The quotient $x^3 - 4x^2 - x + 4$ is called a **reduced polynomial,** or a **depressed polynomial,** of $P(x)$ because it is a factor of $P(x)$ and its degree is 1 less than the degree of $P(x)$. Reduced polynomials play an important role in Sections 3.3 and 3.4.

**Alternative to Example 5**
Verify that $(x - 2)$ is a factor of $P(x) = 3x^3 - 2x^2 + 9x - 34$, and write $P(x)$ as the product of $(x - 2)$ and the reduced polynomial $Q(x)$.
■ $P(x) = (x - 2)(3x^2 + 4x + 17)$

## EXAMPLE 5 Find a Reduced Polynomial

Verify that $(x - 3)$ is a factor of $P(x) = 2x^3 - 3x^2 - 4x - 15$, and write $P(x)$ as the product of $(x - 3)$ and the reduced polynomial $Q(x)$.

### Solution

| | | | | |
|---|---|---|---|---|
| 3 | 2 | −3 | −4 | −15 |
| | | 6 | 9 | 15 |
| | 2 | 3 | 5 | 0 |

This 0 indicates that $(x - 3)$ is a factor of $P(x)$.

Coefficients of the reduced polynomial $Q(x)$

Thus $(x - 3)$ and the reduced polynomial $2x^2 + 3x + 5$ are both factors of $P$. That is,

$$P(x) = 2x^3 - 3x^2 - 4x - 15 = (x - 3)(2x^2 + 3x + 5)$$

▸ Try Exercise 58, page 268

# EXERCISE SET 3.1

## Concept Check

**In Exercises 1 to 4, determine whether the given division can be performed using synthetic division.**

1. $(x^3 + 2x^2 - 5x + 3) \div (x - 2)$ Yes

2. $(2x^4 - 3x^2 + 6x - 5) \div (x + 3)$ Yes

3. $(x^4 - x^2 - 3) \div (x^2 + 1)$ No

4. $\left(x^5 - \frac{1}{2}\right) \div (x + 8)$ Yes

5. If a polynomial $P(x)$ is divided by $(x - 4)$ and the remainder is 7, then what do we know about $P(4)$? $P(4) = 7$

6. If $(x - 3)$ is a factor of the polynomial $P(x)$, then what do we know about $P(3)$? $P(3) = 0$

**In Exercises 7 to 14, use long division to divide the first polynomial by the second.**

7. $5x^3 + 6x^2 - 17x + 20, \quad x + 3$ $\quad 5x^2 - 9x + 10 - \frac{10}{x + 3}$

8. $6x^3 + 15x^2 - 8x + 2, \quad x + 4$ $\quad 6x^2 - 9x + 28 - \frac{110}{x + 4}$

9. $x^4 - 5x^2 + 3x - 1, \quad x - 2$ $\quad x^3 + 2x^2 - x + 1 + \frac{1}{x - 2}$

10. $x^4 - 5x^3 + x - 4, \quad x - 1$ $\quad x^3 - 4x^2 - 4x - 3 - \frac{7}{x - 1}$

11. $2x^4 + 5x^3 - 6x^2 + 4x + 3, \quad 2x^2 - x + 1$ $\quad x^2 + 3x - 2 + \frac{-x + 5}{2x^2 - x + 1}$

12. $3x^3 + x^2 - 5x + 2, \quad x^2 - 2x + 2$ $\quad 3x + 7 + \frac{3x - 12}{x^2 - 2x + 2}$

13. $2x^5 - x^3 + 5x^2 - 9x + 6, \quad 2x^2 + 2x - 3$ $\quad x^3 - x^2 + 2x - 1 + \frac{-x + 3}{2x^2 + 2x - 3}$

14. $x^5 + 3x^4 - 2x^3 - 7x^2 - x + 4, \quad x^2 + 1$ $\quad x^3 + 3x^2 - 3x - 10 + \frac{2x + 14}{x^2 + 1}$

**In Exercises 15 to 28, use synthetic division to divide the first polynomial by the second.**

15. $4x^3 - 5x^2 + 6x - 7, \quad x - 2$ $\quad 4x^2 + 3x + 12 + \frac{17}{x - 2}$

16. $5x^3 + 6x^2 - 8x + 1, \quad x - 5$ $\quad 5x^2 + 31x + 147 + \frac{736}{x - 5}$

17. $4x^3 - 2x + 3, \quad x + 1$ $\quad 4x^2 - 4x + 2 + \frac{1}{x + 1}$

18. $6x^3 - 4x^2 + 17, \quad x + 3$ $\quad 6x^2 - 22x + 66 - \frac{181}{x + 3}$

19. $x^5 - 10x^3 + 5x - 1, \quad x - 4$ $\quad x^4 + 4x^3 + 6x^2 + 24x + 101 + \frac{403}{x - 4}$

20. $6x^4 - 2x^3 - 3x^2 - x, \quad x - 5$ $\quad 6x^3 + 28x^2 + 137x + 684 + \frac{3420}{x - 5}$

21. $x^5 - 1, \quad x - 1$ $\quad x^4 + x^3 + x^2 + x + 1$

22. $x^4 + 1, \quad x + 1$ $\quad x^3 - x^2 + x - 1 + \frac{2}{x + 1}$

■ Indicates Try It Exercises

23. $8x^3 - 4x^2 + 6x - 3, \quad x - \frac{1}{2}$ $8x^2 + 6$

24. $12x^3 + 5x^2 + 5x + 6, \quad x + \frac{3}{4}$ $12x^2 - 4x + 8$

25. $x^8 + x^6 + x^4 + x^2 + 4, \quad x - 2$
$x^7 + 2x^6 + 5x^5 + 10x^4 + 21x^3 + 42x^2 + 85x + 170 + \frac{344}{x-2}$

26. $-x^7 - x^5 - x^3 - x - 5, \quad x + 1$
$-x^6 + x^5 - 2x^4 + 2x^3 - 3x^2 + 3x - 4 - \frac{1}{x+1}$

27. $x^6 + x - 10, \quad x + 3$
$x^5 - 3x^4 + 9x^3 - 27x^2 + 81x - 242 + \frac{716}{x+3}$

28. $2x^5 - 3x^4 - 5x^2 - 10, \quad x - 4$
$2x^4 + 5x^3 + 20x^2 + 75x + 300 + \frac{1190}{x-4}$

**In Exercises 29 to 38, use synthetic division and the Remainder Theorem to find *P*(*c*).**

29. $P(x) = 5x^3 + 2x^2 - x - 7, \quad c = 3$ 143

30. $P(x) = 2x^3 - x^2 + 3x - 1, \quad c = 3$ 53

31. $P(x) = 3x^4 - 5x^2 + 7, \quad c = -3$ 205

32. $P(x) = 8x^3 - 4x^2 + 3x, \quad c = -5$ $-1115$

33. $P(x) = -4x^3 - x^2 + 3x - 11, \quad c = 8$ $-2099$

34. $P(x) = -3x^3 + 6x^2 - 7x + 42, \quad c = 10$ $-2428$

35. $P(x) = -x^4 + x - 2, \quad c = 6$ $-1292$

36. $P(x) = x^5 - 1, \quad c = 1$ 0

37. $P(x) = -x^6 - 8x^4 - 3x + 5, \quad c = 2$ $-193$

38. $P(x) = x^5 + 20x^2 - 1, \quad c = -4$ $-705$

**In Exercises 39 to 48, use synthetic division and the Factor Theorem to determine whether the given binomial is a factor of *P*(*x*).**

39. $P(x) = x^3 + 2x^2 - 5x - 6, \quad x - 2$ Yes

40. $P(x) = x^3 + 4x^2 - 27x - 90, \quad x + 6$ Yes

41. $P(x) = 2x^3 + x^2 - 3x - 1, \quad x + 1$ No

42. $P(x) = 2x^3 - 8x^2 - 5x + 33, \quad x - 3$ Yes

43. $P(x) = x^4 + x^3 - 2x^2 + 5x - 140, \quad x + 4$ Yes

44. $P(x) = x^4 - 12x^2 + 36, \quad x - 3$ No

45. $P(x) = x^5 + 2x^4 - 22x^3 - 50x^2 - 75x, \quad x - 5$ Yes

46. $P(x) = 9x^4 - 6x^3 - 23x^2 - 4x + 4, \quad x + 1$ Yes

47. $P(x) = 16x^4 - 8x^3 + 9x^2 + 14x + 4, \quad x - \frac{1}{4}$ No

48. $P(x) = 10x^4 + 9x^3 - 4x^2 + 9x + 6, \quad x + \frac{1}{2}$ Yes

**In Exercises 49 to 56, use synthetic division to show that *c* is a zero of *P*.**

49. $P(x) = 3x^3 - 8x^2 - 10x + 28, \quad c = 2$

50. $P(x) = 4x^3 - 10x^2 - 8x + 6, \quad c = 3$

51. $P(x) = x^4 - 1, \quad c = 1$

52. $P(x) = x^3 + 8, \quad c = -2$

53. $P(x) = 3x^4 + 8x^3 + 10x^2 + 2x - 20, \quad c = -2$

54. $P(x) = x^4 - 2x^2 - 100x - 75, \quad c = 5$

55. $P(x) = 2x^3 - 18x^2 - 50x + 66, \quad c = 11$

56. $P(x) = 2x^4 - 34x^3 + 70x^2 - 153x + 45, \quad c = 15$

**In Exercises 57 to 60, verify that the given binomial is a factor of *P*(*x*), and write *P*(*x*) as the product of the binomial and its reduced polynomial *Q*(*x*).**

57. $P(x) = x^3 + x^2 + x - 14, \quad x - 2$ $(x - 2)(x^2 + 3x + 7)$

58. $P(x) = x^4 + 5x^3 + 3x^2 - 5x - 4, \quad x + 1$
$(x + 1)(x^3 + 4x^2 - x - 4)$

59. $P(x) = x^4 - x^3 - 9x^2 - 11x - 4, \quad x - 4$
$(x - 4)(x^3 + 3x^2 + 3x + 1)$

60. $P(x) = 2x^5 - x^4 - 7x^3 + x^2 + 7x - 10, \quad x - 2$
$(x - 2)(2x^4 + 3x^3 - x^2 - x + 5)$

61. **Selection of Cards** The number of ways you can select three cards from a stack of $n$ cards, in which the order of selection is important, is given by

$$P(n) = n^3 - 3n^2 + 2n, \quad n \geq 3$$

a. Use the Remainder Theorem to determine the number of ways you can select three cards from a stack of $n = 8$ cards. 336

b. Evaluate $P(n)$ for $n = 8$ by substituting 8 for $n$. How does this result compare with the result obtained in **a**? 336; they are the same.

62. **Selection of Bridesmaids** A bride-to-be has many girlfriends, but she has decided to have only five bridesmaids, including the maid of honor. The number of different ways $n$ girlfriends can be chosen and assigned a position, such as maid of honor, first bridesmaid, second bridesmaid, and so on, is given by the polynomial function

$$P(n) = n^5 - 10n^4 + 35n^3 - 50n^2 + 24n, \quad n \geq 5$$

a. Use the Remainder Theorem to determine the number of ways the bride can select her bridesmaids if she chooses from $n = 7$ girlfriends. 2520

b. Evaluate $P(n)$ for $n = 7$ by substituting 7 for $n$. How does this result compare with the result obtained in **a**? 2520; they are the same.

63. **House of Cards** The number of cards $C$ needed to build a house of cards with $r$ rows (levels) is given by the polynomial function $C(r) = 1.5r^2 + 0.5r$.

Topham/The Image Works

Use the Remainder Theorem to determine the number of cards needed to build a house of cards with

a. $r = 7$ rows 77 cards

b. $r = 12$ rows 222 cards

64. **Display of Soda Cans** The number $S$ of soda cans needed to build a square pyramid display with $n$ levels is given by the polynomial function

$$S(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$


A square pyramid display with $n^2$ soda cans in level $n$

Use the Remainder Theorem to determine the number of soda cans needed to build a square pyramid display with

a. $n = 6$ levels 91 cans

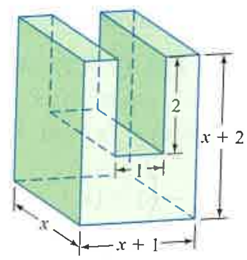
b. $n = 12$ levels 650 cans

65. **Election of Class Officers** The number of ways a class of $n$ students can elect a president, a vice president, a secretary, and a treasurer is given by $P(n) = n^4 - 6n^3 + 11n^2 - 6n$, where $n \geq 4$. Use the Remainder Theorem to determine the number of ways the class can elect officers if the class consists of

a. $n = 8$ students 1680 ways

b. $n = 18$ students 73,440 ways

66. **Volume of a Solid** The volume, in cubic inches, of the following solid is given by $V(x) = x^3 + 3x^2$.

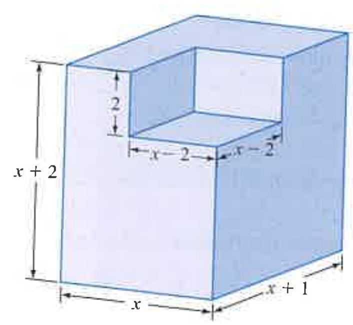



Use the Remainder Theorem to determine the volume of the solid if

a. $x = 7$ inches 490 in.$^3$

b. $x = 11$ inches 1694 in.$^3$

67. **Volume of a Solid** The volume, in cubic inches, of the following solid is given by $V(x) = x^3 + x^2 + 10x - 8$.

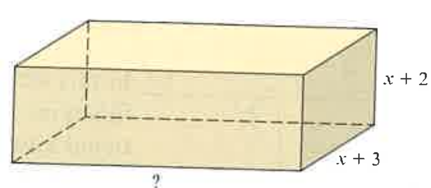



Use the Remainder Theorem to determine the volume of the solid if

a. $x = 6$ inches 304 in.$^3$

b. $x = 9$ inches 892 in.$^3$

68. **Volume of a Box** A rectangular box has a volume of $V(x) = x^3 + 10x^2 + 31x + 30$ cubic inches. The height of the box is $x + 2$ inches. The width of the box is $x + 3$ inches. Find the length of the box in terms of $x$. $(x + 5)$ in.



## Enrichment Exercises

**69.** Use synthetic division to divide each of the following polynomials by $x - 1$.

$$x^3 - 1, x^5 - 1, x^7 - 1$$

Use the pattern suggested by these quotients to write the quotient of $(x^9 - 1) \div (x - 1)$.
$x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

**In Exercises 70 to 73, determine the value of $k$ so that the divisor is a factor of the dividend.**

**70.** $(x^3 - x^2 - 14x + k) \div (x - 2)$ 24

**71.** $(2x^3 + x^2 - 25x + k) \div (x - 3)$ 12

**72.** $(3x^3 + 14x^2 + kx - 6) \div (x + 2)$ 13

**73.** $(x^4 + 3x^3 - 8x^2 + kx + 16) \div (x + 4)$ −12

**74.** Use the Factor Theorem to show that for any positive integer $n$

$$P(x) = x^n - 1$$

has $x - 1$ as a factor. $P(1) = (1)^n - 1 = 0$
Thus, by the Factor Theorem, $(x - 1)$ is a factor of $P(x)$ for any positive integer $n$.

**75.** Find the remainder of

$$5x^{48} + 6x^{10} - 5x + 7$$

divided by $x - 1$. 13

**76.** Find the remainder of

$$18x^{80} - 6x^{50} + 4x^{20} - 2$$

divided by $x + 1$. 14

**77.** Determine whether $i$ is a zero of

$$P(x) = x^3 - 3x^2 + x - 3$$ Yes

**78.** Determine whether $-2i$ is a zero of

$$P(x) = x^4 - 2x^3 + x^2 - 8x - 12$$ Yes

## SECTION 3.2

- Far-Left and Far-Right Behavior
- Maximum and Minimum Values
- Real Zeros of a Polynomial Function
- Intermediate Value Theorem
- Real Zeros, $x$-Intercepts, and Factors of a Polynomial Function
- Even and Odd Powers of $(x - c)$ Theorem
- Procedure for Graphing Polynomial Functions
- Cubic and Quartic Regression Functions

# Polynomial Functions of Higher Degree

### PREPARE FOR THIS SECTION

**Prepare for this section by completing the following exercises. The answers can be found on page A18.**

**PS1.** Find the minimum value of $P(x) = x^2 - 4x + 6$. [2.4] 2

**PS2.** Find the maximum value of $P(x) = -2x^2 - x + 1$. [2.4] $\frac{9}{8}$

**PS3.** Find the interval on which $P(x) = x^2 + 2x + 7$ is increasing. [2.4] $[-1, \infty)$

**PS4.** Find the interval on which $P(x) = -2x^2 + 4x + 5$ is decreasing. [2.4] $[1, \infty)$

**PS5.** Factor: $x^4 - 5x^2 + 4$ [P.4] $(x + 1)(x - 1)(x + 2)(x - 2)$

**PS6.** Find the $x$-intercepts of the graph of $P(x) = 6x^2 - x - 2$. [2.4] $\left(\frac{2}{3}, 0\right), \left(-\frac{1}{2}, 0\right)$

Table 3.1 summarizes information developed in Chapter 2 about graphs of polynomial functions of degree 0, 1, or 2.

### Table 3.1

| Polynomial Function $P(x)$ | Graph |
|---|---|
| $P(x) = a$ (degree 0) | Horizontal line through $(0, a)$ |
| $P(x) = ax + b$ (degree 1), $a \neq 0$ | Line with $y$-intercept $(0, b)$ and slope $a$ |
| $P(x) = ax^2 + bx + c$ (degree 2), $a \neq 0$ | Parabola with vertex $\left(-\frac{b}{2a}, P\left(-\frac{b}{2a}\right)\right)$ |

In this section, we will focus on polynomial functions of degree 3 or higher. These functions can be graphed by the technique of plotting points; however, some additional knowledge about polynomial functions will make graphing easier.

All polynomial functions have graphs that are **smooth continuous curves**. The terms *smooth* and *continuous* are defined rigorously in calculus, but for the

CALCULUS CONNECTION