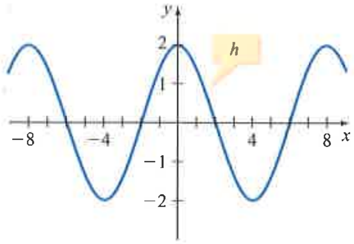
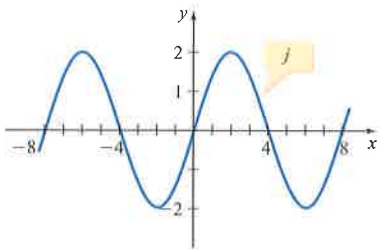
69. Use the graph of $y = h(x)$ to sketch the graph of

a. $y = h(2x)$ **b.** $y = h\left(\frac{1}{2}x\right)$

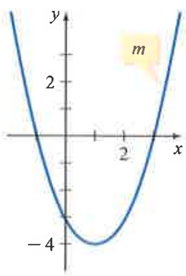


70. Use the graph of $y = j(x)$ to sketch the graph of

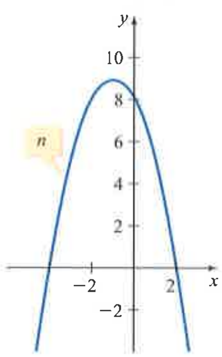
a. $y = j(2x)$ **b.** $y = j\left(\frac{1}{3}x\right)$



71. Use the graph of m to sketch the graph of $y = -\frac{1}{2}m(x) + 3$.

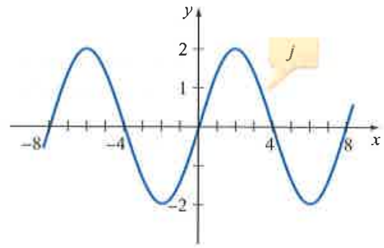


72. Use the graph of n to sketch the graph of $y = \frac{1}{2}n(x) + 1$.



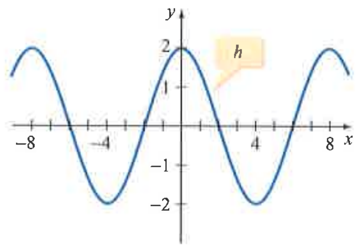
73. Use the graph of $y = j(x)$ to sketch the graph of

a. $y = -\frac{1}{2}j(x) + 1$ **b.** $y = 2j(x) - 1$



74. Use the graph of $y = h(x)$ to sketch the graph of

a. $y = \frac{1}{2}h(x) - 1$ **b.** $y = -2h(x) + 1$



In Exercises 75 to 78, use a graphing utility to graph a function containing a variable *c*. One way to do this is to use a list, a very powerful tool in many branches of mathematics. A list is enclosed in braces. For instance, to graph $y = x^2 + c$ for $c = -2, 0$, and 3, enter Y1 = x^2 + {−2, 0, 3}. Then press Graph.

75. On the same coordinate axes, graph

$$G(x) = \sqrt[3]{x} + c$$

for $c = 0, -1$, and 3.

76. On the same coordinate axes, graph

$$H(x) = \sqrt[3]{x + c}$$

for $c = 0, -1$, and 3.

77. On the same coordinate axes, graph

$$L(x) = cx^2$$

for $c = 1, \frac{1}{2}$, and 2.

78. On the same coordinate axes, graph

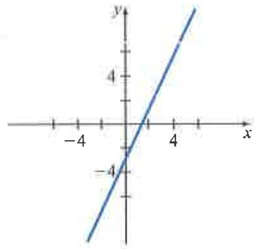
$$M(x) = c\sqrt{x^2 - 4}$$

for $c = 1, \frac{1}{3}$, and 3.

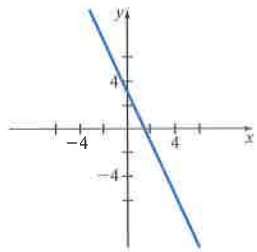
Enrichment Exercises

In Exercises 79 to 82, draw the graph that results from applying the operations to the given graph.

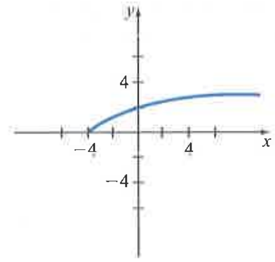
79. Reflect the graph about the y-axis and then about the origin.



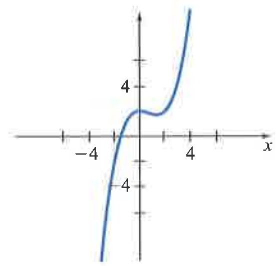
80. Reflect the graph about the origin and then about the y-axis.



81. Reflect the graph about the y-axis and then about the x-axis.



82. Reflect the graph about the x-axis and then about the origin.



SECTION 2.6 Algebra of Functions

Operations on Functions
Difference Quotient
Composition of Functions

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A13.

PS1. Subtract: $(2x^2 + 3x - 4) - (x^2 + 3x - 5)$ [P.3]

PS2. Multiply: $(3x^2 - x + 2)(2x - 3)$ [P.3]

In Exercises PS3 and PS4, find each of the following for $f(x) = 2x^2 - 5x + 2$.

PS3. $f(3a)$ [2.2]

PS4. $f(2 + h)$ [2.2]

In Exercises PS5 and PS6, find the domain of each function.

PS5. $F(x) = \dfrac{x}{x - 1}$ [2.2]

PS6. $r(x) = \sqrt{2x - 8}$ [2.2]

Operations on Functions

Functions can be defined in terms of other functions. For example, the function defined by $h(x) = x^2 + 8x$ is the sum of

$$f(x) = x^2 \qquad \text{and} \qquad g(x) = 8x$$

Thus, if we are given any two functions f and g, we can define the four new functions $f + g$, $f - g$, fg, and $\dfrac{f}{g}$ as follows.

Definitions of Operations on Functions

If f and g are functions with domains D_f and D_g, then we define the sum, difference, product, and quotient of f and g as

Sum	$(f + g)(x) = f(x) + g(x)$	Domain: $D_f \cap D_g$
Difference	$(f - g)(x) = f(x) - g(x)$	Domain: $D_f \cap D_g$

Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	Domain: $D_f \cap D_g$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	Domain: $D_f \cap D_g, g(x) \neq 0$

EXAMPLE

Let $f(x) = 3x - 2$ and $g(x) = x^2 + 6$. Then,

$(f + g)(x) = f(x) + g(x) = (3x - 2) + (x^2 + 6) = x^2 + 3x + 4$

$(f - g)(x) = f(x) - g(x) = (3x - 2) - (x^2 + 6) = -x^2 + 3x - 8$

$(f \cdot g)(x) = f(x) \cdot g(x) = (3x - 2)(x^2 + 6) = 3x^3 - 2x^2 + 18x - 12$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{3x - 2}{x^2 + 6}$$

EXAMPLE 1 Determine the Domain of a Function

If $f(x) = \sqrt{x - 1}$ and $g(x) = x^2 - 4$, find the domains of $f + g$, $f - g$, fg, and $\frac{f}{g}$.

Solution

Note that f has the domain $\{x \mid x \geq 1\}$ and g has the domain of all real numbers. Therefore, the domain of $f + g$, $f - g$, and fg is $\{x \mid x \geq 1\}$. Because $g(x) = 0$ when $x = -2$ or $x = 2$, neither -2 nor 2 is in the domain of $\frac{f}{g}$. The domain of $\frac{f}{g}$ is $\{x \mid x \geq 1 \text{ and } x \neq 2\}$.

▸ Try Exercise 18, page 233

EXAMPLE 2 Evaluate Functions

Let $f(x) = x^2 - 9$ and $g(x) = 2x + 6$. Find the following.

a. $(f + g)(5)$

b. $(fg)(-1)$

c. $\left(\frac{f}{g}\right)(4)$

Solution

a. $(f + g)(x) = f(x) + g(x) = (x^2 - 9) + (2x + 6) = x^2 + 2x - 3$

Therefore, $(f + g)(5) = (5)^2 + 2(5) - 3 = 25 + 10 - 3 = 32$.

b. $(fg)(x) = f(x) \cdot g(x) = (x^2 - 9)(2x + 6) = 2x^3 + 6x^2 - 18x - 54$

Therefore, $(fg)(-1) = 2(-1)^3 + 6(-1)^2 - 18(-1) - 54$

$= -2 + 6 + 18 - 54 = -32$.

c. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 9}{2x + 6} = \frac{\cancel{(x + 3)}(x - 3)}{2\cancel{(x + 3)}} = \frac{x - 3}{2}, \quad x \neq -3$

Therefore, $\left(\frac{f}{g}\right)(4) = \frac{4 - 3}{2} = \frac{1}{2}$.

▸ Try Exercise 22, page 233

∫CALCULUS CONNECTION

Difference Quotient

The expression

$$\frac{f(x+h)-f(x)}{h}, \quad h \neq 0$$

is called the **difference quotient** of f. It enables us to study the manner in which a function changes in value as the independent variable changes.

EXAMPLE 3 Determine a Difference Quotient

Determine the difference quotient of $f(x) = x^2 + 7$.

Solution

$$\frac{f(x+h)-f(x)}{h} = \frac{[(x+h)^2+7]-[x^2+7]}{h}$$ • Apply the difference quotient.

$$= \frac{[x^2+2xh+h^2+7]-[x^2+7]}{h}$$

$$= \frac{x^2+2xh+h^2+7-x^2-7}{h}$$

$$= \frac{2xh+h^2}{h} = \frac{h(2x+h)}{h} = 2x+h$$

Try Exercise 38, page 234

The difference quotient $2x + h$ of $f(x) = x^2 + 7$ from Example 3 is the slope of the secant line through the points

$$(x, f(x)) \quad \text{and} \quad (x+h, f(x+h))$$

For instance, let $x = 1$ and $h = 1$. Then the difference quotient is

$$2x + h = 2(1) + 1 = 3$$

This is the slope of the secant line l_2 through (1, 8) and (2, 11), as shown in Figure 2.98. If we let $x = 1$ and $h = 0.1$, then the difference quotient is

$$2x + h = 2(1) + 0.1 = 2.1$$

This is the slope of the secant line l_1 through (1, 8) and (1.1, 8.21).

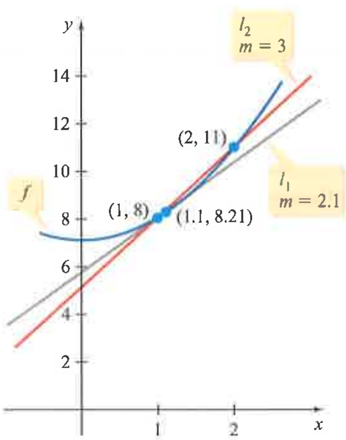


Figure 2.98

The difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

can be used to compute *average velocities.* In such cases, it is traditional to replace f with s (for distance), the variable x with the variable a (for the time at the start of an observed interval), and the variable h with Δt (read as "delta t"), where Δt is the difference between the time at the end of an interval and the time at the start of the interval. For example, if an experiment is observed over the time interval from $t = 3$ seconds to $t = 5$ seconds, then the time interval is denoted as [3, 5] with $a = 3$ and $\Delta t = 5 - 3 = 2$. Thus if the distance traveled by a ball that rolls down a ramp is given by $s(t)$, where t is the time in seconds after the ball is released (see Figure 2.99), then the **average velocity** of the ball over the interval $t = a$ to $t = a + \Delta t$ is the difference quotient

Figure 2.99

$$\frac{s(a+\Delta t)-s(a)}{\Delta t}$$

EXAMPLE 4 Evaluate Average Velocities

The distance traveled by a ball rolling down a ramp is given by $s(t) = 4t^2$, where t is the time in seconds after the ball is released and $s(t)$ is measured in feet. Evaluate the average velocity of the ball for each time interval.

a. [3, 5] **b.** [3, 4] **c.** [3, 3.5] **d.** [3, 3.01]

Solution

a. In this case, $a = 3$ and $\Delta t = 5 - 3 = 2$. Thus the average velocity over this interval is

$$\frac{s(a + \Delta t) - s(a)}{\Delta t} = \frac{s(3 + 2) - s(3)}{2} = \frac{s(5) - s(3)}{2} = \frac{100 - 36}{2}$$
$$= 32 \text{ feet per second}$$

b. Let $a = 3$ and $\Delta t = 4 - 3 = 1$.

$$\frac{s(a + \Delta t) - s(a)}{\Delta t} = \frac{s(3 + 1) - s(3)}{1} = \frac{s(4) - s(3)}{1} = \frac{64 - 36}{1}$$
$$= 28 \text{ feet per second}$$

c. Let $a = 3$ and $\Delta t = 3.5 - 3 = 0.5$.

$$\frac{s(a + \Delta t) - s(a)}{\Delta t} = \frac{s(3 + 0.5) - s(3)}{0.5} = \frac{49 - 36}{0.5} = 26 \text{ feet per second}$$

d. Let $a = 3$ and $\Delta t = 3.01 - 3 = 0.01$.

$$\frac{s(a + \Delta t) - s(a)}{\Delta t} = \frac{s(3 + 0.01) - s(3)}{0.01} = \frac{36.2404 - 36}{0.01}$$
$$= 24.04 \text{ feet per second}$$

Try Exercise 46, page 234

Composition of Functions

Composition of functions is another way in which functions can be combined. This method of combining functions uses the output of one function as the input for a second function.

Suppose that the spread of oil from a leak in a tanker can be approximated by a circle, with the tanker at its center. The radius r (in feet) of the spill t hours after the leak began is given by $r(t) = 150\sqrt{t}$. The area of the spill is the area of a circle and is given by the formula $A(r) = \pi r^2$. To find the area of the spill 4 hours after the leak began, we first find the radius of the spill and then use that number to find the area of the spill.

$$\begin{aligned} r(t) &= 150\sqrt{t} \\ r(4) &= 150\sqrt{4} \quad \bullet\ t = 4 \text{ hours} \\ &= 150(2) \\ &= 300 \end{aligned}$$

$$\begin{aligned} A(r) &= \pi r^2 \\ A(300) &= \pi(300^2) \quad \bullet\ r = 300 \text{ feet} \\ &= 90{,}000\pi \\ &\approx 283{,}000 \end{aligned}$$

The area of the spill after 4 hours is approximately 283,000 square feet.

There is an alternative way to solve this problem. Because the area of the spill depends on the radius and the radius depends on the time, there is a relationship between area and time. We can determine this relationship by evaluating the formula for the area of a circle using $r(t) = 150\sqrt{t}$. This will give the area of the spill as a function of time.

$$A(r) = \pi r^2$$

$$A[r(t)] = \pi[r(t)]^2 \qquad \text{• Replace } r \text{ with } r(t).$$

$$= \pi(150\sqrt{t})^2 \qquad \text{• } r(t) = 150\sqrt{t}$$

$$A(t) = 22{,}500\pi t \qquad \text{• Simplify.}$$

The area of the spill as a function of time is $A(t) = 22{,}500\pi t$. To find the area of the oil spill after 4 hours, evaluate this function at $t = 4$.

$$A(t) = 22{,}500\pi t$$

$$A(4) = 22{,}500\pi(4) \qquad \text{• } t = 4 \text{ hours}$$

$$= 90{,}000\pi$$

$$\approx 283{,}000$$

This is the same as the result we calculated earlier.

The function $A(t) = 22{,}500\pi t$ is referred to as the *composition* of A with r. The notation $A \circ r$ is used to denote this composition of functions. That is,

$$(A \circ r)(t) = 22{,}500\pi t$$

Note

The requirement in the definition of the composition of two functions that $g(x)$ be in the domain of f for all x in the domain of g is important. For instance, let

$$f(x) = \frac{1}{x-1} \quad \text{and}$$

$$g(x) = 3x - 5$$

When $x = 2$,

$$g(2) = 3(2) - 5 = 1$$

$$f[g(2)] = f(1)$$

$$= \frac{1}{1-1}$$

$$= \frac{1}{0} \qquad \text{• Undefined}$$

In this case, $g(2)$ is not in the domain of f. Thus the composition $(f \circ g)(x)$ is not defined at 2.

Definition of the Composition of Two Functions

Let f and g be two functions such that $g(x)$ is in the domain of f for all x in the domain of g. Then the composition of the two functions, denoted by $f \circ g$, is the function whose value at x is given by $(f \circ g)(x) = f[g(x)]$.

The function defined by $(f \circ g)(x)$ is also called the *composite* of f and g. We read $(f \circ g)(x)$ as "f circle g of x" and $f[g(x)]$ as "f of g of x."

Consider the functions $f(x) = 2x - 1$ and $g(x) = x^2 - 3$. The expression $(f \circ g)(-1)$—or, equivalently, $f[g(-1)]$—means to evaluate the function f at $g(-1)$.

$$g(x) = x^2 - 3$$

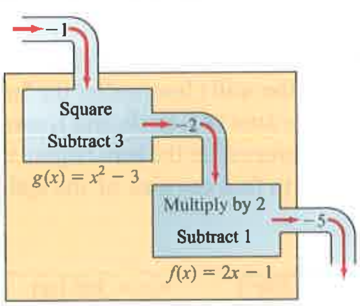
$$g(-1) = (-1)^2 - 3 \qquad \text{• Evaluate } g \text{ at } -1.$$

$$g(-1) = -2$$

$$f(x) = 2x - 1$$

$$f[g(-1)] = 2[g(-1)] - 1 \qquad \text{• Replace } x \text{ with } g(-1).$$

$$f(-2) = 2(-2) - 1 = -5 \qquad \text{• } g(-1) = -2$$



$(f \circ g)(-1) = f[g(-1)] = -5$

Figure 2.100

A graphical depiction of the composition $(f \circ g)(-1)$ would look something like Figure 2.100.

We can find a general expression for $f[g(x)]$ by evaluating f at $g(x)$.

$$f(x) = 2x - 1$$

$$f[g(x)] = 2[g(x)] - 1 \qquad \text{• Replace } x \text{ with } g(x).$$

$$= 2[x^2 - 3] - 1 \qquad \text{• Replace } g(x) \text{ with } x^2 - 3.$$

$$= 2x^2 - 7 \qquad \text{• Simplify.}$$

Thus $f[g(x)] = 2x^2 - 7$. If we evaluate this function at -1, we have

$$f[g(x)] = 2x^2 - 7$$

$$f[g(-1)] = 2(-1)^2 - 7 = 2 - 7 = -5$$

This is the same as the result that was obtained in Figure 2.100.

In general, the composition of functions is not a commutative operation. That is, $(f \circ g)(x) \neq (g \circ f)(x)$. To verify this, we will compute $(g \circ f)(x) = g[f(x)]$, again using the functions $f(x) = 2x - 1$ and $g(x) = x^2 - 3$.

$$g(x) = x^2 - 3$$

$$g[f(x)] = [f(x)]^2 - 3 \quad \text{• Replace } x \text{ with } f(x).$$

$$= [2x - 1]^2 - 3 \quad \text{• Replace } f(x) \text{ with } 2x - 1.$$

$$= 4x^2 - 4x - 2 \quad \text{• Simplify.}$$

Thus $g[f(x)] = 4x^2 - 4x - 2$, which is not equal to $f[g(x)] = 2x^2 - 7$. Therefore, $(f \circ g)(x) \neq (g \circ f)(x)$ and composition is not a commutative operation.

Question • Let $f(x) = x - 1$ and $g(x) = x + 1$. Then $f[g(x)] = g[f(x)]$. (You should verify this statement.) Does this contradict the statement we made that composition is not a commutative operation?

EXAMPLE 5 Form Composite Functions

If $f(x) = x^2 - 3x$ and $g(x) = 2x + 1$, find the following.

a. $(g \circ f)(x)$ **b.** $(f \circ g)(x)$

Solution

a. $(g \circ f)(x) = g[f(x)] = 2(f(x)) + 1$ • Substitute $f(x)$ for x in g.

$= 2(x^2 - 3x) + 1$ • $f(x) = x^2 - 3x$

$= 2x^2 - 6x + 1$

b. $(f \circ g)(x) = f[g(x)] = (g(x))^2 - 3(g(x))$ • Substitute $g(x)$ for x in f.

$= (2x + 1)^2 - 3(2x + 1)$ • $g(x) = 2x + 1$

$= 4x^2 - 2x - 2$

▶ **Try Exercise 48, page 234**

Some care must be used when forming the composition of functions. For instance, if $f(x) = x + 1$ and $g(x) = \sqrt{x - 4}$, then

$$(g \circ f)(2) = g[f(2)] = g(3) = \sqrt{3 - 4} = \sqrt{-1}$$

which is not a real number. We can avoid this problem by imposing suitable restrictions on the domain of f so that the range of f is part of the domain of g. If the domain of f is restricted to $[3, \infty)$, then the range of f is $[4, \infty)$. But this is precisely the domain of g. Note that $2 \notin [3, \infty)$. Thus we avoid the problem of $(g \circ f)(2)$ not being a real number.

To evaluate $(f \circ g)(c)$ for some constant c, you can use either of the following methods.

Method 1 First evaluate $g(c)$. Then substitute this result for x in $f(x)$.

Method 2 First determine $f[g(x)]$. Then substitute c for x.

Answer • No. When we say that composition is not a commutative operation, we mean that generally, given any two functions, $(f \circ g)(x) \neq (g \circ f)(x)$. However, there may be particular instances in which $(f \circ g)(x) = (g \circ f)(x)$. It turns out that these particular instances are quite important, as we shall see later.

EXAMPLE 6 Evaluate a Composite Function

Evaluate $(f \circ g)(3)$, where $f(x) = 2x - 7$ and $g(x) = x^2 + 4$.

Solution

Method 1 $(f \circ g)(3) = f[g(3)]$

$= f[(3)^2 + 4]$ • Evaluate $g(3)$.

$= f(13)$

$= 2(13) - 7 = 19$ • Substitute 13 for x in f.

Method 2 $(f \circ g)(x) = 2[g(x)] - 7$ • Form $f[g(x)]$.

$= 2[x^2 + 4] - 7$

$= 2x^2 + 1$

$(f \circ g)(3) = 2(3)^2 + 1 = 19$ • Substitute 3 for x.

▸ Try Exercise 60, page 234

Study tip

In Example 6, both Method 1 and Method 2 produce the same result. Method 2 is the better method if you must evaluate $(f \circ g)(x)$ for several values of x.

EXAMPLE 7 Use a Composite Function to Solve an Application

A graphic artist has drawn a 3-inch by 2-inch rectangle on a computer screen. The artist has been scaling the size of the rectangle for t seconds in such a way that the upper right corner of the original rectangle is moving to the right at the rate of 0.5 inch per second and downward at the rate of 0.2 inch per second. See Figure 2.101.

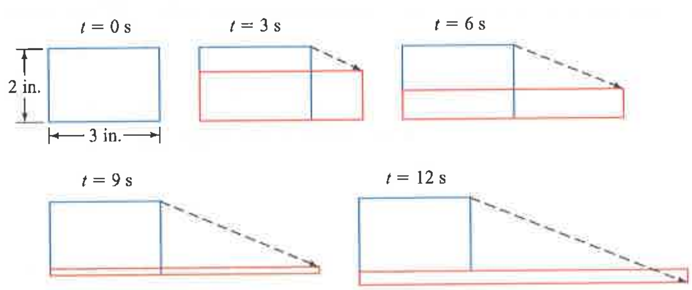


Figure 2.101

a. Write the length l and the width w of the scaled rectangles as functions of t.

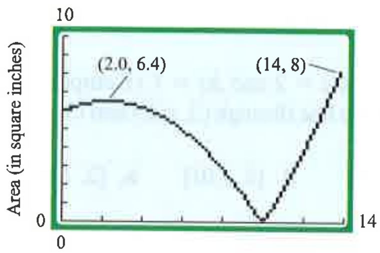
b. Write the area A of the scaled rectangle as a function of t.

c. Find the intervals on which A is an increasing function for $0 \le t \le 14$. Also find the intervals on which A is a decreasing function.

d. Find the value of t (where $0 \le t \le 14$) that maximizes $A(t)$.

Solution

a. Because *Distance* = *Rate* · *Time*, we see that the change in l is given by $0.5t$. Therefore, the length at any time t is $l = 3 + 0.5t$. For $0 \le t \le 10$, the width is given by $w = 2 - 0.2t$. For $10 < t \le 14$, the width is $w = -2 + 0.2t$. In either case, the width can be determined by finding $w = |2 - 0.2t|$. (The absolute value symbol is needed to keep the width positive for $10 < t \le 14$.)



$A = (3 + 0.5t)|2 - 0.2t|$

Figure 2.102

b. $A = lw = (3 + 0.5t)|2 - 0.2t|$

c. Use a graphing utility to determine that A is increasing on [0, 2] and on [10, 14] and that A is decreasing on [2, 10]. See Figure 2.102.

d. The highest point on the graph of A occurs when $t = 14$ seconds. See Figure 2.102.

Try Exercise 86, page 235

You may be inclined to think that if the area of a rectangle is decreasing then its perimeter is also decreasing, but this is not always the case. For example, the area of the scaled rectangle in Example 7 was shown to decrease on [2, 10] even though its perimeter was always increasing.

EXERCISE SET 2.6

Concept Check

In Exercises 1 to 4, let $f(-2) = 3$ and $g(-2) = -6$. Find each of the following.

1. $(f + g)(-2)$

2. $(f - g)(-2)$

3. $(f \cdot g)(-2)$

4. $\frac{f}{g}(-2)$

5. Suppose $f(-5) = 7$ and $g(7) = -2$. What is the value of $g[f(-5)]$?

6. If $f[g(0)] = 3$ and $g(0) = 4$, what is the value of $f(4)$?

In Exercises 7 and 8, find $f(2 + h)$.

7. $f(x) = 3x - 4$

8. $f(x) = x^2 + 1$

In Exercises 9 to 20, use the given functions f and g to find $f + g$, $f - g$, fg, and $\frac{f}{g}$. State the domain of each.

9. $f(x) = x^2 - 2x - 15, \quad g(x) = x + 3$

10. $f(x) = x^2 - 25, \quad g(x) = x - 5$

11. $f(x) = 2x + 8, \quad g(x) = x + 4$

12. $f(x) = 5x - 15, \quad g(x) = x - 3$

13. $f(x) = x^3 - 2x^2 + 7x, \quad g(x) = x$

14. $f(x) = x^2 - 5x - 8, \quad g(x) = -x$

15. $f(x) = 4x - 7, \quad g(x) = 2x^2 + 3x - 5$

16. $f(x) = 6x + 10, \quad g(x) = 3x^2 + x - 10$

17. $f(x) = \sqrt{x - 3}, \quad g(x) = x$

18. $f(x) = \sqrt{x - 4}, \quad g(x) = -x$

19. $f(x) = \sqrt{4 - x^2}, \quad g(x) = 2 + x$

20. $f(x) = \sqrt{x^2 - 9}, \quad g(x) = x - 3$

In Exercises 21 to 36, evaluate the indicated function, where $f(x) = x^2 - 3x + 2$ and $g(x) = 2x - 4$.

21. $(f + g)(5)$

22. $(f + g)(-7)$

23. $(f + g)\left(\frac{1}{2}\right)$

24. $(f + g)\left(\frac{2}{3}\right)$

25. $(f - g)(-3)$

26. $(f - g)(24)$

27. $(f - g)(-1)$

28. $(f - g)(0)$

29. $(fg)(7)$

30. $(fg)(-3)$

31. $(fg)\left(\frac{2}{5}\right)$

32. $(fg)(-100)$

33. $\left(\frac{f}{g}\right)(-4)$

34. $\left(\frac{f}{g}\right)(11)$

35. $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$

36. $\left(\frac{f}{g}\right)\left(\frac{1}{4}\right)$

Indicates Try It Exercises

In Exercises 37 to 44, find the difference quotient of the given function.

37. $f(x) = 2x + 4$

38. $f(x) = 4x - 5$

39. $f(x) = x^2 - 6$

40. $f(x) = x^2 + 11$

41. $f(x) = 2x^2 + 4x - 3$

42. $f(x) = 2x^2 - 5x + 7$

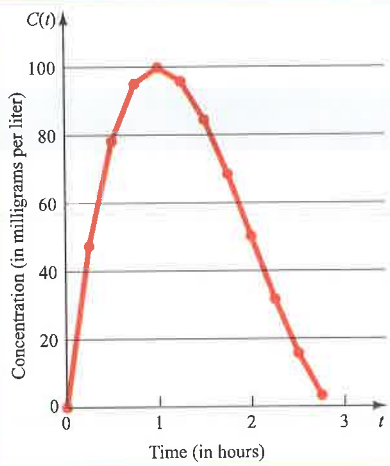
43. $f(x) = -4x^2 + 6$

44. $f(x) = -5x^2 - 4x$

45. **Concentration of a Medication** The concentration $C(t)$ (in milligrams per liter) of a medication in a patient's blood is given by the data in the following table.

Concentration of Medication in Patient's Blood

t (h)	$C(t)$ (mg/L)
0	0
0.25	47.3
0.50	78.1
0.75	94.9
1.00	99.8
1.25	95.7
1.50	84.4
1.75	68.4
2.00	50.1
2.25	31.6
2.50	15.6
2.75	4.3



The **average rate of change** of the concentration over the time interval from $t = a$ to $t = a + \Delta t$ is

$$\frac{C(a + \Delta t) - C(a)}{\Delta t}$$

Use the data in the table to evaluate the average rate of change for each of the following time intervals.

a. $[0, 1]$ (*Hint:* In this case, $a = 0$ and $\Delta t = 1$.) Compare this result to the slope of the line through $(0, C(0))$ and $(1, C(1))$.

b. $[0, 0.5]$ **c.** $[1, 2]$ **d.** $[1, 1.5]$ **e.** $[1, 1.25]$

f. The data in the table can be modeled by the function $Con(t) = 25t^3 - 150t^2 + 225t$. Use $Con(t)$ to verify that the average rate of change over $[1, 1 + \Delta t]$ is given by $-75(\Delta t) + 25(\Delta t)^2$. What does the average rate of change over $[1, 1 + \Delta t]$ seem to approach as Δt approaches 0?

46. **Ball Rolling on a Ramp** The distance traveled by a ball rolling down a ramp is given by $s(t) = 6t^2$, where t is the time in seconds after the ball is released and $s(t)$ is measured in feet. The ball travels 6 feet in 1 second and 24 feet in 2 seconds. Use the difference quotient for average velocity given on page 228 to evaluate the average velocity for each of the following time intervals.

a. $[2, 3]$ (*Hint:* In this case, $a = 2$ and $\Delta t = 1$.) Compare this result to the slope of the line through $(2, s(2))$ and $(3, s(3))$.

b. $[2, 2.5]$ **c.** $[2, 2.1]$ **d.** $[2, 2.01]$ **e.** $[2, 2.001]$

f. Verify that the average velocity over $[2, 2 + \Delta t]$ is $24 + 6(\Delta t)$. What does the average velocity seem to approach as Δt approaches 0?

In Exercises 47 to 58, find $(g \circ f)(x)$ and $(f \circ g)(x)$ for the given functions f and g.

47. $f(x) = 3x + 5, \quad g(x) = 2x - 7$

48. $f(x) = 2x - 7, \quad g(x) = 3x + 2$

49. $f(x) = x^2 + 4x - 1, \quad g(x) = x + 2$

50. $f(x) = x^2 - 11x, \quad g(x) = 2x + 3$

51. $f(x) = x^3 + 2x, \quad g(x) = -5x$

52. $f(x) = -x^3 - 7, \quad g(x) = x + 1$

53. $f(x) = \dfrac{2}{x + 1}, \quad g(x) = 3x - 5$

54. $f(x) = \sqrt{x + 4}, \quad g(x) = \dfrac{1}{x}$

55. $f(x) = \dfrac{1}{x^2}, \quad g(x) = \sqrt{x - 1}$

56. $f(x) = \dfrac{6}{x - 2}, \quad g(x) = \dfrac{3}{5x}$

57. $f(x) = \dfrac{3}{|5 - x|}, \quad g(x) = -\dfrac{2}{x}$

58. $f(x) = |2x + 1|, \quad g(x) = 3x^2 - 1$

In Exercises 59 to 74, evaluate each composite function, where $f(x) = 2x + 3$, $g(x) = x^2 - 5x$, and $h(x) = 4 - 3x^2$.

59. $(g \circ f)(4)$

60. $(f \circ g)(4)$

61. $(f \circ g)(-3)$

62. $(g \circ f)(-1)$

63. $(g \circ h)(0)$

64. $(h \circ g)(0)$

65. $(f \circ f)(8)$

66. $(f \circ f)(-8)$

67. $(h \circ g)\left(\dfrac{2}{5}\right)$

68. $(g \circ h)\left(-\dfrac{1}{3}\right)$

69. $(g \circ f)(\sqrt{3})$

70. $(f \circ g)(\sqrt{2})$

71. $(g \circ f)(2c)$

72. $(f \circ g)(3k)$

73. $(g \circ h)(k + 1)$

74. $(h \circ g)(k - 1)$

In Exercises 75 to 78, show that $(f \circ g)(x) = (g \circ f)(x)$.

75. $f(x) = 2x + 3$; $g(x) = 5x + 12$

76. $f(x) = 4x - 2$; $g(x) = 7x - 4$

77. $f(x) = \dfrac{6x}{x - 1}$; $g(x) = \dfrac{5x}{x - 2}$

78. $f(x) = \dfrac{5x}{x + 3}$; $g(x) = -\dfrac{2x}{x - 4}$

In Exercises 79 to 82, show that

$$(g \circ f)(x) = x \quad \text{and} \quad (f \circ g)(x) = x$$

79. $f(x) = 2x + 3$, $g(x) = \dfrac{x - 3}{2}$

80. $f(x) = 4x - 5$, $g(x) = \dfrac{x + 5}{4}$

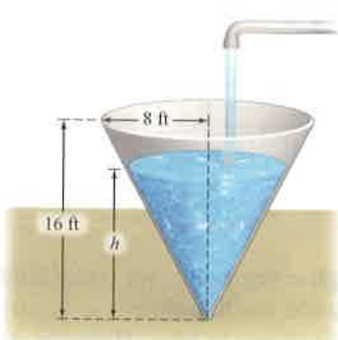
81. $f(x) = \dfrac{4}{x + 1}$, $g(x) = \dfrac{4 - x}{x}$

82. $f(x) = \dfrac{2}{1 - x}$, $g(x) = \dfrac{x - 2}{x}$

83. **Conversion Functions** The function $F(x) = \dfrac{x}{12}$ converts x inches to feet. The function $Y(x) = \dfrac{x}{3}$ converts x feet to yards. Explain the meaning of $(Y \circ F)(x)$.

84. **Conversion Functions** The function $F(x) = 3x$ converts x yards to feet. The function $I(x) = 12x$ converts x feet to inches. Explain the meaning of $(I \circ F)(x)$.

85. **Water Tank** A water tank has the shape of a right circular cone with height 16 feet and radius 8 feet. Water is running into the tank so that the radius r (in feet) of the surface of the water is given by $r = 1.5t$, where t is the time (in minutes) that the water has been running. See the accompanying diagram.



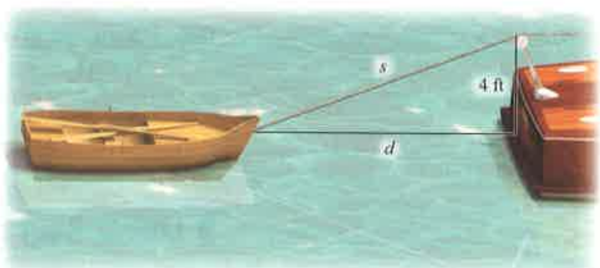
a. The area A of the surface of the water is $A = \pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t = 2$ minutes.

b. The volume V of the water is given by $V = \dfrac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t = 3$ minutes. (*Hint:* The height of the water in the cone is always twice the radius of the surface of the water.)

86. **Scaling a Rectangle** Rework Example 7 of this section with the scaling as follows: The upper right corner of the original rectangle is pulled to the *left* at 0.5 inch per second and downward at 0.2 inch per second.

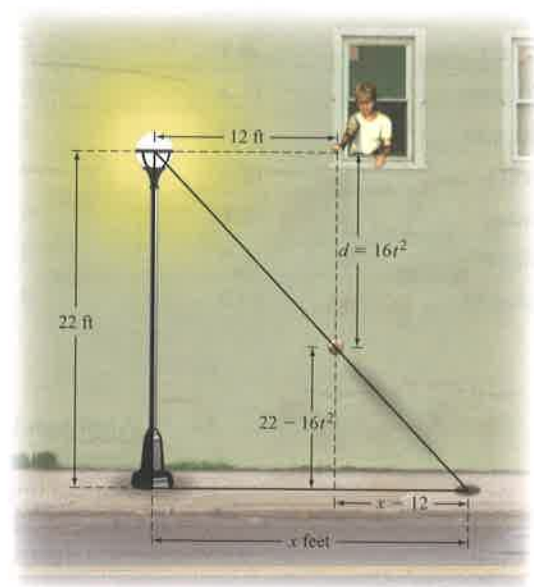
Enrichment Exercises

87. **Towing a Boat** A boat is towed by a rope that runs through a pulley that is 4 feet above the point where the rope is tied to the boat. The length (in feet) of the rope from the boat to the pulley is given by $s = 48 - t$, where t is the time in seconds that the boat has been in tow. The horizontal distance from the pulley to the boat is d.



a. Find $d(t)$. **b.** Evaluate $s(35)$ and $d(35)$.

88. **Shadow Position** The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 feet above the ground, as shown in the diagram. The distance d, in feet, the ball has dropped t seconds after it is released is given by $d = 16t^2$. Find the distance x, in feet, of the shadow from the base of the lamppost as a function of time t.



SECTION 2.7

Modeling Data Using Regression

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A14.

PS1. Find the slope and the y-intercept of the graph of $y = -\frac{x}{3} + 4$. [2.3]

PS2. Find the slope and the y-intercept of the graph of $3x - 4y = 12$. [2.3]

PS3. Find the equation of the line that has a slope of -0.45 and a y-intercept of $(0, 2.3)$. [2.3]

PS4. Find the equation of the line that passes through the point $P(3, -4)$ and has a slope of $-\frac{2}{3}$. [2.3]

PS5. If $f(x) = 3x^2 + 4x - 1$, find $f(2)$. [2.2]

PS6. You are given $P_1(2, -1)$ and $P_2(4, 14)$. If $f(x) = x^2 - 3$, find $|f(x_1) - y_1| + |f(x_2) - y_2|$. [2.2]

Linear Regression Models

The data in the table given below show the population of selected states and the number of professional sports teams (Major League Baseball, National Football League, National Basketball Association, Women's National Basketball Association, National Hockey League) in those states. A scatter plot of the data is shown in Figure 2.103 on page 237.

Number of Professional Sports Teams for Selected States

State	Population (millions)	Number of Teams	State	Population (millions)	Number of Teams
Arizona	6.4	5	Minnesota	5.3	5
California	37.3	16	New Jersey	8.8	3
Colorado	5.0	4	New York	19.4	9
Florida	18.8	9	North Carolina	9.5	3
Illinois	12.8	6	Ohio	11.5	7
Indiana	6.5	3	Pennsylvania	12.7	7
Michigan	10.0	4	Texas	25.1	10

Although there is no one line that passes through every point, we could find an approximate linear model for these data. For instance, the line shown in Figure 2.104 in red approximates the data better than the line shown in blue. However, as Figure 2.105 shows, there are many other lines we could draw that seem to approximate the data.