1.4 Solving Linear Systems

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Learning Target Solve linear systems in three variables.

Success Criteria
- I can visualize solutions of linear systems in three variables.
- I can solve linear systems in three variables algebraically.
- I can solve real-life problems using systems of equations in three variables.

EXPLORE IT! Solving Three-Variable Systems

Work with a partner. Consider the system shown.

$$y = 4x \quad \text{Equation 1}$$
$$x + 2y - z = 7 \quad \text{Equation 2}$$
$$3x - 4y + 2z = -9 \quad \text{Equation 3}$$

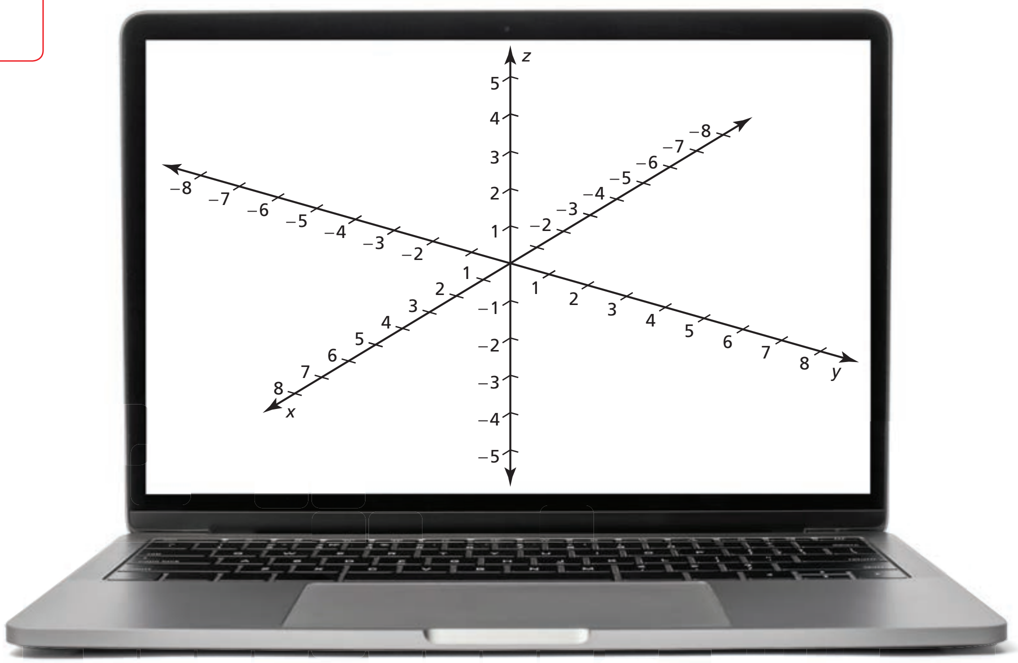
a. How is this linear system different from linear systems you have solved in previous courses?

Math Practice

Use Prior Results
How can you use previously established methods of solving linear systems to help you solve systems with three variables?

b. Explain a method you can use to solve the system. Solve the system and show how you can represent the solution.

c. The graph of each equation in the system is a plane in three-dimensional space. A three-dimensional coordinate system is shown below. How can the solution of this system be represented in the graph?



d. Can a linear system in three variables have no solution? infinitely many solutions? If so, sketch an example of what each type of solution might look like in three-dimensional space. Explain your reasoning.

Vocabulary

linear equation in three variables, *p. 28*
system of three linear equations, *p. 28*
solution of a system of three linear equations, *p. 28*
ordered triple, *p. 28*

Visualizing Solutions of Systems

A **linear equation in three variables** x, y, and z is an equation of the form $ax + by + cz = d$, where a, b, and c are not all zero.

Here is an example of a **system of three linear equations** in three variables.

$3x + 4y - 8z = -3$ Equation 1

$x + y + 5z = -12$ Equation 2

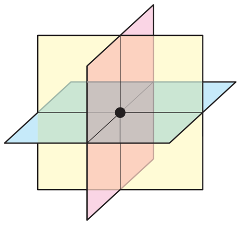
$4x - 2y + z = 10$ Equation 3

A **solution** of such a system is an **ordered triple** (x, y, z) whose coordinates make each equation true.

The graph of a linear equation in three variables is a plane in three-dimensional space. The graphs of three such equations that form a system are three planes whose intersection determines the number of solutions of the system.

Exactly One Solution

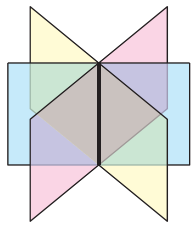
The planes intersect in a single point, which is the solution of the system.



Infinitely Many Solutions

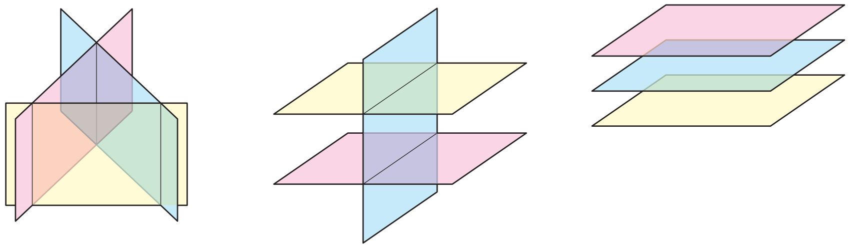
The planes intersect in a line. Every point on the line is a solution of the system.

The planes can also be the same plane. Every point in the plane is a solution of the system.



No Solution

There are no points in common with all three planes.



Solving Systems of Equations Algebraically

The algebraic methods you used to solve systems of linear equations in two variables can be extended to solve a system of linear equations in three variables.

KEY IDEA

Solving a Three-Variable System

Step 1 Rewrite the linear system in three variables as a linear system in two variables by using the substitution or elimination method.

Step 2 Solve the new linear system for both of its variables.

Step 3 Substitute the values found in Step 2 into one of the original equations and solve for the remaining variable.

When you obtain a false equation, such as $0 = 1$, in any of the steps, the system has *no solution*. When you do not obtain a false equation, but obtain an identity such as $0 = 0$, the system has *infinitely many solutions*.

EXAMPLE 1 Solving a Three-Variable System (One Solution)

WATCH

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Math Practice

Look for Structure
Why is y a convenient variable to eliminate? Are any other variables convenient to eliminate? Explain.

Solve the system.

$4x + 2y + 3z = 12$ Equation 1

$2x - 3y + 5z = -7$ Equation 2

$6x - y + 4z = -3$ Equation 3

SOLUTION

Step 1 Rewrite the system as a linear system in *two* variables.

$$\begin{array}{rl} 4x + 2y + 3z &= 12 \\ 12x - 2y + 8z &= -6 \\ \hline 16x \quad + 11z &= 6 \end{array}$$

Add 2 times Equation 3 to Equation 1 (to eliminate y).

New Equation 1

$$\begin{array}{rl} 2x - 3y + 5z &= -7 \\ -18x + 3y - 12z &= 9 \\ \hline -16x \quad - 7z &= 2 \end{array}$$

Add -3 times Equation 3 to Equation 2 (to eliminate y).

New Equation 2

Step 2 Solve the new linear system for both of its variables.

$$\begin{array}{rl} 16x + 11z &= 6 \\ -16x - 7z &= 2 \\ \hline 4z &= 8 \end{array}$$

Add new Equation 1 and new Equation 2.

$z = 2$ Solve for z.

$x = -1$ Substitute into new Equation 1 or 2 to find x.

Step 3 Substitute $x = -1$ and $z = 2$ into an original equation and solve for y.

$6x - y + 4z = -3$ Write original Equation 3.

$6(-1) - y + 4(2) = -3$ Substitute -1 for x and 2 for z.

$y = 5$ Solve for y.

▶ The solution is $x = -1$, $y = 5$, and $z = 2$, or the ordered triple $(-1, 5, 2)$.

Check

Equation 1

$4x + 2y + 3z = 12$

$4(-1) + 2(5) + 3(2) \stackrel{?}{=} 12$

$12 = 12$ ✓

Equation 2

$2x - 3y + 5z = -7$

$2(-1) - 3(5) + 5(2) \stackrel{?}{=} -7$

$-7 = -7$ ✓

Equation 3

$6x - y + 4z = -3$

$6(-1) - 5 + 4(2) \stackrel{?}{=} -3$

$-3 = -3$ ✓

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

1. **VOCABULARY** Describe two different ways you can represent the solution of a system of three linear equations in three variables.

Solve the system. Check your solution.

2. $x - 2y + z = -11$
$3x + 2y - z = 7$
$-x + 2y + 4z = -9$

3. $5x - 3y + 2z = 18$
$-2x + 4y - z = -11$
$-3x - 2y + 3z = 14$

4. $5x + 2y - 4z = -6$
$4x - 3y + 2z = 20$
$-x + 4y + 6z = 8$

EXAMPLE 2 Solving a Three-Variable System (No Solution)

WATCH

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Solve the system.

$$x + y + z = 2 \quad \text{Equation 1}$$
$$5x + 5y + 5z = 3 \quad \text{Equation 2}$$
$$4x + y - 3z = -6 \quad \text{Equation 3}$$

SOLUTION

Rewrite the system as a linear system in *two* variables.

$$\begin{array}{r} -5x - 5y - 5z = -10 \\ 5x + 5y + 5z = 3 \\ \hline 0 = -7 \end{array}$$

Add −5 times Equation 1 to Equation 2.

▶ Because you obtain a false equation, the original system has no solution.

EXAMPLE 3 Solving a Three-Variable System (Many Solutions)

WATCH

Solve the system.

$$x - y + z = -3 \quad \text{Equation 1}$$
$$x - y - z = -3 \quad \text{Equation 2}$$
$$5x - 5y + z = -15 \quad \text{Equation 3}$$

SOLUTION

ANOTHER WAY

Subtracting Equation 2 from Equation 1 gives $z = 0$. After substituting 0 for z in each equation, you can see that each is equivalent to $y = x + 3$.

Step 1 Rewrite the system as a linear system in *two* variables.

$$\begin{array}{r} x - y + z = -3 \\ x - y - z = -3 \\ \hline 2x - 2y \quad = -6 \end{array}$$

Add Equation 1 to Equation 2 (to eliminate z).

New Equation 2

$$\begin{array}{r} x - y - z = -3 \\ 5x - 5y + z = -15 \\ \hline 6x - 6y \quad = -18 \end{array}$$

Add Equation 2 to Equation 3 (to eliminate z).

New Equation 3

Step 2 Solve the new linear system for both of its variables.

$$\begin{array}{r} -6x + 6y = 18 \\ 6x - 6y = -18 \\ \hline 0 = 0 \end{array}$$

Add −3 times new Equation 2 to new Equation 3.

Because you obtain the identity $0 = 0$, the system has infinitely many solutions.

Step 3 Describe the solutions of the system using an ordered triple. One way to do this is to solve new Equation 2 for y to obtain $y = x + 3$. Then substitute $x + 3$ for y in original Equation 1 to obtain $z = 0$.

▶ So, any ordered triple of the form $(x, x + 3, 0)$ is a solution of the system.

SELF-ASSESSMENT

1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

Solve the system. Check your solution, if possible.

5. $x + y - z = -1$
$4x + 4y - 4z = -2$
$3x + 2y + z = 0$

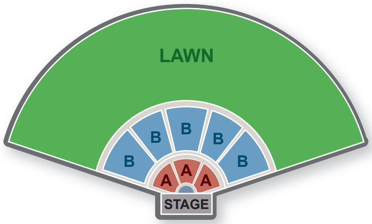
6. $-2x - 6y - 3z = 15$
$3x + 9y - 3z = 0$
$4x - 2y - 5z = 3$

7. $x + y + z = 8$
$x - y + z = 8$
$2x + y + 2z = 16$

8. Describe the solutions of the system in Example 3 using an ordered triple in terms of y.

Solving Real-Life Problems

EXAMPLE 4 **Modeling Real Life** WATCH



An amphitheater charges \$75 for each seat in Section A, \$55 for each seat in Section B, and \$30 for each lawn seat. There are three times as many seats in Section B as in Section A. The revenue from selling all 23,000 seats is \$870,000. How many seats are in each section of the amphitheater?

SOLUTION

Step 1 Write a verbal model for the situation.

[Number of seats in B, y] = 3 • [Number of seats in A, x]

[Number of seats in A, x] + [Number of seats in B, y] + [Number of lawn seats, z] = [Total number of seats]

75 • [Number of seats in A, x] + 55 • [Number of seats in B, y] + 30 • [Number of lawn seats, z] = [Total revenue]

Step 2 Write a system of equations.

$y = 3x$ Equation 1

$x + y + z = 23{,}000$ Equation 2

$75x + 55y + 30z = 870{,}000$ Equation 3

Step 3 Rewrite the system in Step 2 as a linear system in *two* variables by substituting $3x$ for y in Equations 2 and 3.

$x + y + z = 23{,}000$ Write Equation 2.

$x + 3x + z = 23{,}000$ Substitute $3x$ for y.

$4x + z = 23{,}000$ New Equation 2

$75x + 55y + 30z = 870{,}000$ Write Equation 3.

$75x + 55(3x) + 30z = 870{,}000$ Substitute $3x$ for y.

$240x + 30z = 870{,}000$ New Equation 3

Step 4 Solve the new linear system for both of its variables.

$$\begin{aligned} -120x - 30z &= -690{,}000 \\ 240x + 30z &= 870{,}000 \\ \hline 120x \qquad &= 180{,}000 \end{aligned}$$

Add −30 times new Equation 2 to new Equation 3.

$x = 1500$ Solve for x.

$y = 4500$ Substitute into Equation 1 to find y.

$z = 17{,}000$ Substitute into Equation 2 to find z.

STUDY TIP

When substituting to find values of other variables, choose the equations that are easiest to use.

▶ The solution is $x = 1500$, $y = 4500$, and $z = 17{,}000$, or $(1500, 4500, 17{,}000)$. So, there are 1500 seats in Section A, 4500 seats in Section B, and 17,000 lawn seats.

SELF-ASSESSMENT 1 I do not understand. 2 I can do it with help. 3 I can do it on my own. 4 I can teach someone else.

9. **MP REASONING** Another concert is held in the amphitheater. On the first day, 10,000 tickets are sold for \$356,000. On the second day, 7640 tickets are sold for \$284,300. How many seats remain in each section after the first two days of ticket sales when equal numbers of tickets for Sections A and B are sold on the first day, and twice as many tickets are sold for Section B as for Section A on the second day?

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In Exercises 1–12, solve the system using the elimination method. *Examples 1, 2, and 3*

1. $x + y - 2z = 5$
$-x + 2y + z = 2$
$2x + 3y - z = 9$

2. $x + 4y - 6z = -1$
$2x - y + 2z = -7$
$-x + 2y - 4z = 5$

3. $3x - y + 2z = 4$
$6x - 2y + 4z = -8$
$2x - y + 3z = 10$

4. $5x + y - z = 6$
$x + y + z = 2$
$12x + 4y = 10$

5. $2x + y - z = 9$
$-x + 6y + 2z = -17$
$5x + 7y + z = 4$

6. $3x + 2y - z = 8$
$-3x + 4y + 5z = -14$
$x - 3y + 4z = -14$

7. $x + 3y - z = 2$
$x + y - z = 0$
$3x + 2y - 3z = -1$

8. $x + 2y - z = 3$
$-2x - y + z = -1$
$6x - 3y - z = -7$

9. $2x + 2y + 5z = 6$
$2x - y + z = 2$
$2x + 4y - 3z = 14$

10. $-2x - 3y + z = -6$
$x + y - z = 5$
$7x + 8y - 6z = 31$

11. $x + 2y + 3z = 4$
$-3x + 2y - z = 12$
$-2x - 2y - 4z = -14$

12. $3x + 2y - 3z = -2$
$7x - 2y + 5z = -14$
$2x + 4y + z = 6$

ERROR ANALYSIS In Exercises 13 and 14, describe and correct the error in the first step of solving the system of linear equations.

$4x - y + 2z = -18$
$-x + 2y + z = 11$
$3x + 3y - 4z = 44$

13.
✗
$$\begin{array}{r} 4x - y + 2z = -18 \\ -4x + 2y + z = 11 \\ \hline y + 3z = -7 \end{array}$$

14.
✗
$$\begin{array}{r} 12x - 3y + 6z = -18 \\ 3x + 3y - 4z = 44 \\ \hline 15x + 2z = 26 \end{array}$$

15. MODELING REAL LIFE Three orders are placed at a pizza shop. Two small pizzas, a liter of soda, and a salad cost $14; one small pizza, a liter of soda, and three salads cost $15; and three small pizzas, a liter of soda, and two salads cost $22. How much does each item cost?

16. MODELING REAL LIFE Sam's Furniture Store places the following advertisement in the local newspaper. What is the price of each piece of furniture?

In Exercises 17–26, solve the system of linear equations using the substitution method. *Example 4*

17. $2x - 3y + z = 10$
$y + 2z = 13$
$z = 5$

18. $x = 4$
$x + y = -6$
$4x - 3y + 2z = 26$

19. $y = 2x - 6z + 1$
$3x + 2y + 5z = 16$
$7x + 3y - 4z = 11$

20. $-x + 5y + 3z = 2$
$x = 6y + 2z - 8$
$3x - 2y - 4z = 18$

21. $x + y + z = 4$
$5x + 5y + 5z = 12$
$x - 4y + z = 9$

22. $x + 2y = -1$
$-x + 3y + 2z = -4$
$-x + y - 4z = 10$

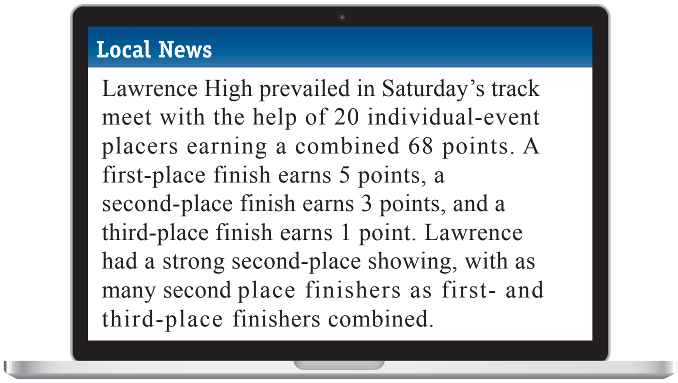
23. $x + y - z = 4$
$3x + 2y + 4z = 17$
$-x + 5y + z = 8$

24. $2x - y - z = 15$
$4x + 5y + 2z = 10$
$-x - 4y + 3z = -20$

25. $4x + y + 5z = 5$
$8x + 2y + 10z = 10$
$x - y - 2z = -2$

26. $x + 2y - z = 3$
$2x + 4y - 2z = 6$
$-x - 2y + z = -6$

27. MODELING REAL LIFE The results of a track meet are described in the article below. How many athletes from Lawrence High finished in each place?



28. MP **PROBLEM SOLVING** The percent of left-handed people in the world is one-tenth the percent of right-handed people. The percent of right-handed people is nine times the percent of left-handed people and ambidextrous people combined. What percent of people are ambidextrous?

29. **WRITING** Explain when it might be more convenient to use the elimination method than the substitution method to solve a linear system. Give an example to support your claim.

30. **COLLEGE PREP** Which of the following systems has infinitely many solutions?

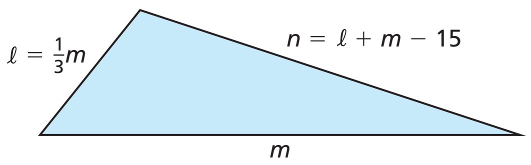
System A	System B
$3x + 2y - z = -5$	$x + 3y + 5z = 13$
$-2x + 5y + 3z = 12$	$-x - 6y + z = -4$
$-12x - 8y + 4z = -20$	$x + 5y + z = 7$

Ⓐ System A Ⓑ System B

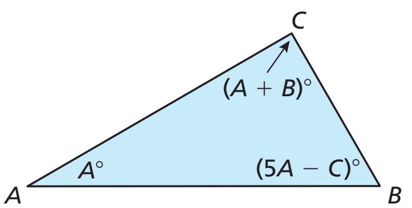
Ⓒ both Ⓓ neither

CONNECTING CONCEPTS In Exercises 31 and 32, write and use a linear system to answer the question.

31. The triangle has a perimeter of 65 feet. What are the lengths of sides ℓ, m, and n?



32. What are the measures of angles A, B, and C?



33. MP **REPEATED REASONING** Using what you know about solving linear systems in two and three variables, plan a strategy for how to solve a system that has *four* linear equations in *four* variables.

34. **CRITICAL THINKING** Find the values of a, b, and c so that the linear system shown has $(-1, 2, -3)$ as its only solution. Explain your reasoning.

$$x + 2y - 3z = a$$
$$-x - y + z = b$$
$$2x + 3y - 2z = c$$

35. **MAKING AN ARGUMENT** A linear system in three variables has no solution. Your friend concludes that it is not possible for two of the three equations to have any points in common. Is your friend correct? Explain your reasoning.

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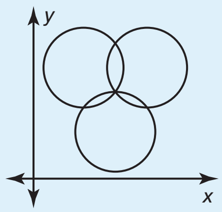
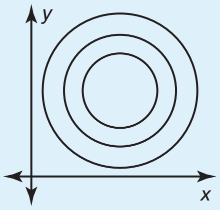
36. **HOW DO YOU SEE IT?**
Determine whether the system of equations that represents the circles has *no solution*, *one solution*, or *infinitely many solutions*. Explain your reasoning.

a.



b.



37. **OPEN-ENDED** Consider the system of linear equations below. Choose nonzero values for a, b, and c so the system satisfies each condition. Explain your reasoning.

$$x + y + z = 2$$
$$ax + by + cz = 10$$
$$x - 2y + z = 4$$

a. The system has no solution.

b. The system has exactly one solution.

c. The system has infinitely many solutions.

38. MP **PROBLEM SOLVING** A florist must make 5 identical bouquets. The budget is \$160, and each bouquet must have 12 flowers. Roses cost \$2.50 each, lilies cost \$4 each, and irises cost \$2 each. The florist wants twice as many roses as the other two types of flowers combined.

a. Assuming the florist plans to use the entire budget, how many of each type of flower should be in each bouquet?

b. Suppose there is no limitation on the total cost of the bouquets. Does the problem still have exactly one solution? If so, find the solution. If not, give three possible solutions.

39. **ANALYZING RELATIONSHIPS** Determine which arrangement(s) of the integers −5, 2, and 3 produce a linear system with a solution that consists of only integers. Justify your answer.

$$x - 3y + 6z = 21$$
$$\square x + \square y + \square z = -30$$
$$2x - 5y + 2z = -6$$

40. THOUGHT PROVOKING Solve the system.

$$-\frac{1}{x} - \frac{1}{y} + \frac{6}{z} = -3$$
$$\frac{6}{x} + \frac{5}{y} - \frac{12}{z} = 11$$
$$\frac{3}{x} + \frac{2}{y} - \frac{2}{z} = 4$$

41. DIG DEEPER The scales shown are used to compare the weights of apples, tangerines, grapefruits, and bananas. How many tangerines will balance one apple? Justify your answer using a linear system.

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REVIEW & REFRESH

WATCH

In Exercises 42–45, write a function g described by the given transformation of $f(x) = |x| - 5$.

42. translation 2 units to the left

43. reflection in the x-axis

44. translation 4 units up

45. vertical stretch by a factor of 3

In Exercises 46 and 47, find the product.

46. $(x - 2)^2$ **47.** $(3m + 1)^2$

In Exercises 48–53, solve the inequality. Graph the solution.

48. $6 + w > -15$ **49.** $5y - 2 \le 28$

50. $6.5 \ge -\frac{n}{3}$ **51.** $2(x - 4) > 6x - 16$

52. $|2h + 3| - 3 < -1$

53. $4t + 21 < -7$ *or* $-\frac{1}{2}t \le 2$

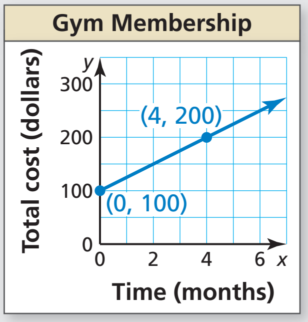
54. MODELING REAL LIFE The table shows the total distance a new car travels each month after it is purchased. What type of function can you use to model the data? Estimate the mileage after 1 year.

Time (months), x	Distance (miles), y
0	0
2	2300
5	5750
6	6900
9	10,350

55. Solve $\frac{4}{x} = \frac{16}{10}$.

56. OPEN-ENDED Create a nonnumerical data set that has more than one mode.

57. Use the graph to write an equation of the line and interpret the slope.

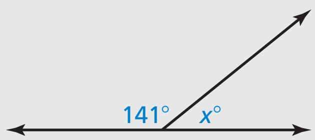


In Exercises 58–61, simplify the expression. Write your answer using only positive exponents.

58. $\frac{k^{10}}{k^2}$ **59.** $b^{-12} \bullet b^9$

60. $(2c)^6$ **61.** $\left(\frac{z^7}{3}\right)^{-4}$

62. Find the value of x.



In Exercises 63 and 64, solve the system.

63. $-3x + y - 2z = 8$
$2x + 2y + 4z = 13$
$x = y - 5$

64. $4x + 3y - 5z = 7$
$x + 2y - z = 1$
$-2x - 4y + z = -4$

65. Explain how to determine the number of real solutions of $x^2 = 100$ without solving.