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CHAPTER 3 Polynomial and Rational Functions

Queuing Theory

We have all experienced the frustration of waiting in line. Many businesses employ *queuing theory*, the mathematical study of waiting lines (queues), to estimate the expected average waiting time customers will wait in line, and the expected number of customers that will be waiting in line under different situations. One of the goals of these businesses is to determine the minimum resources that are needed to provide their customers with satisfactory service.

In Exercise 89, page 322, you will use a rational function to estimate the expected average waiting time customers will wait in line, dependent on the average number of people that a business can serve in a given time period.

SECTION 3.1

- Division of Polynomials
- Synthetic Division
- Remainder Theorem
- Factor Theorem
- Reduced Polynomials

Remainder Theorem and Factor Theorem

If P is a polynomial function, then the values of x for which $P(x)$ is equal to 0 are called the **zeros** of P. For instance, -1 is a zero of $P(x) = 2x^3 - x + 1$ because

$$\begin{aligned} P(-1) &= 2(-1)^3 - (-1) + 1 \\ &= -2 + 1 + 1 \\ &= 0 \end{aligned}$$

Question • Is 0 a zero of $P(x) = 2x^3 - x + 1$?

Much of this chapter concerns finding the zeros of polynomial functions. Sometimes the zeros of a polynomial function are determined by dividing one polynomial by another.

Division of Polynomials

Recall

A fraction bar acts as a grouping symbol. Division of a polynomial by a monomial is an application of the distributive property.

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial. For instance,

$$\frac{16x^3 - 8x^2 + 12x}{4x} = \frac{16x^3}{4x} - \frac{8x^2}{4x} + \frac{12x}{4x}$$ • Divide each term in the numerator by the denominator.

$$= 4x^2 - 2x + 3$$ • Simplify.

To divide a polynomial by a binomial, we use a method similar to that used to divide natural numbers. For instance, consider $(6x^3 - 16x^2 + 23x - 5) \div (3x - 2)$.

$$3x - 2\overline{)6x^3 - 16x^2 + 23x - 5}$$

$$\begin{array}{r} 2x^2 \\ 3x - 2\overline{)6x^3 - 16x^2 + 23x - 5} \\ \underline{6x^3 - 4x^2} \\ -12x^2 + 23x \end{array}$$

• Think $\frac{6x^3}{3x} = 2x^2$.

• Multiply: $2x^2(3x - 2) = 6x^3 - 4x^2$

• Subtract and bring down the next term, $23x$.

$$\begin{array}{r} 2x^2 - 4x \\ 3x - 2\overline{)6x^3 - 16x^2 + 23x - 5} \\ \underline{6x^3 - 4x^2} \\ -12x^2 + 23x \\ \underline{-12x^2 + 8x} \\ 15x - 5 \end{array}$$

• Think $\frac{-12x^2}{3x} = -4x$.

• Multiply: $-4x(3x - 2) = -12x^2 + 8x$

• Subtract and bring down the next term, -5.

$$\begin{array}{r} 2x^2 - 4x + 5 \\ 3x - 2\overline{)6x^3 - 16x^2 + 23x - 5} \\ \underline{6x^3 - 4x^2} \\ -12x^2 + 23x \\ \underline{-12x^2 + 8x} \\ 15x - 5 \\ \underline{15x - 10} \\ 5 \end{array}$$

Divisor → $3x - 2$; Quotient ← $2x^2 - 4x + 5$; Dividend ← $6x^3 - 16x^2 + 23x - 5$

• Think $\frac{15x}{3x} = 5$.

• Multiply: $5(3x - 2) = 15x - 10$

• Subtract to produce the remainder, 5.

The division process ends when the expression in the bottom row is of less degree than the divisor. The expression in the bottom row is the **remainder**

Answer • No. $P(0) = 2(0)^3 - 0 + 1 = 1$. Because $P(0) \neq 0$, we know that 0 is not

> **Note**
>
> $\frac{20}{3}$ written as a mixed number is $6\frac{2}{3}$. Recall, however, that $6\frac{2}{3}$ means $6 + \frac{2}{3}$, which is in the form Quotient + $\frac{\text{Remainder}}{\text{Divisor}}$.

the polynomial in the top row is the **quotient.** Thus $(6x^3 - 16x^2 + 23x - 5) \div (3x - 2) = 2x^2 - 4x + 5$ with a remainder of 5.

Although there is nothing wrong with writing the answer as we did above, it is more common to write the answer as the quotient plus the remainder divided by the divisor. (See the note at the left.) Using this method, we write

$$\frac{\overbrace{6x^3 - 16x^2 + 23x - 5}^{\text{Dividend}}}{\underbrace{3x - 2}_{\text{Divisor}}} = \overbrace{2x^2 - 4x + 5}^{\text{Quotient}} + \frac{5 \leftarrow \text{Remainder}}{3x - 2 \leftarrow \text{Divisor}}$$

In every division, the dividend is equal to the product of the divisor and quotient, plus the remainder. That is,

$$\underbrace{6x^3 - 16x^2 + 23x - 5}_{\text{Dividend}} = \underbrace{(3x - 2)}_{\text{Divisor}} \cdot \underbrace{(2x^2 - 4x + 5)}_{\text{Quotient}} + \underbrace{5}_{\text{Remainder}}$$

The preceding polynomial division concepts are summarized by the following theorem.

Division Algorithm for Polynomials

Let $P(x)$ and $D(x)$ be polynomials, with $D(x)$ of lower degree than $P(x)$ and $D(x)$ of degree 1 or more. Then there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = D(x) \cdot Q(x) + R(x)$$

where $R(x)$ is either 0 or of degree less than the degree of $D(x)$. The polynomial $P(x)$ is called the **dividend,** $D(x)$ is the **divisor,** $Q(x)$ is the **quotient,** and $R(x)$ is the **remainder.**

Before dividing polynomials, make sure that each polynomial is written in descending order. It is helpful to insert a 0 in the dividend for a missing term (one whose coefficient is 0) so that like terms align in the same column. This is demonstrated in Example 1.

Question • What is the first step you should perform to find the quotient of $(2x + 1 + x^2) \div (x - 1)$?

EXAMPLE 1 Divide Polynomials

Divide: $\dfrac{-5x^2 - 8x + x^4 + 3}{x - 3}$

Solution

Write the numerator in descending order. Then divide.

$$\frac{-5x^2 - 8x + x^4 + 3}{x - 3} = \frac{x^4 - 5x^2 - 8x + 3}{x - 3}$$

(continued)

Answer • Write the dividend in descending order as $x^2 + 2x + 1$.

$$
\begin{array}{r}
x^3 + 3x^2 + 4x + 4 \\
x - 3\overline{)x^4 + 0x^3 - 5x^2 - 8x + 3} \\
\underline{x^4 - 3x^3} \qquad\qquad\qquad\quad \\
3x^3 - 5x^2 \qquad\qquad \\
\underline{3x^3 - 9x^2} \qquad\qquad \\
4x^2 - 8x \qquad \\
\underline{4x^2 - 12x} \qquad \\
4x + 3 \\
\underline{4x - 12} \\
15
\end{array}
$$

• Inserting $0x^3$ for the missing term helps align like terms in the same column.

Thus $\dfrac{-5x^2 - 8x + x^4 + 3}{x - 3} = x^3 + 3x^2 + 4x + 4 + \dfrac{15}{x - 3}$.

Try Exercise 12, page 267

Synthetic Division

A procedure called **synthetic division** can expedite the division process. To apply the synthetic division procedure, the divisor must be a polynomial of the form $x - c$, where c is a constant. In the synthetic division procedure, the variables that occur in the polynomials are not listed. To understand how synthetic division is performed, examine the following **long division** on the left and the related synthetic division on the right.

Long Division

Coefficients of the quotient

$$
\begin{array}{r}
4x^2 + 3x + 8 \\
x - 2\overline{)4x^3 - 5x^2 + 2x - 10} \\
\underline{4x^3 - 8x^2} \qquad\qquad\quad \\
3x^2 + 2x \qquad\quad \\
\underline{3x^2 - 6x} \qquad\quad \\
8x - 10 \\
\underline{8x - 16} \\
6
\end{array}
$$

Remainder

Synthetic Division

2	4	−5	2	−10	← First row
		8	6	16	← Second row
	4	3	8	6	← Third row

4, 3, 8: Coefficients of the quotient; 6: Remainder

In the long division, the dividend is $4x^3 - 5x^2 + 2x - 10$ and the divisor is $x - 2$. Because the divisor is of the form $x - c$, with $c = 2$, the division can be performed by the synthetic division procedure. Observe that in the above synthetic division:

1. The constant c is listed as the first number in the first row, followed by the coefficients of the dividend.
2. The first number in the third row is the leading coefficient of the dividend.
3. Each number in the second row is determined by computing the product of c and the number in the third row of the preceding column.
4. Each of the numbers in the third row, other than the first number, is determined by adding the numbers directly above it.

The following explanation illustrates the steps used to find the quotient and remainder of $(2x^3 - 8x + 7) \div (x + 3)$ using synthetic division. The divisor $x + 3$ is written in $x - c$ form as $x - (-3)$, which indicates that $c = -3$. The dividend $2x^3 - 8x + 7$ is missing an x^2 term. If we insert $0x^2$ for the missing term, the dividend becomes $2x^3 + 0x^2 - 8x + 7$.

Coefficients of the dividend

$$\begin{array}{r|rrrr} -3 & 2 & 0 & -8 & 7 \\ & \downarrow & & & \\ \hline & 2 & & & \end{array}$$

- Write the constant c, -3, followed by the coefficients of the dividend. Bring down the first coefficient in the first row, 2, as the first number of the third row.

$$\begin{array}{r|rrrr} -3 & 2 & 0 & -8 & 7 \\ & & -6 & & \\ \hline & 2 & -6 & & \end{array}$$

- Multiply c times the first number in the third row, 2, to produce the first number of the second row, -6. Add the 0 and the -6 to produce the next number of the third row, -6.

$$\begin{array}{r|rrrr} -3 & 2 & 0 & -8 & 7 \\ & & -6 & 18 & \\ \hline & 2 & -6 & 10 & \end{array}$$

- Multiply c times the second number in the third row, -6, to produce the next number of the second row, 18. Add the -8 and the 18 to produce the next number of the third row, 10.

$$\begin{array}{r|rrrr} -3 & 2 & 0 & -8 & 7 \\ & & -6 & 18 & -30 \\ \hline & 2 & -6 & 10 & -23 \end{array}$$

- Multiply c times the third number in the third row, 10, to produce the next number of the second row, -30. Add the 7 and the -30 to produce the last number of the third row, -23.

Coefficients of the quotient — Remainder

The last number in the bottom row, -23, is the remainder. The other numbers in the bottom row are the coefficients of the quotient. The quotient of a synthetic division always has a degree that is *one less than* the degree of the dividend. Thus the quotient in this example is $2x^2 - 6x + 10$. The results of the synthetic division can be expressed in **fractional form** as

$$\frac{2x^3 - 8x + 7}{x + 3} = 2x^2 - 6x + 10 + \frac{-23}{x + 3}$$

or as

$$2x^3 - 8x + 7 = (x + 3)(2x^2 - 6x + 10) - 23$$

Note

$2x^2 - 6x + 10 + \dfrac{-23}{x + 3}$ can also be written as $2x^2 - 6x + 10 - \dfrac{23}{x + 3}$

In Example 2, we illustrate the compact form of synthetic division, obtained by condensing the process explained here.

EXAMPLE 2 Use Synthetic Division to Divide Polynomials

Use synthetic division to divide $x^4 - 4x^2 + 7x + 15$ by $x + 4$.

Solution

Because the divisor is $x + 4$, we perform synthetic division with $c = -4$.

$$\begin{array}{r|rrrrr} -4 & 1 & 0 & -4 & 7 & 15 \\ & & -4 & 16 & -48 & 164 \\ \hline & 1 & -4 & 12 & -41 & 179 \end{array}$$

The quotient is $x^3 - 4x^2 + 12x - 41$, and the remainder is 179.

$$\frac{x^4 - 4x^2 + 7x + 15}{x + 4} = x^3 - 4x^2 + 12x - 41 + \frac{179}{x + 4}$$

Try Exercise 16, page 267

Integrating Technology

A TI-83/TI-83 Plus/TI-84 Plus synthetic division program called SYDIV is available on the Internet at www.cengagebrain.com. The program prompts you to enter the degree of the dividend, the coefficients of the dividend, and the constant c from the divisor $x - c$. For instance, to perform the synthetic division in Example 2, enter 4 for the degree of the dividend, followed by the coefficients 1, 0, -4, 7, and 15. See Figure 3.1. Press ENTER followed by -4 to produce the display in Figure 3.2. Press ENTER to produce the display in Figure 3.3. Press ENTER again to produce the display in Figure 3.4.

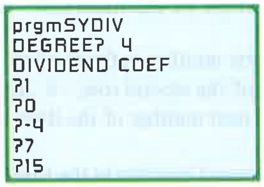


Figure 3.1

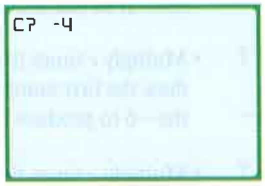


Figure 3.2

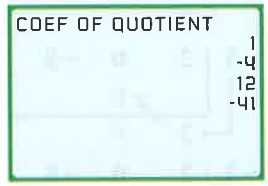


Figure 3.3

Figure 3.4

Remainder Theorem

The following theorem shows that synthetic division can be used to determine the value $P(c)$ for a given polynomial $P(x)$ and constant c.

Remainder Theorem

If a polynomial $P(x)$ is divided by $x - c$, then the remainder equals $P(c)$.

Proof of the Remainder Theorem

Because the degree of the remainder must be less than the degree of the divisor $(x - c)$, we know that the remainder must be a constant. If we call the constant remainder r, then by the division algorithm we have

$$P(x) = (x - c) \cdot Q(x) + r$$

Setting $x = c$ produces

$$P(c) = (c - c) \cdot Q(c) + r$$

$$P(c) = 0 + r$$

$$P(c) = r$$

■

The following example shows that the remainder of $P(x) = x^2 + 9x - 16$ divided by $x - 3$ is the same as $P(3)$.

$$\begin{array}{r} x + 12 \\ x - 3\,\overline{)\,x^2 + 9x - 16} \\ \underline{x^2 - 3x} \\ 12x - 16 \\ \underline{12x - 36} \\ 20 \end{array}$$

Let $x = 3$ and $P(x) = x^2 + 9x - 16$.
Then $P(3) = 3^2 + 9(3) - 16$
$= 9 + 27 - 16$
$= 20$

The remainder of $P(x)$ divided by $x - 3$ is equal to $P(3)$.

In Example 3, we use synthetic division and the Remainder Theorem to evaluate a polynomial function.

EXAMPLE 3 Use the Remainder Theorem to Evaluate a Polynomial Function

Let $P(x) = 2x^3 + 3x^2 + 2x - 2$. Use the Remainder Theorem to find $P(c)$ for $c = -2$ and $c = \frac{1}{2}$.

Algebraic Solution

Perform synthetic division with $c = -2$ and $c = \frac{1}{2}$ and examine the remainders.

−2	2	3	2	−2
		−4	2	−8
	2	−1	4	−10

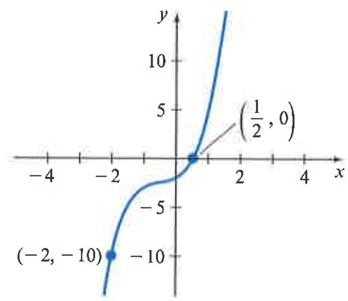
The remainder is −10. Therefore, $P(-2) = -10$.

$\frac{1}{2}$	2	3	2	−2
		1	2	2
	2	4	4	0

The remainder is 0. Therefore, $P\left(\frac{1}{2}\right) = 0$.

Visualize the Solution

The points $(-2, -10)$ and $\left(\frac{1}{2}, 0\right)$ are on the graph of P.



$P(x) = 2x^3 + 3x^2 + 2x - 2$

▶ Try Exercise 30, page 268

Using the Remainder Theorem to evaluate a polynomial function is often faster than evaluating the polynomial function by direct substitution. For instance, evaluating $P(x) = x^5 - 10x^4 + 35x^3 - 50x^2 + 24x$ by substituting 7 for x requires the following work.

$$\begin{aligned} P(7) &= (7)^5 - 10(7)^4 + 35(7)^3 - 50(7)^2 + 24(7) \\ &= 16{,}807 - 10(2401) + 35(343) - 50(49) + 24(7) \\ &= 16{,}807 - 24{,}010 + 12{,}005 - 2450 + 168 \\ &= 2520 \end{aligned}$$

Using the Remainder Theorem to evaluate $P(7)$ requires only the following work.

7	1	−10	35	−50	24	0	
		7	−21	98	336	2520	
	1	−3	14	48	360	2520	⟵ $P(7)$

Caution

Because P has a constant term of 0, we must include 0 as the last number in the first row of the synthetic division at the right.

Factor Theorem

Note from Example 3 that $P\left(\frac{1}{2}\right) = 0$. Recall that $\frac{1}{2}$ is a zero of P because $P(x) = 0$ when $x = \frac{1}{2}$.

The following theorem shows the important relationship between a zero of a given polynomial function and a factor of the polynomial.

Factor Theorem

A polynomial $P(x)$ has a factor $(x - c)$ if and only if $P(c) = 0$. That is, $(x - c)$ is a factor of $P(x)$ if and only if c is a zero of P.

Proof of the Factor Theorem

If $(x - c)$ is a factor of $P(x)$, then

$$P(x) = (x - c) \cdot Q(x)$$

and

$$\begin{aligned} P(c) &= (c - c) \cdot Q(c) \\ &= 0 \cdot Q(c) \\ &= 0 \end{aligned}$$

Conversely, if $P(c) = 0$, then, by the Remainder Theorem, $R(x) = 0$ and

$$P(x) = (x - c) \cdot Q(x) + 0 = (x - c) \cdot Q(x)$$ ■

EXAMPLE 4 Apply the Factor Theorem

Use synthetic division and the Factor Theorem to determine whether $(x + 5)$ or $(x - 2)$ is a factor of $P(x) = x^4 + x^3 - 21x^2 - x + 20$.

Solution

−5	1	1	−21	−1	20
		−5	20	5	−20
	1	−4	−1	4	0

The remainder of 0 indicates that -5 is a zero of P and $(x + 5)$ is a factor of $P(x)$.

2	1	1	−21	−1	20
		2	6	−30	−62
	1	3	−15	−31	−42

The remainder of -42 indicates that $(x - 2)$ is not a factor of $P(x)$.

▶ Try Exercise 40, page 268

Here is a summary of the important role played by the remainder in the division of a polynomial by $(x - c)$.

Remainder of a Polynomial Division

In the division of the polynomial $P(x)$ by $(x - c)$, the remainder is

- equal to $P(c)$.
- 0 if and only if $(x - c)$ is a factor of $P(x)$.
- 0 if and only if c is a zero of P.

If c is a real number, then the remainder of $P(x) \div (x - c)$ is 0 if and only if $(c, 0)$ is an x-intercept of the graph of P.

Reduced Polynomials

In Example 4 we showed that $(x + 5)$ is a factor of $P = x^4 + x^3 - 21x^2 - x + 20$ and that the quotient of P divided by $(x + 5)$ is $x^3 - 4x^2 - x + 4$. Thus

$$P(x) = (x + 5)(x^3 - 4x^2 - x + 4)$$

The quotient $x^3 - 4x^2 - x + 4$ is called a **reduced polynomial,** or a **depressed polynomial,** of $P(x)$ because it is a factor of $P(x)$ and its degree is 1 less than the degree of $P(x)$. Reduced polynomials play an important role in Sections 3.3 and 3.4.

EXAMPLE 5 Find a Reduced Polynomial

Verify that $(x - 3)$ is a factor of $P(x) = 2x^3 - 3x^2 - 4x - 15$, and write $P(x)$ as the product of $(x - 3)$ and the reduced polynomial $Q(x)$.

Solution

3	2	−3	−4	−15
		6	9	15
	2	3	5	0

This 0 indicates that $(x - 3)$ is a factor of $P(x)$.

Coefficients of the reduced polynomial $Q(x)$

Thus $(x - 3)$ and the reduced polynomial $2x^2 + 3x + 5$ are both factors of P. That is,

$$P(x) = 2x^3 - 3x^2 - 4x - 15 = (x - 3)(2x^2 + 3x + 5)$$

▶ Try Exercise 58, page 268

EXERCISE SET 3.1

Concept Check

In Exercises 1 to 4, determine whether the given division can be performed using synthetic division.

1. $(x^3 + 2x^2 - 5x + 3) \div (x - 2)$

2. $(2x^4 - 3x^2 + 6x - 5) \div (x + 3)$

3. $(x^4 - x^2 - 3) \div (x^2 + 1)$

4. $\left(x^5 - \frac{1}{2}\right) \div (x + 8)$

5. If a polynomial $P(x)$ is divided by $(x - 4)$ and the remainder is 7, then what do we know about $P(4)$?

6. If $(x - 3)$ is a factor of the polynomial $P(x)$, then what do we know about $P(3)$?

In Exercises 7 to 14, use long division to divide the first polynomial by the second.

7. $5x^3 + 6x^2 - 17x + 20, \quad x + 3$

8. $6x^3 + 15x^2 - 8x + 2, \quad x + 4$

9. $x^4 - 5x^2 + 3x - 1, \quad x - 2$

10. $x^4 - 5x^3 + x - 4, \quad x - 1$

11. $2x^4 + 5x^3 - 6x^2 + 4x + 3, \quad 2x^2 - x + 1$

12. $3x^3 + x^2 - 5x + 2, \quad x^2 - 2x + 2$

13. $2x^5 - x^3 + 5x^2 - 9x + 6, \quad 2x^2 + 2x - 3$

14. $x^5 + 3x^4 - 2x^3 - 7x^2 - x + 4, \quad x^2 + 1$

In Exercises 15 to 28, use synthetic division to divide the first polynomial by the second.

15. $4x^3 - 5x^2 + 6x - 7, \quad x - 2$

16. $5x^3 + 6x^2 - 8x + 1, \quad x - 5$

17. $4x^3 - 2x + 3, \quad x + 1$

18. $6x^3 - 4x^2 + 17, \quad x + 3$

19. $x^5 - 10x^3 + 5x - 1, \quad x - 4$

20. $6x^4 - 2x^3 - 3x^2 - x, \quad x - 5$

21. $x^5 - 1, \quad x - 1$

22. $x^4 + 1, \quad x + 1$

Indicates Try It Exercises

23. $8x^3 - 4x^2 + 6x - 3, \quad x - \frac{1}{2}$

24. $12x^3 + 5x^2 + 5x + 6, \quad x + \frac{3}{4}$

25. $x^8 + x^6 + x^4 + x^2 + 4, \quad x - 2$

26. $-x^7 - x^5 - x^3 - x - 5, \quad x + 1$

27. $x^6 + x - 10, \quad x + 3$

28. $2x^5 - 3x^4 - 5x^2 - 10, \quad x - 4$

In Exercises 29 to 38, use synthetic division and the Remainder Theorem to find *P*(*c*).

29. $P(x) = 5x^3 + 2x^2 - x - 7, \quad c = 3$

30. $P(x) = 2x^3 - x^2 + 3x - 1, \quad c = 3$

31. $P(x) = 3x^4 - 5x^2 + 7, \quad c = -3$

32. $P(x) = 8x^3 - 4x^2 + 3x, \quad c = -5$

33. $P(x) = -4x^3 - x^2 + 3x - 11, \quad c = 8$

34. $P(x) = -3x^3 + 6x^2 - 7x + 42, \quad c = 10$

35. $P(x) = -x^4 + x - 2, \quad c = 6$

36. $P(x) = x^5 - 1, \quad c = 1$

37. $P(x) = -x^6 - 8x^4 - 3x + 5, \quad c = 2$

38. $P(x) = x^5 + 20x^2 - 1, \quad c = -4$

In Exercises 39 to 48, use synthetic division and the Factor Theorem to determine whether the given binomial is a factor of *P*(*x*).

39. $P(x) = x^3 + 2x^2 - 5x - 6, \quad x - 2$

40. $P(x) = x^3 + 4x^2 - 27x - 90, \quad x + 6$

41. $P(x) = 2x^3 + x^2 - 3x - 1, \quad x + 1$

42. $P(x) = 2x^3 - 8x^2 - 5x + 33, \quad x - 3$

43. $P(x) = x^4 + x^3 - 2x^2 + 5x - 140, \quad x + 4$

44. $P(x) = x^4 - 12x^2 + 36, \quad x - 3$

45. $P(x) = x^5 + 2x^4 - 22x^3 - 50x^2 - 75x, \quad x - 5$

46. $P(x) = 9x^4 - 6x^3 - 23x^2 - 4x + 4, \quad x + 1$

47. $P(x) = 16x^4 - 8x^3 + 9x^2 + 14x + 4, \quad x - \frac{1}{4}$

48. $P(x) = 10x^4 + 9x^3 - 4x^2 + 9x + 6, \quad x + \frac{1}{2}$

In Exercises 49 to 56, use synthetic division to show that *c* is a zero of *P*.

49. $P(x) = 3x^3 - 8x^2 - 10x + 28, \quad c = 2$

50. $P(x) = 4x^3 - 10x^2 - 8x + 6, \quad c = 3$

51. $P(x) = x^4 - 1, \quad c = 1$

52. $P(x) = x^3 + 8, \quad c = -2$

53. $P(x) = 3x^4 + 8x^3 + 10x^2 + 2x - 20, \quad c = -2$

54. $P(x) = x^4 - 2x^2 - 100x - 75, \quad c = 5$

55. $P(x) = 2x^3 - 18x^2 - 50x + 66, \quad c = 11$

56. $P(x) = 2x^4 - 34x^3 + 70x^2 - 153x + 45, \quad c = 15$

In Exercises 57 to 60, verify that the given binomial is a factor of *P*(*x*), and write *P*(*x*) as the product of the binomial and its reduced polynomial *Q*(*x*).

57. $P(x) = x^3 + x^2 + x - 14, \quad x - 2$

58. $P(x) = x^4 + 5x^3 + 3x^2 - 5x - 4, \quad x + 1$

59. $P(x) = x^4 - x^3 - 9x^2 - 11x - 4, \quad x - 4$

60. $P(x) = 2x^5 - x^4 - 7x^3 + x^2 + 7x - 10, \quad x - 2$

61. **Selection of Cards** The number of ways you can select three cards from a stack of n cards, in which the order of selection is important, is given by

$$P(n) = n^3 - 3n^2 + 2n, \quad n \geq 3$$

   a. Use the Remainder Theorem to determine the number of ways you can select three cards from a stack of $n = 8$ cards.

   b. Evaluate $P(n)$ for $n = 8$ by substituting 8 for n. How does this result compare with the result obtained in **a**?

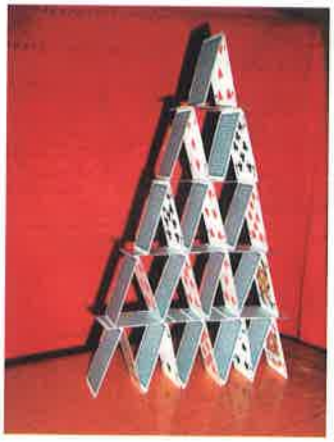
62. **Selection of Bridesmaids** A bride-to-be has many girlfriends, but she has decided to have only five bridesmaids, including the maid of honor. The number of different ways n girlfriends can be chosen and assigned a position, such as maid of honor, first bridesmaid, second bridesmaid, and so on, is given by the polynomial function

$$P(n) = n^5 - 10n^4 + 35n^3 - 50n^2 + 24n, \quad n \geq 5$$

   a. Use the Remainder Theorem to determine the number of ways the bride can select her bridesmaids if she chooses from $n = 7$ girlfriends.

   b. Evaluate $P(n)$ for $n = 7$ by substituting 7 for n. How does this result compare with the result obtained in **a**?

63. House of Cards The number of cards C needed to build a house of cards with r rows (levels) is given by the polynomial function $C(r) = 1.5r^2 + 0.5r$.



Topham/The Image Works

Use the Remainder Theorem to determine the number of cards needed to build a house of cards with

a. $r = 7$ rows

b. $r = 12$ rows

64. Display of Soda Cans The number S of soda cans needed to build a square pyramid display with n levels is given by the polynomial function

$$S(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

A square pyramid display with n^2 soda cans in level n

Use the Remainder Theorem to determine the number of soda cans needed to build a square pyramid display with

a. $n = 6$ levels

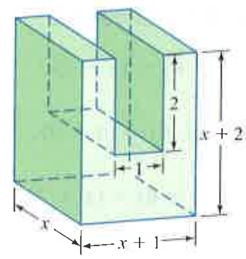
b. $n = 12$ levels

65. Election of Class Officers The number of ways a class of n students can elect a president, a vice president, a secretary, and a treasurer is given by $P(n) = n^4 - 6n^3 + 11n^2 - 6n$, where $n \geq 4$. Use the Remainder Theorem to determine the number of ways the class can elect officers if the class consists of

a. $n = 8$ students

b. $n = 18$ students

66. Volume of a Solid The volume, in cubic inches, of the following solid is given by $V(x) = x^3 + 3x^2$.

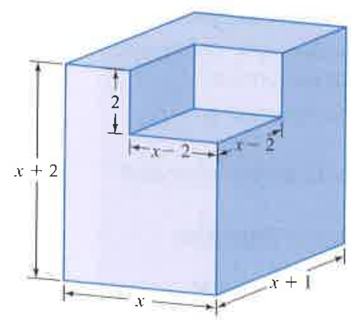


Use the Remainder Theorem to determine the volume of the solid if

a. $x = 7$ inches

b. $x = 11$ inches

67. Volume of a Solid The volume, in cubic inches, of the following solid is given by $V(x) = x^3 + x^2 + 10x - 8$.

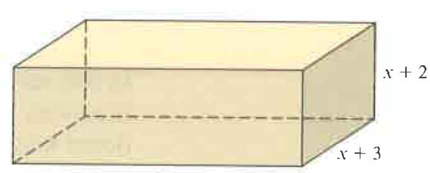


Use the Remainder Theorem to determine the volume of the solid if

a. $x = 6$ inches

b. $x = 9$ inches

68. Volume of a Box A rectangular box has a volume of $V(x) = x^3 + 10x^2 + 31x + 30$ cubic inches. The height of the box is $x + 2$ inches. The width of the box is $x + 3$ inches. Find the length of the box in terms of x.



Enrichment Exercises

69. Use synthetic division to divide each of the following polynomials by $x - 1$.

$$x^3 - 1, x^5 - 1, x^7 - 1$$

Use the pattern suggested by these quotients to write the quotient of $(x^9 - 1) \div (x - 1)$.

In Exercises 70 to 73, determine the value of k so that the divisor is a factor of the dividend.

70. $(x^3 - x^2 - 14x + k) \div (x - 2)$

71. $(2x^3 + x^2 - 25x + k) \div (x - 3)$

72. $(3x^3 + 14x^2 + kx - 6) \div (x + 2)$

73. $(x^4 + 3x^3 - 8x^2 + kx + 16) \div (x + 4)$

74. Use the Factor Theorem to show that for any positive integer n

$$P(x) = x^n - 1$$

has $x - 1$ as a factor.

75. Find the remainder of

$$5x^{48} + 6x^{10} - 5x + 7$$

divided by $x - 1$.

76. Find the remainder of

$$18x^{80} - 6x^{50} + 4x^{20} - 2$$

divided by $x + 1$.

77. Determine whether i is a zero of

$$P(x) = x^3 - 3x^2 + x - 3$$

78. Determine whether $-2i$ is a zero of

$$P(x) = x^4 - 2x^3 + x^2 - 8x - 12$$

SECTION 3.2

- Far-Left and Far-Right Behavior
- Maximum and Minimum Values
- Real Zeros of a Polynomial Function
- Intermediate Value Theorem
- Real Zeros, x-Intercepts, and Factors of a Polynomial Function
- Even and Odd Powers of $(x - c)$ Theorem
- Procedure for Graphing Polynomial Functions
- Cubic and Quartic Regression Functions

Polynomial Functions of Higher Degree

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A18.

PS1. Find the minimum value of $P(x) = x^2 - 4x + 6$. [2.4]

PS2. Find the maximum value of $P(x) = -2x^2 - x + 1$. [2.4]

PS3. Find the interval on which $P(x) = x^2 + 2x + 7$ is increasing. [2.4]

PS4. Find the interval on which $P(x) = -2x^2 + 4x + 5$ is decreasing. [2.4]

PS5. Factor: $x^4 - 5x^2 + 4$ [P.4]

PS6. Find the x-intercepts of the graph of $P(x) = 6x^2 - x - 2$. [2.4]

Table 3.1 summarizes information developed in Chapter 2 about graphs of polynomial functions of degree 0, 1, or 2.

Table 3.1

Polynomial Function $P(x)$	Graph
$P(x) = a$ (degree 0)	Horizontal line through $(0, a)$
$P(x) = ax + b$ (degree 1), $a \neq 0$	Line with y-intercept $(0, b)$ and slope a
$P(x) = ax^2 + bx + c$ (degree 2), $a \neq 0$	Parabola with vertex $\left(-\frac{b}{2a}, P\left(-\frac{b}{2a}\right)\right)$

In this section, we will focus on polynomial functions of degree 3 or higher. These functions can be graphed by the technique of plotting points; however, some additional knowledge about polynomial functions will make graphing easier.

CALCULUS CONNECTION

All polynomial functions have graphs that are **smooth continuous curves.** The terms *smooth* and *continuous* are defined rigorously in calculus, but for the