

Algebraic Solution

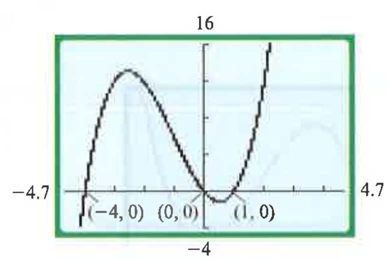
P can be factored as shown below.

$$\begin{aligned} P(x) &= x^3 + 3x^2 - 4x \\ &= x(x^2 + 3x - 4) && \text{• Factor out the common factor } x. \\ &= x(x-1)(x+4) && \text{• Factor the trinomial } x^2 + 3x - 4. \end{aligned}$$

Thus by the Factor Theorem we know that the real zeros of $P(x)$ are $x = 0$, $x = 1$, and $x = -4$.

Visualize the Solution

The graph of P has x-intercepts at (0, 0), (1, 0), and (−4, 0).



$P(x) = x^3 + 3x^2 - 4x$

▶ Try Exercise 22, page 283

Intermediate Value Theorem

The following theorem states an important property of polynomial functions.

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Intermediate Value Theorem

If P is a polynomial function and $P(a) \neq P(b)$ for $a < b$, then P takes on every value between $P(a)$ and $P(b)$ in the interval $[a, b]$.

The Intermediate Value Theorem is often used to verify the existence of a zero of a polynomial function in an interval. The essential idea is to find two values a and b such that the polynomial function is *positive* at one of the values and *negative* at the other. Then you can conclude by the Intermediate Value Theorem that the function has a zero between a and b. Stated in geometric terms, if the points $(a, P(a))$ and $(b, P(b))$ are on opposite sides of the x-axis, then the graph of the polynomial function P must cross the x-axis *at least once* between a and b. See Figure 3.14.

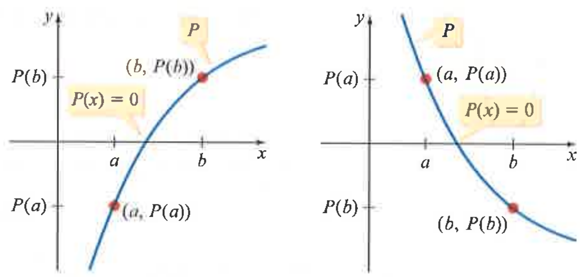


Figure 3.14

EXAMPLE 4 Apply the Intermediate Value Theorem

Use the Intermediate Value Theorem to verify that $P(x) = x^3 - x - 2$ has a real zero between 1 and 2.

Algebraic Solution

Use substitution or synthetic division to evaluate $P(1)$ and $P(2)$.

$$P(x) = x^3 - x - 2$$
$$\begin{aligned} P(1) &= (1)^3 - (1) - 2 \\ &= 1 - 1 - 2 \\ &= -2 \end{aligned}$$

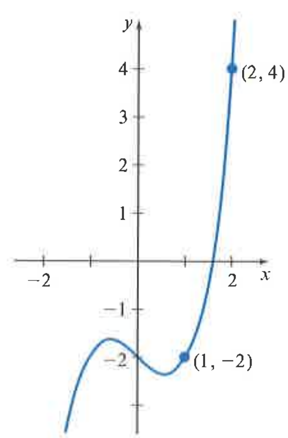
and

$$P(x) = x^3 - x - 2$$
$$\begin{aligned} P(2) &= (2)^3 - (2) - 2 \\ &= 8 - 2 - 2 \\ &= 4 \end{aligned}$$

Because $P(1)$ and $P(2)$ have opposite signs, we know by the Intermediate Value Theorem that the polynomial function P has at least one real zero between 1 and 2.

Visualize the Solution

The graph of P crosses the x-axis between $x = 1$ and $x = 2$. Thus P has a real zero between 1 and 2.



$P(x) = x^3 - x - 2$

▶ Try Exercise 28, page 283

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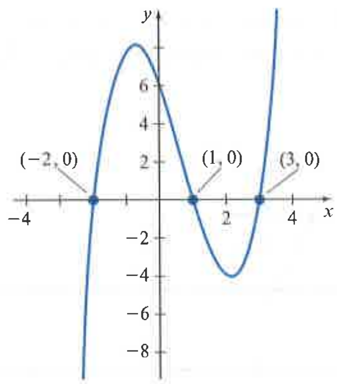
Real Zeros, x-Intercepts, and Factors of a Polynomial Function

The following theorem summarizes important relationships among the real zeros of a polynomial function, the x-intercepts of its graph, and its factors; this theorem can be written in the form $(x - c)$, where c is a real number.

Polynomial Functions, Real Zeros, Graphs, and Factors $(x - c)$

If P is a polynomial function and c is a real number, then all of the following statements are equivalent in the following sense: If any one statement is true, then they are all true, and if any one statement is false, then they are all false.

- $(x - c)$ is a factor of P.
- $x = c$ is a real solution of $P(x) = 0$.
- $x = c$ is a real zero of P.
- $(c, 0)$ is an x-intercept of the graph of $y = P(x)$.



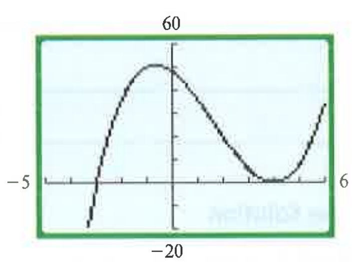
$S(x) = x^3 - 2x^2 - 5x + 6$

Figure 3.15

Sometimes it is possible to make use of the preceding theorem and a graph of a polynomial function to find factors of the function. For example, the graph of

$$S(x) = x^3 - 2x^2 - 5x + 6$$

is shown in Figure 3.15. The x-intercepts are $(-2, 0)$, $(1, 0)$, and $(3, 0)$. Hence -2, 1, and 3 are zeros of S, and $[x - (-2)]$, $(x - 1)$, and $(x - 3)$ are all factors of S.



$y = (x + 3)(x - 4)^2$

Figure 3.16

Even and Odd Powers of $(x - c)$ Theorem

Use a graphing utility to graph $P(x) = (x + 3)(x - 4)^2$. Compare your graph with Figure 3.16. Examine the graph near the x-intercepts $(-3, 0)$ and $(4, 0)$. Observe that the graph of P

- crosses the x-axis at $(-3, 0)$.
- intersects but does not cross the x-axis at $(4, 0)$.

The following theorem can be used to determine at which x-intercepts the graph of a polynomial function will cross the x-axis and at which x-intercepts the graph will intersect but not cross the x-axis.

Even and Odd Powers of $(x - c)$ Theorem

If c is a real number and the polynomial function P has $(x - c)$ as a factor exactly k times, then the graph of P will

- intersect but not cross the x-axis at $(c, 0)$, provided k is an even positive integer.
- cross the x-axis at $(c, 0)$, provided k is an odd positive integer.

EXAMPLE 5 Apply the Even and Odd Powers of $(x - c)$ Theorem

Determine where the graph of $P(x) = (x + 3)(x - 2)^2(x - 4)^3$ crosses the x-axis and where the graph intersects but does not cross the x-axis.

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Solution

The exponents of the factors $(x + 3)$ and $(x - 4)$ are odd integers. Therefore, the graph of P will cross the x-axis at the x-intercepts $(-3, 0)$ and $(4, 0)$.

The exponent of the factor $(x - 2)$ is an even integer. Therefore, the graph of P will intersect but not cross the x-axis at $(2, 0)$.

Use a graphing utility to check these results.

Try Exercise 38, page 284

Procedure for Graphing Polynomial Functions

You may find that you can sketch the graph of a polynomial function just by plotting several points; however, the following procedure will help you sketch the graph of many polynomial functions in an efficient manner.

TO REVIEW

Factoring of Polynomials
See Section P.4.

Procedure for Graphing Polynomial Functions

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

To graph P, follow these steps.

1. ***Determine the far-left and far-right behavior*** Examine the leading coefficient $a_n x^n$ to determine the far-left and far-right behavior of the graph.
2. ***Find the y-intercept*** Determine the y-intercept by evaluating $P(0)$.
3. ***Find the x-intercept or x-intercepts and determine the behavior of the graph near the x-intercept or x-intercepts*** If possible, find the x-intercepts by factoring. If $(x - c)$, where c is a real number, is a factor of P, then $(c, 0)$ is an x-intercept of the graph. Use the Even and Odd Powers of $(x - c)$ Theorem to determine where the graph crosses the x-axis and where the graph intersects but does not cross the x-axis.
4. ***Find additional points on the graph*** Find a few additional points (in addition to the intercepts).
5. ***Check for symmetry***
   a. The graph of an even function is symmetric with respect to the y-axis.
   b. The graph of an odd function is symmetric with respect to the origin.
6. ***Sketch the graph*** Use all the information previously obtained to sketch the graph of the polynomial function. The graph should be a smooth continuous curve that passes through the points determined in steps 2 through 4. The graph should have a maximum of $n - 1$ turning points.

EXAMPLE 6 Graph a Polynomial Function

Sketch the graph of $P(x) = x^3 - 4x^2 + 4x$.

Solution

1. ***Determine the far-left and far-right behavior*** The leading term is $1x^3$. Because the leading coefficient, 1, is positive and the degree of the polynomial, 3, is odd, the graph of P goes down to the far left and up to the far right.
2. ***Find the y-intercept*** $P(0) = 0^3 - 4(0)^2 + 4(0) = 0$. The y-intercept is $(0, 0)$.
3. ***Find the x-intercept or x-intercepts and determine the behavior of the graph near the x-intercept or x-intercepts*** Try to factor $x^3 - 4x^2 + 4x$.

(continued)

$$x^3 - 4x^2 + 4x = x(x^2 - 4x + 4)$$
$$= x(x - 2)(x - 2)$$
$$= x(x - 2)^2$$

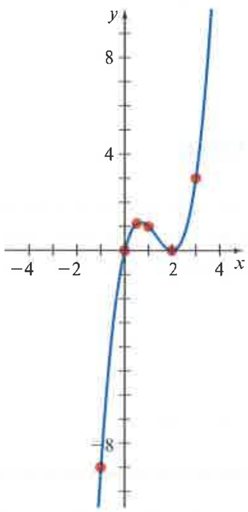
Because $(x - 2)$ is a factor of P, the point $(2, 0)$ is an x-intercept of the graph of P. Because x is a factor of P (think of x as $x - 0$), the point $(0, 0)$ is an x-intercept of the graph of P. Applying the Even and Odd Powers of $(x - c)$ Theorem allows us to determine that the graph of P crosses the x-axis at $(0, 0)$ and intersects but does not cross the x-axis at $(2, 0)$.

4. ***Find additional points on the graph***

x	$P(x)$
−1	−9
0.5	1.125
1	1
3	3

5. ***Check for symmetry*** The function P is neither an even nor an odd function, so the graph of P is *not* symmetric to either the y-axis or the origin.

6. ***Sketch the graph*** See Figure 3.17.



$P(x) = x^3 - 4x^2 + 4x$

Figure 3.17

Try Exercise 46, page 284

Cubic and Quartic Regression Functions

In Section 2.7 we used linear and quadratic functions to model several data sets. In many applications, data sets can be modeled more closely using cubic and quartic regression functions. In the following example, a TI-83/84 calculator was used to find the cubic regression function; however, you can also use WolframAlpha to find cubic and quartic regression functions as shown in the *Exploring Concepts with Technology* feature on page 324.

EXAMPLE 7 Model an Application with a Cubic Function

The following table lists the number of United States movie screens for the years from 2001 to 2011.

a. Use a graphing utility to construct a scatter plot of the data.

b. Find the cubic regression function that models the data and examine the coefficient of determination.

Losevsky Pavel/Shutterstock.com

Number of U.S. Movie Screens

Year	Number of Screens	Year	Number of Screens
2001	36,764	2007	38,974
2002	35,280	2008	38,834
2003	35,786	2009	39,233
2004	36,594	2010	39,547
2005	38,852	2011	39,641
2006	38,415		

Source: The World Almanac and Book of Facts 2013, p. 274.

Note

During recent years many cinema sites have been replaced with multiscreen complexes, or multiplexes. Thus there is an upward trend in the number of movie screens and a downward trend in the number of cinema sites.

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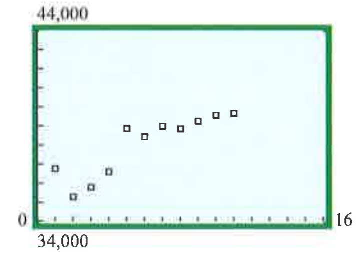
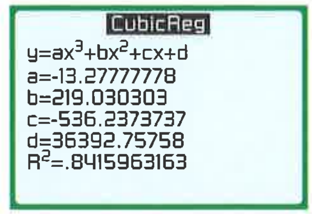
Solution

a. Construct a scatter plot.
On a TI-83/TI-83 Plus/TI-84 Plus calculator, select **1:EDIT** in the STAT menu to enter the data and construct a scatter plot as explained on page 238. We have used $x = 1$ to represent 2001, $x = 2$ to represent 2002, . . . , and $x = 11$ to represent 2011. The WINDOW settings are Xmin = 0, Xmax = 16, Xscl = 1, Ymin = 34000, Ymax = 44000, and Yscl = 1000.



b. Find the cubic regression function and examine the coefficient of determination.
To find the cubic regression function that models the data, select **6: CubicReg** in the STAT CALC menu.



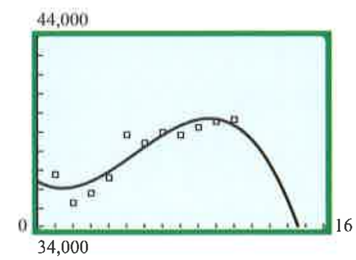
The cubic regression function is

$$P(x) = -13.27777778x^3 + 219.030303x^2 - 536.2373737x + 36392.75758$$

The coefficient of determination, R^2, is

$$0.8415963163$$

which is relatively close to 1. Thus the cubic regression function provides a fair model of the data, as shown by the graph at the right.

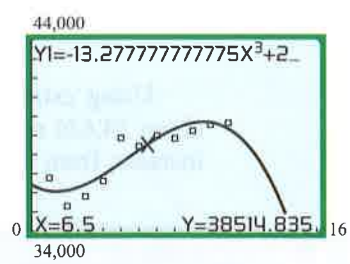


▶ **Try Exercise 72, page 285**

> ***Recall***
> The value of the coefficient of determination will not be displayed on a TI-83/TI-83 Plus/TI-84 Plus calculator unless the DiagnosticOn command is enabled. The DiagnosticOn command is in the CATALOG menu.

> ***Note***
> See the *Exploring Concepts with Technology* feature, on page 324, for information on how to use WolframAlpha to find the cubic and quartic regression functions for a data set.

Regression functions are often used to estimate a data point between two known data points. This procedure is referred to as **interpolation**. For instance, if we evaluate the cubic regression function in Example 7 at $x = 6.5$, then we get approximately 38,515 as an estimate of the number of movie screens halfway through 2006. See the following graph.



Regression functions are sometimes used to estimate a data point that lies to the left or to the right of the given data points. This procedure is referred to as **extrapolation**. The results of extrapolations are often subject to greater uncertainty than the results obtained by interpolation. For instance, the cubic regression function in Example 7 estimates about 35,381 movie screens in 2014, as shown by the following graph. This is fewer screens than in any of the years from 2003 to 2011. This scenario is unlikely.

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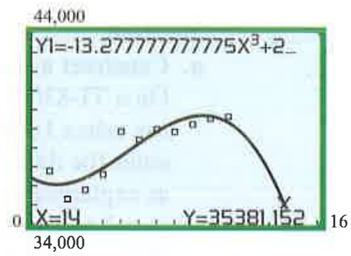
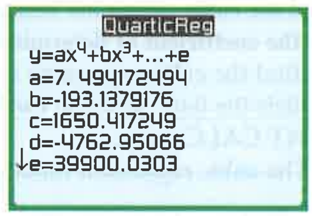
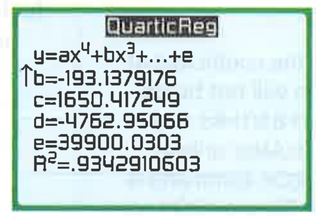


A **quartic regression** function will model the data in an application at least as well as does a cubic regression function. To illustrate this, find the quartic regression function for the movie screen data in Example 7. On a TI-83/TI-83 Plus/TI-84 Plus calculator, this can be accomplished by selecting **7: QuartReg** in the STAT CALC menu.



The coefficients of the quartic regression function

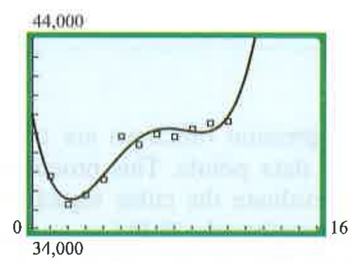


R^2 is the coefficient of determination

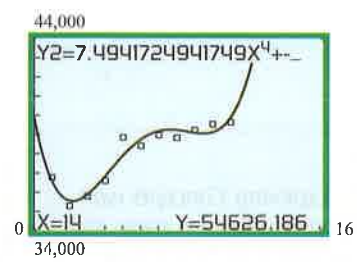
The quartic model is

$$P(x) = 7.494172494x^4 - 193.1379176x^3 + 1650.417249x^2 - 4762.95066x + 39900.0303$$

The coefficient of determination ($R^2 = 0.9342910603$) and the graphs of the quartic regression function, shown below, confirm that the quartic function does provide a better fit to the data than does the cubic regression function.



A graph of the quartic regression function and the data



The quartic regression function estimates about 54,626 movie screens in 2014.

Using extrapolation, we find that the quartic regression function estimates about 54,626 movie screens in 2014. See the graph on the right above. This large increase, from 39,641 screens in 2011, is also unlikely.

EXERCISE SET 3.2

Concept Check

1. What is the degree of

$$P(x) = 4x^3 - x^2 + 3x - 5$$

2. What is the leading term of

$$P(x) = 5x - 3x^2 + 2x^3 - 1$$

3. What is the largest possible number of turning points for the graph of a polynomial function of degree 4?

Indicates Try It Exercises

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4. The factored form of $P(x) = x^3 + 2x^2 - 15x$ is

$$P(x) = x(x - 3)(x + 5)$$

What are the zeros of P?

In Exercises 5 to 12, examine the leading term and determine the far-left and far-right behavior of the graph of the polynomial function.

5. $P(x) = 3x^4 - 2x^2 - 7x + 1$

6. $P(x) = -2x^3 - 6x^2 + 5x - 1$

7. $P(x) = 5x^5 - 4x^3 - 17x^2 + 2$

8. $P(x) = -6x^4 - 3x^3 + 5x^2 - 2x + 5$

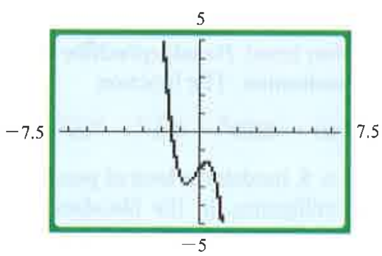
9. $P(x) = -450 + x^6$

10. $P(x) = 5 - 2x - 4x^2 - 3x^3$

11. $P(x) = -x^5 + 4x^2 + 9$

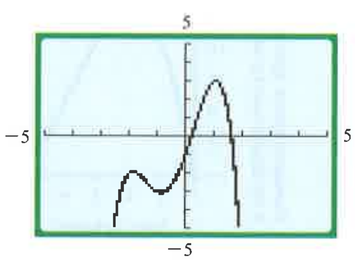
12. $P(x) = \frac{3}{4}x^4 - 5x^3 + 6x^2 - 3x - 7$

13. The following graph is the graph of a third-degree (cubic) polynomial function. What does the far-left and far-right behavior of the graph say about the leading coefficient a?



$P(x) = ax^3 + bx^2 + cx + d$

14. The following graph is the graph of a fourth-degree (quartic) polynomial function. What does the far-left and far-right behavior of the graph say about the leading coefficient a?



$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

In Exercises 15 to 20, use a graphing utility to graph each polynomial. Use the maximum and minimum features of the graphing utility to estimate, to the nearest tenth, the coordinates of the points where $P(x)$ has a relative maximum or a relative minimum. For each point, indicate whether the y value is a relative maximum or a relative minimum. The number in parentheses to the right of the polynomial is the total number of relative maxima and minima.

15. $P(x) = x^3 + x^2 - 9x - 9$ (2)

16. $P(x) = x^3 + 4x^2 - 4x - 16$ (2)

17. $P(x) = x^3 - 3x^2 - 24x + 3$ (2)

18. $P(x) = -2x^3 - 3x^2 + 12x + 1$ (2)

19. $P(x) = x^4 - 4x^3 - 2x^2 + 12x - 5$ (3)

20. $P(x) = x^4 - 10x^2 + 9$ (3)

In Exercises 21 to 26, find the real zeros of each polynomial function by factoring. The number in parentheses to the right of each polynomial indicates the number of real zeros of the given polynomial function.

21. $P(x) = x^3 - 2x^2 - 15x$ (3)

22. $P(x) = x^3 - 6x^2 + 8x$ (3)

23. $P(x) = x^4 - 13x^2 + 36$ (4)

24. $P(x) = 4x^4 - 37x^2 + 9$ (4)

25. $P(x) = x^5 - 5x^3 + 4x$ (5)

26. $P(x) = x^5 - 25x^3 + 144x$ (5)

In Exercises 27 to 36, use the Intermediate Value Theorem to verify that P has a zero between a and b.

27. $P(x) = 2x^3 + 3x^2 - 23x - 42;\ a = 3, b = 4$

28. $P(x) = 4x^3 - x^2 - 6x + 1;\ a = 0, b = 1$

29. $P(x) = 3x^3 + 7x^2 + 3x + 7;\ a = -3, b = -2$

30. $P(x) = 2x^3 - 21x^2 - 2x + 25;\ a = 1, b = 2$

31. $P(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15;\ a = 1, b = 1\frac{1}{2}$

32. $P(x) = 5x^3 - 16x^2 - 20x + 64;\ a = 3, b = 3\frac{1}{2}$

33. $P(x) = x^4 - x^2 - x - 4;\ a = 1.7, b = 1.8$

34. $P(x) = x^3 - x - 8;\ a = 2.1, b = 2.2$

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35. $P(x) = -x^4 + x^3 + 5x - 1; \quad a = 0.1, b = 0.2$

36. $P(x) = -x^3 - 2x^2 + x - 3; \quad a = -2.8, b = -2.7$

In Exercises 37 to 44, determine the x-intercepts of the graph of P. For each x-intercept, use the Even and Odd Powers of $(x - c)$ Theorem to determine whether the graph of P crosses the x-axis or intersects but does not cross the x-axis.

37. $P(x) = (x - 1)(x + 1)(x - 3)$

38. $P(x) = (x + 2)(x - 6)^2$

39. $P(x) = x(x - 5)^2(x - 3)$

40. $P(x) = -(2x - 8)(x - 7)^2$

41. $P(x) = x^2(x - 15)(2x - 7)^2$

42. $P(x) = x(x + 4)(x - 5)^2$

43. $P(x) = x^3 - 6x^2 + 9x$

44. $P(x) = x^4 + 3x^3 + 4x^2$

In Exercises 45 to 60, sketch the graph of the polynomial function. Do not use a graphing utility.

45. $P(x) = x^3 - x^2 - 2x$

46. $P(x) = x^3 + 2x^2 - 3x$

47. $P(x) = -x^3 - 2x^2 + 5x + 6$
(In factored form, $P(x) = -(x + 3)(x + 1)(x - 2)$.)

48. $P(x) = -x^3 - 3x^2 + x + 3$
(In factored form, $P(x) = -(x + 3)(x + 1)(x - 1)$.)

49. $P(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$
(In factored form, $P(x) = (x + 1)(x - 1)^2(x - 3)$.)

50. $P(x) = x^4 - 6x^3 + 8x^2$

51. $P(x) = x^3 + 6x^2 + 5x - 12$
(In factored form, $P(x) = (x - 1)(x + 3)(x + 4)$.)

52. $P(x) = -x^3 + 4x^2 + x - 4$

53. $P(x) = -x^3 + 7x - 6$

54. $P(x) = x^3 - 6x^2 + 9x$
(In factored form, $P(x) = x(x - 3)^2$.)

55. $P(x) = -x^3 + 4x^2 - 4x$
(In factored form, $P(x) = -x(x - 2)^2$.)

56. $P(x) = -x^4 + 2x^3 + 3x^2 - 4x - 4$
(In factored form, $P(x) = -(x - 2)^2(x + 1)^2$.)

57. $P(x) = -x^4 + 3x^3 + x^2 - 3x$

58. $P(x) = \frac{1}{2}x^4 + x^3 - 2x^2 - x + \frac{3}{2}$
(In factored form, $P(x) = \frac{1}{2}(x - 1)^2(x + 1)(x + 3)$.)

59. $P(x) = x^5 - x^4 - 5x^3 + x^2 + 8x + 4$
(In factored form, $P(x) = (x + 1)^3(x - 2)^2$.)

60. $P(x) = 2x^5 - 3x^4 - 4x^3 + 3x^2 + 2x$

In Exercises 61 to 66, use translation, reflection, or both concepts to explain how the graph of P can be used to produce the graph of Q.

61. $P(x) = x^3 + x; \quad Q(x) = x^3 + x + 2$

62. $P(x) = x^4; \quad Q(x) = x^4 - 3$

63. $P(x) = x^4; \quad Q(x) = (x - 1)^4$

64. $P(x) = x^3; \quad Q(x) = (x + 3)^3$

65. $P(x) = x^5; \quad Q(x) = -(x - 2)^5 + 3$

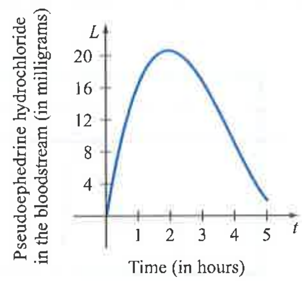
66. $P(x) = x^6; \quad Q(x) = (x + 4)^6 - 5$

67. **Medication Level** Pseudoephedrine hydrochloride is an allergy medication. The function

$$L(t) = 0.03t^4 + 0.4t^3 - 7.3t^2 + 23.1t$$

where $0 \le t \le 5$, models the level of pseudoephedrine hydrochloride, in milligrams, in the bloodstream of a patient t hours after 30 milligrams of the medication have been taken.

a. Use a graphing utility and the function $L(t)$ to determine the maximum level of pseudoephedrine hydrochloride in the patient's bloodstream. Round your result to the nearest 0.01 milligram.



b. At what time t, to the nearest minute, is this maximum level of pseudoephedrine hydrochloride reached?

68. **Profit** A software company produces a computer game. The company has determined that its profit P, in dollars, from the manufacture and sale of x games is given by

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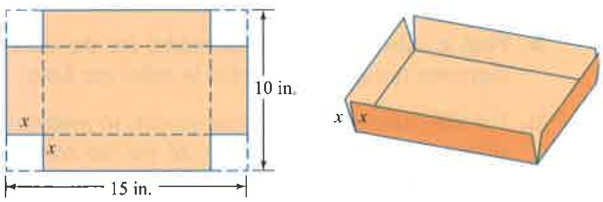
$$P(x) = -0.000001x^3 + 96x - 98{,}000$$

where $0 < x \leq 9000$.

a. What is the maximum profit, to the nearest thousand dollars, the company can expect from the sale of its game?

b. How many games, to the nearest unit, does the company need to produce and sell to obtain the maximum profit?

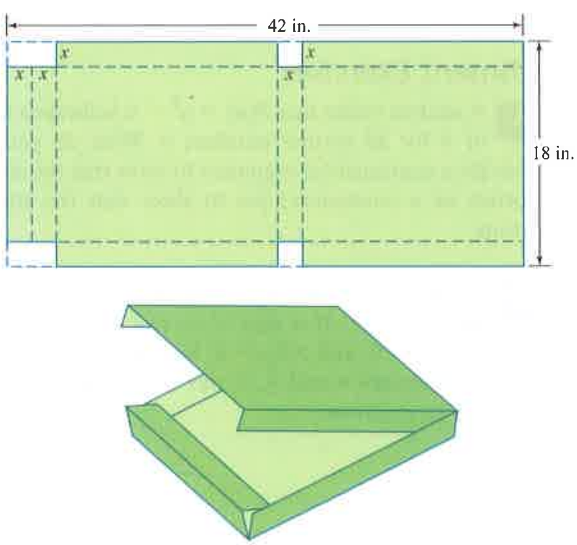
69. **Construction of a Box** A company constructs boxes from rectangular pieces of cardboard that measure 10 inches by 15 inches. An open box is formed by cutting squares that measure x inches by x inches from each corner of the cardboard and folding up the sides, as shown in the following figure.



a. Express the volume V of the box as a function of x.

b. Determine (to the nearest hundredth of an inch) the x value that maximizes the volume of the box.

70. **Maximizing Volume** A closed box is to be constructed from a rectangular sheet of cardboard that measures 18 inches by 42 inches. The box is made by cutting rectangles that measure x inches by $2x$ inches from two of the corners and by cutting two squares that measure x inches by x inches from the top and from the bottom of the rectangle, as shown in the following figure. What value of x (to the nearest thousandth of an inch) will produce a box with maximum volume?



71. **Wind Turbine Power** The power P, in watts, generated by a particular wind turbine with winds blowing at v meters per second is given by the cubic polynomial function

$$P(v) = 4.95v^3$$

a. Find the power generated, to the nearest 10 watts, when the wind speed is 8 meters per second.

b. What wind speed, in meters per second, is required to generate 10,000 watts? Round to the nearest tenth.

c. If the wind speed is doubled, what effect does this have on the power generated by the turbine?

d. If the wind speed is tripled, what effect does this have on the power generated by the turbine?

72. **Weight of a Tiger Shark** The following table shows the estimated weights of tiger sharks of various lengths.

Estimated Weight of Tiger Sharks

Length (in feet, x)	Weight (in pounds)
4	35
5	73
6	132
7	219
8	338
9	496
10	699
11	954

Source: National Oceanic and Atmospheric Administration.

a. Find the cubic regression function that models the data.

b. What is the coefficient of determination for this cubic regression?

c. What does the coefficient of determination indicate concerning how well the cubic regression function models the data?

d. Use the cubic regression function to estimate the weight of a tiger shark that has a length of 7.75 feet and the weight of a tiger shark that has a length of 12 feet. Round to the nearest pound.

73. **A Cat's Age in Human Years** Many people think cats age about 7 years, in terms of human years, for each calendar year. However, the data in the following table, from *The Cat Owner's Manual,* indicate that the "7-to-1" rule is not accurate for estimating a young cat's age in terms of human years.

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Cat's Age in Human Years

Calendar Age, x (in months)	Approximate age (in human years)
2	3
4	6
6	9
8	11
10	13
12	15
18	20
24	24

Source: The Cat Owner's Manual, by Quirk Books.

a. Find a cubic regression function P that models the data, where x represents the age of the cat in months and $P(x)$ represents the cat's approximate age in human years.

b. What is the coefficient of determination for this cubic regression?

c. What does the coefficient of determination indicate concerning how well the cubic regression function models the data?

d. Use the cubic regression function to estimate the age, in human years, of a 5-month-old cat. Round to the nearest tenth of a year.

74. **Box-Office Receipts** The United States and Canada movie theater box-office receipts are given in the following table, for 2002 to 2011.

Movie Theater Box-Office Receipts

Year	Receipts (billions of dollars)	Year	Receipts (billions of dollars)
2002	9.1	2007	9.6
2003	9.2	2008	9.6
2004	9.3	2009	10.6
2005	8.8	2010	10.6
2006	9.2	2011	10.2

Source: The Motion Picture Association of America.

a. Find a quartic regression function for the data. Use $x = 2$ to represent 2002, $x = 3$ to represent 2003, . . . , and $x = 11$ to represent 2011.

b. Use the quartic regression function to estimate the movie theater box-office receipts for 2010.

75. **Fuel Efficiency** The fuel efficiency, in miles per gallon, for a midsize car at various speeds, in miles per hour, is given in the following table.

Fuel Efficiency of a Midsize Car

Speed (mph)	Fuel Efficiency (mpg)	Speed (mph)	Fuel Efficiency (mpg)
0	0.0	40	30.2
5	11.1	45	30.6
10	17.2	50	31.7
15	22.4	55	30.8
20	26.2	60	29.5
25	27.1	65	28.2
30	28.3	70	26.3
35	29.4	75	24.1

a. Find a cubic and a quartic model for the data. Let x represent the speed of the car in miles per hour.

b. Use the cubic and the quartic models to predict the fuel efficiency, in miles per gallon, of the car traveling at a speed of 80 miles per hour. Round to the nearest tenth.

c. Which of the fuel efficiency values from **b** is the more realistic value?

76. Use a graph of $P(x) = 4x^4 - 12x^3 + 13x^2 - 12x + 9$ to determine between which two consecutive integers P has a real zero.

77. The point (2, 0) is on the graph of P. What point must be on the graph of $P(x - 3)$?

78. The point (3, 5) is on the graph of P. What point must be on the graph of $P(x + 1) - 2$?

79. Explain how to use the graph of $y = x^3$ to produce the graph of $P(x) = (x - 2)^3 + 1$.

Enrichment Exercises

80. A student thinks that $P(n) = n^3 - n$ is always a multiple of 6 for all natural numbers n. What do you think? Provide a mathematical argument to show that the student is correct or a counterexample to show that the student is wrong.

81. Consider the following conjecture. Let P be a polynomial function. If a and b are real numbers such that $a < b$, $P(a) > 0$, and $P(b) > 0$, then $P(x)$ does not have a real zero between a and b. Is this conjecture true or false? Support your answer.

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